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ON CERTAIN SUBCLASS OF MULTIVALENT ANALYTIC FUNCTIONS ASSOCIATED WITH ERDELYI-KOBER TYPE INTEGRAL OPERATOR

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Abstract. In this paper, we introduce a certain subclasses of multivalent uniformly starlike analytic functions by making use of Erdeyi-Kober type integral operator. Further, we determine coefficient estimates and Holder's inequality results. Also, results for family of class preserving integral operators are obtaind for the class $US_p^*T(n,a,c,\mu;\alpha,\beta)$.

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1. Introduction

Let $A(p,n)$ denote the class of functions of the form

$$f(z)=z^p+\sum_{k=n}^{\infty}a_{k+p}z^{k+p} \qquad (n,p\in\mathbb{N}=\{1,2,...\}), \tag{1}$$

which are analytic and p-valent in open unit disc $U=\{z:z\in\mathbb{C},|z|<1\}$. Also, we note that $A(1,1)=A$, that is the class of analytic univalent functions.

A function $f\in A(p,n)$ is said to be in the class $S(p,n,\alpha)$ of p-valent starlike functions of order α if it satisfies the condition

$$Re\left(\frac{zf'(z)}{f(z)}\right)>\alpha \qquad (z\in U;\ 0\leq\alpha<p). \tag{2}$$

A function $f\in A(p,n)$ is said to be in the class $K(p,n,\alpha)$ of p-valent convex functions of order α if it satisfies the condition

$$Re\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha \qquad (z\in U;\ 0\leq\alpha<p). \tag{3}$$

The classes $S(p,n,\alpha)$ and $K(p,n,\alpha)$ were studied by Owa [18]. The class $S^*(p,\alpha)=S(p,1,\alpha)$ was considered by Patil and Thakare [19].

We denote by $T(p,n)$ the subclass of $A(p,n)$ consisting of functions of the form

$$f(z) = z^p - \sum_{k=n}^{\infty} a_{k+p} z^{k+p} \qquad (a_{k+p} \geq 0;\ n,\ p \in \mathbb{N} = \{1, 2, ...\}), \tag{4}$$

and define two further classes $T^*(p,n,\alpha)$ and $C(p,n,\alpha)$ by

$$T^*(p,n,\alpha) = S(p,n,\alpha) \cap T(p,n), \qquad C(p,n,\alpha) := K(p,n,\alpha) \cap T(p,n).$$

Further, the classes

$$T^*(p,\alpha) = S^*(p,\alpha) \cap T(p,n), \quad C(p,\alpha) := K(p,\alpha) \cap T(p,n).$$

The function $f(z) \in T(p,n)$ given by (4) is said to be $\beta-$uniformly starlike of order α $(-p \leq \alpha < p)$ and $\beta \geq 0$ denote by $US_p^*T(n,\alpha,\beta)$ if and only if

$$Re\left(\frac{zf'(z)}{f(z)} - \alpha\right) > \beta\left|\frac{zf'(z)}{f(z)} - p\right| \quad (z \in U). \tag{5}$$

Also, function $f(z)$ is said to be $\beta-$uniformly convex of order α denoted by $UC_pV(n,\alpha,\beta)$ [10] if and only if

$$Re\left(1 + \frac{zf''(z)}{f'(z)} - \alpha\right) > \beta\left|\frac{zf''(z)}{f'(z)} - (p-1)\right| \quad (z \in U). \tag{6}$$

Note that, the classes $US_1^*T(1,\alpha,\beta) = US^*T(\alpha,\beta)$ and $UC_1V(1,\alpha,\beta) = UCV(\alpha,\beta)$ are introduced and studied by Bharati et al. [4]. In particular, the classes $UCV(0,1)$ and $UCV(0,\beta)$ were introduced by Goodman [7] and Kanas and Wisniowska [9].

Definition 1. [2] *For $f \in A(p,n)$, $p,\ n \in \mathbb{N}$, $\mu > 0$, $a, c \in \mathbb{C}$, $Re(a) \geq -\mu p$ and $Re(c-a) > 0$, El-Ashwah and Drbuk define the differ-integral operator which called Erdelyi-Kober type integral operator $I_{p,\mu}^{a,c} : A(p,n) \longrightarrow A(p,n)$ as follows*

$$I_{p,\mu}^{a,c} f(z) = z^p + \sum_{k=n}^{\infty} \Psi_{p,\mu}^{a,c}(k)\, a_{k+p} z^{k+p}, \tag{7}$$

where

$$\Psi_{p,\mu}^{a,c}(k) = \frac{\Gamma(c+\mu p)}{\Gamma(a+\mu p)} \frac{\Gamma(a+\mu(k+p))}{\Gamma(c+\mu(k+p))}.$$

If $a = c$, then we have $I_{p,\mu}^{a,a} f(z) = f(z)$. It easily to verify that,

$$z\left(I_{p,\mu}^{a,c} f(z)\right)' = \frac{a+\mu p}{\mu} I_{p,\mu}^{a+1,c} f(z) - \frac{a}{\mu} I_{p,\mu}^{a,c} f(z).$$

We also note that the operator $I_{p,\mu}^{a,c} f(z)$ generalizes several previously studied familiar operators and we will mention some of the interesting particular cases as follows:

(1) For $p = 1$, we can obtain the operator $I_{\mu}^{a,c} f(z)$ defined in [11, ch.5] (see also [20] and [21, with $m = 0$]);

(2) For $a = \beta$, $c = \beta+1$ and $\mu = 1$, we obtain the familiar integral operator $I_{\beta,p} f(z)$ $(\beta > -p)$ which studies by Saitoh et al. [23];

(3) For $a = \beta$, $c = \alpha + \beta - \gamma + 1$ and $\mu = 1$, we obtain the operator $R_{\beta,p}^{\alpha,\gamma} f(z)$ ($\gamma > 0$; $\alpha \geq \gamma - 1$; $\beta > -1$) studied by Aouf et al. [1];

(4) For $p = 1$, $a = \beta$, $c = \alpha + \beta$ and $\mu = 1$, we obtain the operator $Q_{\beta}^{\alpha} f(z)$ $(\alpha \geq 0,\ \beta > -1)$ studied by Jung et al. [8];

(5) For $p = 1$, $a = \alpha - 1$, $c = \beta - 1$, and $\mu = 1$, we obtain the operator $l(\alpha, \beta) f(z)$ (α, $\beta \in \mathbb{C} \setminus \mathbb{Z}_0$, $\mathbb{Z}_0 = \{0, -1, -2, ...\}$ studied by Carlson and Shafer [5];

(6) For $p = 1$, $a = \rho - 1, c = \ell$ and $\mu = 1$, we obtain the operator $I_{\rho,\ell} f(z)$ ($\rho > 0$; $\ell > -1$) studied by Choi et al. [6];

(7) For $p = 1$, $a = \alpha$, $c = 0$ and $\mu = 1$, we obtain the operator $D^{\alpha} f(z)$ $(\alpha > 1)$ studied by Ruscheweyh [22];

(8) For $p = 1$, $a = 1$, $c = n$ and $\mu = 1$, we obtain the operator $I_n f(z)$ $(n \in N_0)$ studied by Noor and Noor [17]; and Noor [16];

(10) For $p = 1$, $a = \beta$, $c = \beta + 1$ and $\mu = 1$, we obtain the integral operator $I_{\beta,1}$ which studied by Bernardi [3];

(11) For $p = 1, a = 1$, $c = 2$ and $\mu = 1$, we obtain the integral operator $I_{1,1} = I$ which studied by Libera [12] and Livingston [14].

Now, we introduced a new subclasses of $p-$valent functions and discussed some interesting geometric properties of this generalized function class.

Definition 2. *A function $f \in A(p,n)$ is said to be in the class $US_p^*(n,a,c;\mu,\alpha,\beta)$ if it satisfies the inequality*

$$Re\left(\frac{z\left(I_{p,\mu}^{a,c}f(z)\right)'}{I_{p,\mu}^{a,c}f(z)}-\alpha\right)>\beta\left|\frac{z\left(I_{p,\mu}^{a,c}f(z)\right)'}{I_{p,\mu}^{a,c}f(z)}-p\right|,(\ z\in U),$$

which is equivalent to

$$Re\left(\frac{I_{p,\mu}^{a+1,c}f(z)}{I_{p,\mu}^{a,c}f(z)}-\frac{a+\alpha\mu}{a+\mu p}\right)>\beta\left|\frac{I_{p,\mu}^{a+1,c}f(z)}{I_{p,\mu}^{a,c}f(z)}-1\right|,\quad (z\in U) \tag{8}$$

for some $-p \leq \alpha < p$, $\beta \geq 0, p, n \in \mathbb{N}, \mu > 0, a, c \in \mathbb{C}, Re(a) \geq -\mu p$ and $Re(c-a) > 0$.
Furthermore, we define the class $US_p^*T(n,a,c;\mu,\alpha,\beta)$ by $US_p^*(n,a,c;\mu,\alpha,\beta) \cap T(p,n)$.

The main object of this work is to determine coefficient estimates for the analytic functions class $US_p^*T(n,a,c;\mu,\alpha,\beta)$. We study some interesting Holder's inequality for the class $US_p^*T(n,a,c;\mu,\alpha,\beta)$. Also, the family of class preserving integral operators for functions f in the class $US_p^*T(n,a,c;\mu,\alpha,\beta)$ are obtained.

2. Coefficient Inequalities

Unless otherwise mention, we assume in the reminder of this paper that $\mu > 0$, $a, c \in R$, $a > -\mu p$, $(a-c) > 0$, $-p \leq \alpha < p$, $\beta \geq 0, p, n \in \mathbb{N}$. First, we give a coefficients inequality for the class $US_p^*(n,a,c;\mu,\alpha,\beta)$.

Theorem 1. *A sufficient condition for a function $f(z)$ of the form (1) to be in $US_p^*(n,a,c;\mu,\alpha,\beta)$ is*

$$\sum_{k=n}^{\infty}\left[k(1+\beta)+(p-\alpha)\right]\Psi_{p,\mu}^{a,c}(k)\left|a_{k+p}\right| \leq (p-\alpha) \tag{9}$$

where

$$\Psi_{p,\mu}^{a,c}(k)=\frac{\Gamma(c+\mu p)}{\Gamma(a+\mu p)}\frac{\Gamma(a+\mu(k+p))}{\Gamma(c+\mu(k+p))}.$$

Proof. It is sufficient to show that

$$\beta\left|\frac{z(I_{p,\mu}^{a,c}f(z))'}{I_{p,\mu}^{a,c}f(z)}-p\right|-Re\left(\frac{z(I_{p,\mu}^{a,c}f(z))'}{I_{p,\mu}^{a,c}f(z)}-p\right)\leq p-\alpha.$$

We have

$$\begin{aligned}
& \beta\left|\frac{z(I_{p,\mu}^{a,c}f(z))'}{I_{p,\mu}^{a,c}f(z)}-p\right|-Re\left(\frac{z(I_{p,\mu}^{a,c}f(z))'}{I_{p,\mu}^{a,c}f(z)}-p\right) \\
\leq\ & (1+\beta)\left|\frac{z(I_{p,\mu}^{a,c}f(z))'}{I_{p,\mu}^{a,c}f(z)}-p\right| \\
\leq\ & (1+\beta)\left|\frac{pz^p+\sum_{k=n}^{\infty}(k+p)\Psi_{p,\mu}^{a,c}(k)a_{k+p}z^{k+p}}{z^p+\sum_{k=n}^{\infty}\Psi_{p,\mu}^{a,c}(k)a_{k+p}z^{k+p}}-p\right| \\
\leq\ & (1+\beta)\frac{\sum_{k=n}^{\infty}k\Psi_{p,\mu}^{a,c}(k)a_{k+p}z^k}{1-\sum_{k=n}^{\infty}\Psi_{p,\mu}^{a,c}(k)a_{k+p}z^k}.
\end{aligned}$$

The last expression is bounded by $(p-\alpha)$, if

$$\sum_{k=n}^{\infty}\left[k(1+\beta)+(p-\alpha)\right]\Psi_{p,\mu}^{a,c}(k)\left|a_{k+p}\right|\leq(p-\alpha),$$

and hence the proof is completed.

Theorem 2. *A necessary and sufficient condition for a function $f(z)$ of the form (4) to be in $US_p^*T(n,a,c;\mu,\alpha,\beta)$ is*

$$\sum_{k=n}^{\infty}\left[k(1+\beta)+(p-\alpha)\right]\Psi_{p,\mu}^{a,c}(k)\left|a_{k+p}\right|\leq(p-\alpha).$$

Proof. The sufficient condition follows from Theorem 1. To prove the necessity, let $f\in US_p^*T(n,a,c;\mu,\alpha,\beta)$ and z is real, then

$$\begin{aligned}
& \frac{p-\sum_{k=n}^{\infty}(k+p)\Psi_{p,\mu}^{a,c}(k)a_{k+p}z^k}{1-\sum_{k=n}^{\infty}\Psi_{p,\mu}^{a,c}(k)a_{k+p}z^k}-\alpha \\
\geq\ & \beta\left|\frac{p-\sum_{k=n}^{\infty}(k+p)\Psi_{p,\mu}^{a,c}(k)a_{k+p}z^k-p+\sum_{k=n}^{\infty}p\Psi_{p,\mu}^{a,c}(k)a_{k+p}z^k}{1-\sum_{k=n}^{\infty}\Psi_{p,\mu}^{a,c}(k)a_{k+p}z^k}\right|.
\end{aligned}$$

Let $z\to1^-$, we obtain

$$\frac{p-\sum_{k=n}^{\infty}(k+p)\Psi_{p,\mu}^{a,c}(k)a_{k+p}}{1-\sum_{k=n}^{\infty}\Psi_{p,\mu}^{a,c}(k)a_{k+p}}-\alpha\geq\beta\left|\frac{-\sum_{k=n}^{\infty}k\Psi_{p,\mu}^{a,c}(k)a_{k+p}}{1-\sum_{k=n}^{\infty}\Psi_{p,\mu}^{a,c}(k)a_{k+p}}\right|$$

or, equivalently

$$p-\sum_{k=n}^{\infty}(k+p)\Psi_{p,\mu}^{a,c}(k)\left|a_{k+p}\right|-\alpha\left(1-\sum_{k=n}^{\infty}\Psi_{p,\mu}^{a,c}(k)\left|a_{k+p}\right|\right)\geq\beta\sum_{k=n}^{\infty}k\Psi_{p,\mu}^{a,c}(k)\left|a_{k+p}\right|.$$

Thus, we have

$$\sum_{k=n}^{\infty}\left[k\left(1+\beta\right)+\left(p-\alpha\right)\right]\Psi_{p,\mu}^{a,c}\left(k\right)\left|a_{k+p}\right|\leq\left(p-\alpha\right).$$

Then the proof is completed.

Corollary 3. *If $f\left(z\right)$ of the form (4) is in $US_{p}^{*}T\left(n,a,c;\mu,\alpha,\beta\right)$, then*

$$a_{p+k}\leq\frac{\left(p-\alpha\right)}{\left[k\left(1+\beta\right)+\left(p-\alpha\right)\right]\Psi_{p,\mu}^{a,c}\left(k\right)},\qquad\left(k\geq n,n\in\mathbb{N}\right).\tag{10}$$

with equality only for the function

$$f\left(z\right)=z^{p}-\frac{\left(p-\alpha\right)}{\left[k\left(1+\beta\right)+p-\alpha\right]\Psi_{p,\mu}^{a,c}\left(k\right)}z^{p+k},\qquad\left(k\geq n,n\in\mathbb{N}\right).\tag{11}$$

3. Holder's inequality

For function $f_{j}\left(z\right)\in T\left(p,n\right)$ are given by

$$f_{j}\left(z\right)=z^{p}-\sum_{k=n}^{\infty}a_{k+p,j}z^{k+p}\qquad\left(a_{k+p,j}\geq0,j=1,2,3,...,m\right).$$

Now, we define the modified Hadmard product of $f_{j}\left(z\right)$ and the generalization of the modified Hadmard product as follows

$$G_{m}\left(z\right)=z^{p}-\sum_{k=n}^{\infty}\left(\prod_{j=1}^{m}a_{k+p,j}\right)z^{k+p}$$

and

$$H_{m}\left(z\right)=z^{p}-\sum_{k=n}^{\infty}\left(\prod_{j=1}^{m}a_{k+p,j}^{q_{j}}\right)z^{k+p},\qquad\left(q_{j}>0,\ j=1,2,3,...,m\right).$$

(i) For $m=2$, then $G_{2}\left(z\right)=\left(f_{1}*f_{2}\right)\left(z\right).$

(ii) For $q_{j}=1$, we have $G_{m}\left(z\right)=H_{m}\left(z\right)$.

Further, for functions $f_j(z)$ $(j=1,2,...,m)$, the familiar Holder's inequality assumes the following form

$$\sum_{k=n}^{\infty}\left(\prod_{j=1}^{m} a_{k+p,j}\right) \leq \prod_{j=1}^{m}\left(\sum_{k=n}^{\infty}(a_{k+p,j})^{q_j}\right)^{\frac{1}{q_j}}, \quad \left(q_j>1,\ \sum_{j=1}^{m}\frac{1}{q_j}\geq 1,\ j=1,2,3,...,m\right).$$

Recently, Nishiwaki and Owa [15] have studied some results of Holder's inequalities for a subclass of p-valent starlike and convex function.

Theorem 4. *Let $f_j(z)\in US_p^*T(n,a,c;\mu,\alpha_j,\beta)$ $(j=1,2,3,...,m)$, then $H_m(z)\in US_p^*T(n,a,c;\mu,\eta,\beta)$,*

$$\eta \leq p-\frac{[k(1+\beta)]\prod_{j=1}^{m}(p-\alpha_j)^{s_j}}{\prod_{j=1}^{m}[k(1+\beta)+(p-\alpha_j)]^{s_j}\left[\Psi_{p,\mu}^{a,c}(k)\right]^{s_j-1}-\prod_{j=1}^{m}(p-\alpha_j)^{s_j}},$$

where $k\geq n$, $s_j\geq\frac{1}{q_j}$, $q_j>1$, $\sum_{j=1}^{m}\frac{1}{q_j}\geq 1$; $j=1,2,3,...,m$.

Proof. Let $f_j(z)\in US_p^*T(n,a,c;\mu,\alpha_j,\beta)$, then

$$\sum_{k=n}^{\infty}\frac{[k(1+\beta)+(p-\alpha_j)]\,\Psi_{p,\mu}^{a,c}(k)}{(p-\alpha_j)}a_{k+p,j}\leq 1. \tag{12}$$

which implies

$$\left(\sum_{k=n}^{\infty}\frac{[k(1+\beta)+(p-\alpha_j)]\,\Psi_{p,\mu}^{a,c}(k)}{(p-\alpha_j)}a_{k+p,j}\right)^{\frac{1}{q_j}}\leq 1, \qquad \left(q_j>1,\ \sum_{j=1}^{m}\frac{1}{q_j}\geq 1\right). \tag{13}$$

From (13), we have

$$\prod_{j=1}^{m}\left(\sum_{k=n}^{\infty}\frac{[k(1+\beta)+(p-\alpha_j)]\,\Psi_{p,\mu}^{a,c}(k)}{(p-\alpha_j)}a_{k+p,j}\right)^{\frac{1}{q_j}}\leq 1.$$

Applying Holder's inequality, we find that

$$\sum_{k=n}^{\infty}\left[\prod_{j=1}^{m}\left(\frac{[k(1+\beta)+(p-\alpha_j)]\,\Psi_{p,\mu}^{a,c}(k)}{(p-\alpha_j)}\right)^{\frac{1}{q_j}}(a_{k+p,j})^{\frac{1}{q_j}}\right]\leq 1.$$

Thus, we have to determine the largest η such that

$$\sum_{k=n}^{\infty}\frac{[k(1+\beta)+(p-\eta)]\,\Psi_{p,\mu}^{a,c}(k)}{(p-\eta)}\left(\prod_{j=1}^{m}a_{k+p,j}^{s_j}\right)\leq 1.$$

That is

$$\sum_{k=n}^{\infty}\frac{[k(1+\beta)+(p-\eta)]\,\Psi_{p,\mu}^{a,c}(k)}{(p-\eta)}\left(\prod_{j=1}^{m}a_{k+p,j}^{s_j}\right)\leq\sum_{k=n}^{\infty}\left[\prod_{j=1}^{m}\left(\frac{[k(1+\beta)+(p-\alpha_j)]\,\Psi_{p,\mu}^{a,c}(k)}{(p-\alpha_j)}\right)^{\frac{1}{q_j}}(a_{k+p,j})^{\frac{1}{q_j}}\right].$$

Therefore, we need to find the largest η such that

$$\frac{[k(1+\beta)+(p-\eta)]\,\Psi_{p,\mu}^{a,c}(k)}{(p-\eta)}\left(\prod_{j=1}^{m}(a_{k+p,j})^{s_j-\frac{1}{q_j}}\right)\leq\prod_{j=1}^{m}\left(\frac{[k(1+\beta)+(p-\alpha_j)]\,\Psi_{p,\mu}^{a,c}(k)}{(p-\alpha_j)}\right)^{\frac{1}{q_j}},\quad(k\geq n).$$

Since

$$\prod_{j=1}^{m}\left(\frac{[k(1+\beta)+(p-\alpha_j)]\,\Psi_{p,\mu}^{a,c}(k)}{(p-\alpha_j)}\right)^{s_j-\frac{1}{q_j}}(a_{k+p,j})^{s_j-\frac{1}{q_j}}\leq 1,\quad\left(s_j-\frac{1}{q_j}\geq 0\right).$$

We see that,

$$\prod_{j=1}^{m}(a_{k+p,j})^{s_j-\frac{1}{q_j}}\leq\frac{1}{\prod_{j=1}^{m}\left(\frac{[k(1+\beta)+(p-\alpha_j)]\Psi_{p,\mu}^{a,c}(k)}{(p-\alpha_j)}\right)^{s_j-\frac{1}{q_j}}}.$$

This implies that

$$\frac{[k(1+\beta)+(p-\eta)]\,\Psi_{p,\mu}^{a,c}(k)}{(p-\eta)}\leq\frac{\prod_{j=1}^{m}\left[[k(1+\beta)+(p-\alpha_j)]\,\Psi_{p,\mu}^{a,c}(k)\right]^{s_j}}{\prod_{j=1}^{m}(p-\alpha_j)^{s_j}}.$$

Then

$$\eta\leq p-\frac{k(1+\beta)\prod_{j=1}^{m}(p-\alpha_j)^{s_j}}{\prod_{j=1}^{m}[k(1+\beta)+(p-\alpha_j)]^{s_j}\left[\Psi_{p,\mu}^{a,c}(k)\right]^{s_j-1}-\prod_{j=1}^{m}(p-\alpha_j)^{s_j}}$$

This completes the proof of the theorem.

Remark 1. *Putting $a=c$, $\mu=1$, $\beta=0$ in Theorem 4, we obtain the corresponding result obtained by Nishiwaki and Owa [15];*

Corollary 5. *Let $f_j(z)\in US_p^*T(n,a,c;\mu,\alpha_j,\beta)$ $(j=1,2,3,...,m)$, then $H_m(z)\in US_p^*T(n,a,c;\mu,\eta,\beta)$ with*

$$\eta\leq p-\frac{n(1+\beta)\prod_{j=1}^{m}(p-\alpha_j)^{s_j}}{\left[\Psi_{p,\mu}^{a,c}(n)\right]^{r-1}\prod_{j=1}^{m}[n(1+\beta)+(p-\alpha_j)]^{s_j}-\prod_{j=1}^{m}(p-\alpha_j)^{s_j}}$$

where $r=\sum_{j=1}^{m}s_j>1+\frac{p-\alpha}{n(1+\beta)}$, $s_j\geq\frac{1}{q_j}$, $q_j>1$, $\sum_{j=1}^{m}\frac{1}{q_j}\geq 1$; $j=1,2,3,...,m$.

Putting $\alpha_j = \alpha$ in Corollary 5 we obtain the following corollary.

Corollary 6. *Let* $f_j(z) \in US_p^*T(n,a,c;\mu,\alpha_j,\beta)$ $(j = 1,2,3,...,m)$, *then* $H_m(z) \in US_p^*T(n,a,c;\mu,\eta,\beta)$ *with*

$$\eta \le p - \frac{[n(1+\beta)](p-\alpha)^r}{[n(1+\beta)+(p-\alpha)]^r \left[\Psi_{p,\mu}^{a,c}(n)\right]^{r-1} - (p-\alpha)^r}$$

where $r = \sum_{j=1}^m s_j > 1 + \frac{p-\alpha}{n(1+\beta)}$, $s_j \ge \frac{1}{q_j}$, $q_j > 1$, $\sum_{j=1}^m \frac{1}{q_j} \ge 1$; $j = 1,2,3,...,m$.

Example 1. *Let* $f_j(z)$ $(j = 1,2,3,...,m)$ *define as follows*

$$f_j(z) = z^p - \frac{(p-\alpha)}{[n(1+\beta)+(p-\alpha)]\Psi_{p,\mu}^{a,c}(n)}\epsilon z^{n+p} - \frac{(p-\alpha)}{[(n+j)(1+\beta)+(p-\alpha)]\Psi_{p,\mu}^{a,c}(n+j)}\epsilon_j z^{n+p+j},$$
$$(\epsilon + \epsilon_j \le 1),$$

then $H_m(z) \in US_p^*T(n,a,c;\mu,\eta,\beta)$ *with*

$$\eta \le p - \frac{[n(1+\beta)](p-\alpha)^r}{[n(1+\beta)+(p-\alpha)]^r \left[\Psi_{p,\mu}^{a,c}(n)\right]^{r-1} - (p-\alpha)^r}.$$

Since

$$\begin{aligned} f_j(z) &= z^p - \frac{(p-\alpha)}{[n(1+\beta)+(p-\alpha)]\Psi_{p,\mu}^{a,c}(n)}\epsilon z^{n+p} - \frac{(p-\alpha)}{[(n+j)(1+\beta)+(p-\alpha)]\Psi_{p,\mu}^{a,c}(n+j)}\epsilon_j z^{n+p+j}, \\ &\quad (\epsilon + \epsilon_j \le 1,\ j = 1,2,3,...,m), \end{aligned}$$

we have

$$\begin{aligned} \sum_{k=n}^{\infty} \frac{[k(1+\beta)+(p-\alpha)]\Psi_{p,\mu}^{a,c}(k)}{(p-\alpha)} a_{k+p} &= \frac{[n(1+\beta)+(p-\alpha)]\Psi_{p,\mu}^{a,c}(n)}{(p-\alpha)}\epsilon a_{n+p} \\ &\quad + \frac{[(n+j)(1+\beta)+(p-\alpha)]\Psi_{p,\mu}^{a,c}(n+j)}{(p-\alpha)}\epsilon_j a_{n+p+j} \\ &= \epsilon + \epsilon_j \le 1. \end{aligned}$$

Then $f_j(z) \in US_p^*T(n,a,c;\mu,\alpha,\beta)$ *and we have*

$$H_m(z) = z^p - \left(\frac{(p-\alpha)}{[n(1+\beta)+(p-\alpha)]\Psi_{p,\mu}^{a,c}(n)}\epsilon\right)^r z^{n+p},$$

and $H_m(z) \in US_p^*T(n,a,c;\mu,\eta,\beta)$.

4. Modified Hadamard products

Let the functions $f_i(z)$ $(i = 1,\ 2)$ be defined by

$$f_i(z) = z^p - \sum_{k=n}^{\infty} a_{k+p,i} z^{k+p} \tag{14}$$

The modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) = z^p - \sum_{k=n}^{\infty} a_{k+p,1} a_{k+p,2} z^{k+p}.$$

Corollary 7. *Let the functions $f_i(z)\,(i = 1, 2)$ defined by (14) be in the class $US_p^*T(n, a, c; \mu, \alpha_1, \beta)$ and $US_p^*T(n, a, c; \mu, \alpha_2, \beta)$, then $(f_1 * f_2)(z) \in US_p^*T(n; a, c; \mu, \delta, \beta)$ where*

$$\delta \le p - \frac{n(1+\beta)(p-\alpha_1)(p-\alpha_2)}{[n(1+\beta)+(p-\alpha_1)][n(1+\beta)+(p-\alpha_2)]\Psi_{p,\mu}^{a,c}(n) - (p-\alpha_1)(p-\alpha_2)} \quad (z \in U; n \in \mathbb{N}).$$

Corollary 8. *Let the functions $f_i(z)\,(i = 1,\ 2)$ defined by (14) be in the class $US_p^*T(n, a, c; \mu, \alpha, \beta)$ then $(f_1 * f_2)(z) \in US_p^*T(n, a, c; \mu, \delta, \beta)$ where*

$$\delta \le p - \frac{n(1+\beta)(p-\alpha)^2}{\Psi_{p,\mu}^{a,c}(n)[n(1+\beta)+(p-\alpha)]^2 - (p-\alpha)^2} \qquad (z \in U; n \in \mathbb{N}).$$

Theorem 9. *Let the functions $f_i(z)\,(i = 1, 2)$ defined by (14) be in the class $US_p^*T(n, a, c; \mu, \alpha, \beta)$, then $h(z) = z^p - \sum_{k=p+1}^{\infty}\left(a_{k,1}^2 + a_{k,2}^2\right) z^p$ belongs to the class $US_p^*T(n, a, c; \mu, \delta, \beta)$ where*

$$\delta = \Omega(n) \le p - \frac{2n(1+\beta)(p-\alpha)^2}{[n(1+\beta)+(p-\alpha)]^2 \Psi_{p,\mu}^{a,c}(n) - 2(p-\alpha)^2} \qquad (z \in U, n \in \mathbb{N}).$$

Proof. To prove the theorem, we need to find the largest δ such that

$$\sum_{k=p+1}^{\infty} \frac{[k(1+\beta)+(p-\delta)]\Psi_{p,\mu}^{a,c}(k)}{(p-\delta)}\left(a_{k,1}^2 + a_{k,2}^2\right) \le 1. \tag{15}$$

Hence

$$\sum_{k=n}^{\infty}\left\{\frac{[k(1+\beta)+(p-\alpha)]\Psi_{p,\mu}^{a,c}(k)}{(p-\alpha)}\right\}^2 a_{k,i}^2 \le \left\{\sum_{k=n}^{\infty}\frac{[k(1+\beta)+(p-\alpha)]\Psi_{p,\mu}^{a,c}(k)}{(p-\alpha)} a_{k,i}\right\}^2 \le 1, \quad (i = 1, 2). \tag{16}$$

Then
$$\sum_{k=n}^{\infty}\frac{1}{2}\left[\frac{[k(1+\beta)+(p-\alpha)]}{(p-\alpha)}\Psi_{p,\mu}^{a,c}(k)\right]^2\left(a_{k,1}^2+a_{k,2}^2\right)\leq 1,$$
and (15) is true if
$$\sum_{k=n}^{\infty}\frac{[k(1+\beta)+(p-\delta)]\,\Psi_{p,\mu}^{a,c}(k)}{(p-\delta)}\left(a_{k,1}^2+a_{k,2}^2\right)\leq\sum_{k=n}^{\infty}\frac{1}{2}\left[\frac{[k(1+\beta)+(p-\alpha)]\,\Psi_{p,\mu}^{a,c}(k)}{(p-\alpha)}\right]^2\left(a_{k,1}^2+a_{k,2}^2\right).$$
If
$$\frac{[k(1+\beta)+(p-\delta)]}{(p-\delta)}\leq\frac{[k(1+\beta)+(p-\alpha)]^2}{2(p-\alpha)^2}\Psi_{p,\mu}^{a,c}(k),$$
then
$$\delta\leq\Omega(k)=p-\frac{2k(1+\beta)(p-\alpha)^2}{[k(1+\beta)+(p-\alpha)]^2\,\Psi_{p,\mu}^{a,c}(k)-2(p-\alpha)^2},\qquad (k\geq n, n\in\mathbb{N})$$
which is an increasing function of $k\geq n,\ 0\leq\alpha<p,\ p\in\mathbb{N},\ 0<\beta\leq 1.$

Then
$$\delta=\Omega(n)\leq p-\frac{2n(1+\beta)(p-\alpha)^2}{[n(1+\beta)+(p-\alpha)]^2\,\Psi_{p,\mu}^{a,c}(n)-2(p-\alpha)^2}.$$
The proof is completed.

5. Closure Properties under Integral Operators

In this section, we discuss some preserving integral operators. We recall here the generalized Komatu integral operator (see [13]) define by
$$\begin{aligned}K(z)&=\frac{(\gamma+p)^d}{\Gamma(d)\,z^{\gamma}}\int_0^z t^{\gamma-1}\left(\log\frac{z}{t}\right)^{d-1}f(t)\,dt\quad (f(z)\in T(n,p))\\&=z^p-\sum_{k=n}^{\infty}\left(\frac{\gamma+p}{\gamma+k+p}\right)^d a_{k+p}z^{k+p}\qquad (d\geq 0;\gamma>-p;z\in U).\quad (17)\end{aligned}$$
Also the generalized Jung-Kim-Srivastava operator (see[11]) define by
$$\begin{aligned}I(z)&=Q_{\gamma,p}^{d}f(z)=\begin{pmatrix}p+d+\gamma-1\\p+\gamma-1\end{pmatrix}\frac{d}{z^{\gamma}}\int_0^z t^{\gamma-1}\left(1-\frac{t}{z}\right)^{d-1}f(t)\,dt\qquad (f(z)\in T(n,p))\\&=z^p-\sum_{k=n}^{\infty}\frac{\Gamma(p+k+\gamma)\,\Gamma(p+\gamma+d)}{\Gamma(p+k+\gamma+d)\,\Gamma(p+\gamma)}a_{k+p}z^{k+p}\qquad (d\geq 0;\gamma>-p;z\in U).\quad (18)\end{aligned}$$

Theorem 10. *Let $d > 0$, $\gamma > -p$ and $f(z) \in US_p^*T(n,a,c;\mu,\alpha,\beta)$, then $K(z)$ defined by* (17) *belongs to $US_p^*T(n,a,c;\mu,\alpha,\beta)$.*

Proof. Let $f(z) \in US_p^*T(n,a,c;\mu,\alpha,\beta)$ defined by (4), and $K(z)$ defined by (17) Then $K(z) \in US_p^*T(n,a,c;\mu,\alpha,\beta)$ if

$$\sum_{k=n}^{\infty} \frac{[k(1+\beta)+(p-\alpha)]\,\Psi_{p,\mu}^{a,c}(k)}{(p-\alpha)} \left(\frac{\gamma+p}{\gamma+k+p}\right)^d a_{k+p} \leq 1. \tag{19}$$

Now, from Theorem 2, $f(z) \in US_p^*T(n,a,c;\mu,\alpha,\beta)$ if and only if

$$\sum_{k=n}^{\infty} \frac{[k(1+\beta)+(p-\alpha)]\,\Psi_{p,\mu}^{a,c}(k)}{(p-\alpha)} a_{k+p} \leq 1.$$

Since $\frac{\gamma+p}{\gamma+k+p} \leq 1$, for $k \geq n$, then (19) holds true. Therefore $K(z) \in US_p^*T(n,a,c;\mu,\alpha,\beta)$.

Theorem 11. *Let $d > 0$, $\gamma > -p$ and $f(z) \in US_p^*T(n,a,c;\mu,\alpha,\beta)$, then $K(z)$ defined by* (17) *is $p-$valent in the disk $|z| < R_1$, where*

$$R_1 = \inf_k \left\{ \frac{p[k(1+\beta)+(p-\alpha)](\gamma+k+p)^d}{(k+p)(p-\alpha)(\gamma+p)^d} \Psi_{p,\mu}^{a,c}(k) \right\}^{\frac{1}{k}} \tag{20}$$

Proof. In order to prove the assertion, it is enough to show that

$$\left| \frac{K'(z)}{z^{p-1}} - p \right| \leq p. \tag{21}$$

Now, in view of (21), we get

$$\begin{aligned} \left| \frac{K'(z)}{z^{p-1}} - p \right| &= \left| -\sum_{k=n}^{\infty} (k+p) \left(\frac{\gamma+p}{\gamma+k+p}\right)^d a_{k+p} z^{k+p} \right| \\ &\leq \sum_{k=n}^{\infty} (k+p) \left(\frac{\gamma+p}{\gamma+k+p}\right)^d a_{k+p} \left|z^k\right|. \end{aligned}$$

This expression is bounded by p if

$$\sum_{k=n}^{\infty} \frac{(k+p)}{p} \left(\frac{\gamma+p}{\gamma+k+p}\right)^d a_{k+p} \left|z^k\right| \leq 1. \tag{22}$$

Since $f(z) \in US_p^*T(n,a,c;\mu,\alpha,\beta)$, and from Theorem 2 (22) holds if

$$\frac{(k+p)}{p}\left(\frac{\gamma+p}{\gamma+k+p}\right)^d a_{k+p}\left|z^k\right| \le \frac{[k(1+\beta)+(p-\alpha)]\,\Psi_{p,\mu}^{a,c}(k)}{(p-\alpha)}a_{n+p}, \quad (k\in\mathbb{N}).$$

That is

$$|z| \le \left\{\frac{p[k(1+\beta)+(p-\alpha)](\gamma+k+p)^d}{(k+p)(p-\alpha)(\gamma+p)^d}\Psi_{p,\mu}^{a,c}(k)\right\}^{\frac{1}{k}}.$$

The result follows by setting $|z| = R_1$.

Following similar steps as in the proofs of Theorem 10 and Theorem 11, we have the following results for $I(z)$.

Theorem 12. *Let $d > 0$, $\gamma > -p$ and $f(z) \in US_p^*T(n,a,c;\mu,\alpha,\beta)$, then $I(z)$ defined by* (18) *belongs to $US_p^*T(n,a,c;\mu,\alpha,\beta)$.*

Theorem 13. *Let $d > 0, c > -p$ and $f(z) \in US_p^*T(n;a,c;\mu,\alpha,\beta)$. Then $I(z)$ defined by* (18) *is $p-$valent in the disk $|z| < R_2$, where*

$$R_2 = \inf_k\left\{\frac{p[k(1+\beta)+(p-\alpha)]\Gamma(p+k+\gamma+d)\Gamma(p+\gamma)}{(k+p)(p-\alpha)\Gamma(p+k+\gamma)\Gamma(p+\gamma+d)}\Psi_{p,\mu}^{a,c}(k)\right\}^{\frac{1}{k}}$$

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