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ON $H(X)$-FIBONACCI-EULER AND $H(X)$-LUCAS-EULER NUMBERS AND POLYNOMIALS

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Abstract. Let $h(x)$ be a polynomial with real coefficients. We introduce $h(x)$-Fibonacci-Euler polynomials that generalize both Catalan's Fibonacci polynomials and Byrd's Fibonacci polynomials and also the k-Fibonacci numbers, and we provide properties and summation formulas for these polynomials. We also introduce $h(x)$-Lucas-Euler polynomials that generalize the Lucas and Hermite polynomials and present properties and symmetric identities of these polynomials by applying the generating functions.

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1. Introduction

The Fibonacci numbers F_n are the terms of the sequence $0, 1, 2, 3, 5, \cdots$, where $F_n = F_{n-1} + F_{n-2}, n \geq 2$ with the initial values $F_0 = 0$ and $F_1 = 1$. Fibonacci numbers are ubiquitous in nature: from petal arrangements in flowers to the patterns on the surface of a pineapple (see [1, 7, 8, 13, 14, 19]). They also have many applications, such as the "Fibonacci retracement" in the technical analysis of stock trading. For some more applications (see [2-5]).

Falcon and Plaza [3] introduced a general Fibonacci sequence that generalizes among others both the classical Fibonacci sequence and the pell sequence. These general k-Fibonacci numbers $F_{k,n}$ are defined by $F_{k,n} = kF_{k,n-1} + F_{k,n-2}, \quad n \geq 2$ with the initial values $F_0 = 0$ and $F_1 = 1$. The Pell numbers are the 2-Fibonacci numbers. In [4] the k-Fibonacci numbers were defined in explicit way and many properties were given. In particular, the k-Fibonacci numbers were shown to be related with the so called Pascal 2-triangle.

The polynomials $F_n(x)$ studied by Catalan are defined by the recurrence relation

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x), \qquad n \geq 3, \tag{1.1}$$

where $F_1(x) = 1$, $F_2(x) = x$. The Fibonacci polynomials studied by Jacobsthal are defined by

$$J_n(x) = J_{n-1}(x) + xJ_{n-2}(x), \qquad n \geq 3, \tag{1.2}$$

where $J_1(x) = J_2(x) = 1$. The Fibonacci polynomials studied by P.F.Byrd are defined by

$$\phi_n(x) = 2x\phi_{n-1}(x) + \phi_{n-2}(x), \qquad n \geq 2, \tag{1.3}$$

where $\phi_0(x) = 0$, $\phi_1(x) = 1$. The Lucas polynomials $L_n(x)$ originally studied in 1970 by Bicknell are defined by

$$L_n(x) = xL_{n-1}(x) + L_{n-2}(x), \qquad n \geq 2, \tag{1.4}$$

where $L_0(x) = 2$, $L_1(x) = x$.

The generalized Bernoulli polynomials $B_n^{(\alpha)}(x)$ of order $\alpha \in \mathbb{C}$, the generalized Euler polynomials $E_n^{(\alpha)}(x)$ of order $\alpha \in \mathbb{C}$ and the generalized Genocchi polynomials $G_n^{(\alpha)}(x)$ of order $\alpha \in \mathbb{C}$, each of degree n as well as well as $\alpha \in \mathbb{C}$ are defined respectively by the following generating functions (see [15-18]):

$$\left(\frac{t}{e^t - 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{t^n}{n!}, (|t| < 2\pi, 1^{\alpha} = 1), \tag{1.5}$$

$$\left(\frac{2}{e^t + 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x)\frac{t^n}{n!}, (|t| < \pi, 1^{\alpha} = 1), \tag{1.6}$$

and

$$\left(\frac{2t}{e^t + 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x)\frac{t^n}{n!}, (|t| < \pi, 1^{\alpha} = 1). \tag{1.7}$$

Taking $x = 0$ in generating functions (1.5)-(1.7), we find

$$\left(\frac{t}{e^t - 1}\right)^{\alpha} = \sum_{n=0}^{\infty} B_n^{(\alpha)}\frac{t^n}{n!}, (|t| < 2\pi, 1^{\alpha} = 1), \tag{1.8}$$

$$\left(\frac{2}{e^t + 1}\right)^{\alpha} = \sum_{n=0}^{\infty} E_n^{(\alpha)}\frac{t^n}{n!}, (|t| < \pi, 1^{\alpha} = 1), \tag{1.9}$$

$$\left(\frac{2t}{e^t+1}\right)^{\alpha}=\sum_{n=0}^{\infty}G_n^{(\alpha)}\frac{t^n}{n!},(|t|<\pi,1^{\alpha}=1),\tag{1.10}$$

where

$$B_n^{(\alpha)}=B_n^{(\alpha)}(0);E_n^{(\alpha)}=E_n^{(\alpha)}(0);G_n^{(\alpha)}=G_n^{(\alpha)}(0),\tag{1.11}$$

are the corresponding numbers.

It is easy to see that $B_n(x)$, $E_n(x)$ and $G_n(x)$ are given respectively by

$$B_n^{(1)}(x)=B_n(x);E_n^{(1)}=E_n(x);G_n^{(1)}(x)=G_n(x),n\in\mathbb{N}_0=\mathbb{N}\cup\{0\}.\tag{1.12}$$

The classical Bernoulli numbers B_n, the classical Euler numbers E_n and the classical Genocchi numbers G_n of order n are given as

$$B_n=B_n(0)=B_n^{(1)}(0);E_n=E_n(0)=E_n^{(1)}(0);G_n=G_n(0)=G_n^{(1)}(0),\tag{1.13}$$

respectively.

For each $k\in\mathbb{N}_0$, the sum $M_k(n)=\sum_{i=0}^{n}(-1)^k i^k$ is known as the sum of alternative integer powers defined by the generating relation:

$$\sum_{k=0}^{\infty}M_k(n)\frac{t^k}{k!}=\frac{1-(-e^t)^{(n+1)}}{e^t+1}.\tag{1.14}$$

In [9], Nalli and Haukkanen introduced the $h(x)$-Fibonacci polynomials. That generalize Catalan's Fibonacci polynomials $F_n(x)$ and the k-Fibonacci numbers $F_{k,n}$. In this paper, we introduce Fibonacci-Euler numbers, $h(x)$-Fibonacci-Euler polynomials, Lucas-Euler numbers and $h(x)$-Lucas-Euler polynomials and then we obtain new sums and identities. The resulting formulas allow a considerable unification of various special results which appear in the literature.

2. The $h(x)$-Fibonacci-Euler numbers and polynomials

Define $h(x)$-Fibonacci-Euler polynomials ${}_EF_{h,n}(x)$ by

$$\frac{2t}{(1-h(x)t-t^2)(e^t+1)}=\sum_{n=0}^{\infty}{}_EF_{h,n}(x)\frac{t^n}{n!}.\tag{2.1}$$

For $h(x)=x$, we obtain Catalan's Fibonacci-Euler polynomials and for $h(x)=2x$, we obtain Byrd's Fibonacci-Euler polynomials. For $h(x)=k$, we obtain the k-Fibonacci-Euler numbers. For $k=1$ and $k=2$, we obtain the usual Fibonacci-Euler numbers and the Pell-Euler numbers.

Equation (2.1) is

$$\frac{t}{1-h(x)t-t^2}\frac{2}{e^t+1}=\sum_{n=0}^{\infty}{}_EF_{h,n}(x)\frac{t^n}{n!}=\sum_{n=0}^{\infty}F_{h,n}(x)t^n\sum_{m=0}^{\infty}E_m\frac{t^m}{m!}.$$

Comparing the coefficients of t^n, we get

$${}_EF_{h,n}(x)=n!\sum_{m=0}^{n}\frac{1}{m!}F_{h,n-m}(x)E_m. \tag{2.2}$$

Theorem 2.1. For $n\geq 1$, we have

$$\frac{G_n}{n!}={}_EF_{h,n}(x)\frac{1}{n!}-{}_EF_{h,n-1}(x)\frac{h(x)}{(n-1)!}-{}_EF_{h,n-2}(x)\frac{1}{(n-2)!}. \tag{2.3}$$

$$F_{h,n}(x)=\frac{1}{2}\sum_{m=0}^{n}\binom{n}{m}{}_EF_{h,n-m}(x)+\frac{1}{2}{}_EF_{h,n}(x)\frac{1}{n!}. \tag{2.4}$$

Proof. From (2.1), we have

$$\frac{2t}{e^t+1}=(1-h(x)t-t^2)\sum_{n=0}^{\infty}{}_EF_{h,n}(x)t^n$$

$$\sum_{n=0}^{\infty}G_n\frac{t^n}{n!}=(1-h(x)t-t^2)\sum_{n=0}^{\infty}{}_EF_{h,n}(x)t^n$$

$$\sum_{n=0}^{\infty}G_n\frac{t^n}{n!}=\sum_{n=0}^{\infty}{}_EF_{h,n}(x)\frac{t^n}{n!}-\sum_{n=0}^{\infty}{}_EF_{h,n-1}(x)\frac{h(x)t^n}{(n-1)!}-\sum_{n=0}^{\infty}{}_EF_{h,n-2}(x)\frac{t^n}{(n-2)!}.$$

Comparing the coefficients of t^n, we get the result (2.3).

Again equation (2.1) can be written as

$$\frac{2t}{1-h(x)t-t^2}=(e^t+1)\sum_{n=0}^{\infty}{}_EF_{h,n}(x)\frac{t^n}{n!}$$

$$2\sum_{n=0}^{\infty} F_{h,n}(x)t^n = \sum_{m=0}^{\infty} \frac{t^m}{m!} \sum_{n=0}^{\infty} {}_EF_{h,n}(x)\frac{t^n}{n!} + \sum_{n=0}^{\infty} {}_EF_{h,n}(x)\frac{t^n}{n!}$$

$$2\sum_{n=0}^{\infty} F_{h,n}(x)t^n = \sum_{m=0}^{n} \binom{n}{m} {}_EF_{h,n-m}(x)t^n + \sum_{n=0}^{\infty} {}_EF_{h,n}(x)\frac{t^n}{n!}.$$

Comparing the coefficients of t^n, we get the result (2.4).

Theorem 2.2. For $n \geq 1$, we have

$$ {}_EF_{h,n}(x) = n! \sum_{m=0}^{n} \sum_{i=0}^{[\frac{m-1}{2}]} \binom{m-i-1}{i} \frac{E_{n-m}}{(n-m)!} h^{m-2i-1}(x). \tag{2.5}$$

Proof. From (2.1), we have

$$\frac{t}{1-h(x)t-t^2}\frac{2}{e^t+1} = t\frac{2}{e^t+1}\sum_{n=0}^{\infty}(h(x)t+t^2)^n$$

$$= t\frac{2}{e^t+1}\sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{n}{i}(h(x)t)^{n-i}(t^2)^i \tag{2.6}$$

$$= \frac{2}{e^t+1}\sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{n}{i}(h(x)t)^{n-i}(t^{n+i+1}).$$

On writing $n+i+1=m$ in R.H.S of the above equation, we get

$$\frac{t}{1-h(x)t-t^2}\frac{2}{e^t+1} = \frac{2}{e^t+1}\sum_{m=0}^{\infty}\left[\sum_{i=0}^{[\frac{m-1}{2}]}\binom{m-i-1}{i}h^{m-2i-1}(x)\right]t^m$$

$$\sum_{n=0}^{\infty} {}_EF_{h,n}(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty} E_n\frac{t^n}{n!}\sum_{m=0}^{\infty}\left[\sum_{i=0}^{[\frac{m-1}{2}]}\binom{m-i-1}{i}h^{m-2i-1}(x)\right]t^m.$$

Replace n by $n-m$ and compare the coefficients of t^n to get the result (2.5).

Theorem 2.3. For $n \geq 1$, we have

$$\frac{t(1+t^2)sint + t^2h(x)cost}{(1+t^2)^2+([h(x)t])^2} = \sum_{n=0}^{\infty} {}_EF_{h,2n}(x)(-1)^{n+1}t^{2n}. \tag{2.7}$$

$$\frac{(1+t^2)cost - th(x)sint}{(1+t^2)^2+([h(x)t])^2} = \sum_{n=0}^{\infty} {}_E F_{h,2n+1}(x)(-1)^n t^{2n+1}. \qquad (2.8)$$

Proof. Replacing t by it where $t^2 = -1$ in (2.1), using $e^{it} = cost + isint$,

$$\frac{2}{e^{it}+1} = \frac{1+cost - isint}{1+cost}$$

and simplifying,we get

$$\frac{it(1+t^2+ih(x)t)}{(1+t^2)^2+([h(x)t])^2}\frac{1+cost-isint}{1+cost} = \sum_{n=0}^{\infty} {}_E F_{h,n}(x)(it)^n.$$

Now separating real and imaginary parts, we get the theorem.

If in place of (2.1), we simply consider

$$\frac{t}{(1-h(x)t-t^2)} = \sum_{n=0}^{\infty} F_{h,n}(x)t^n \qquad (2.9)$$

and follow the procedure of the proof of the above theorem, then we get

Corollary 2.1. For $n \geq 1$, we have

$$\frac{t^2h(x)}{(1+t^2)^2+([h(x)t])^2} = \sum_{n=0}^{\infty} F_{h,2n}(x)(-1)^{n+1}t^{2n}. \qquad (2.10)$$

$$\frac{(1+t^2)}{(1+t^2)^2+([h(x)t])^2} = \sum_{n=0}^{\infty} F_{h,2n+1}(x)(-1)^n t^{2n+1}. \qquad (2.11)$$

Theorem 2.4. Representation of Euler polynomials in terms of $h(x)$-Fibonacci-Euler polynomials is

$$\frac{E_n}{n!} = \frac{{}_E F_{h,n+1}(x)}{(n+1)!} - \frac{h(x)_E F_{h,n}(x)}{n!} - \frac{{}_E F_{h,n-1}(x)}{(n-1)!}, \quad n \geq 1. \qquad (2.12)$$

Proof. Writing (2.1) in the form

$$\frac{2}{e^t+1} = (1-h(x)t-t^2)\sum_{n=0}^{\infty} {}_E F_{h,n}(x)\frac{t^{n-1}}{n!}$$

$$\sum_{n=0}^{\infty} E_n \frac{t^n}{n!} = (1 - h(x)t - t^2) \sum_{n=0}^{\infty} {}_E F_{h,n}(x) \frac{t^{n-1}}{n!}.$$

Comparing the coefficients of t^n, we get the result (2.12).

Theorem 2.5. Suppose that $h(x)$ is an odd polynomial (that is $h(-x) = -h(x)$). Then for $n \geq 0$

$$\sum_{m=0}^{n} \binom{n}{m} {}_E F_{h,n-m}(x) = (-1)^n {}_E F_{h,n}(-x) \tag{2.13}$$

$${}_E F_{h,n}(x) = \frac{1}{2} \sum_{m=0}^{n} \left(\begin{array}{c} n \\ m \end{array} \right) E_m [{}_E F_{h,n-m}(x) + (-1)^n {}_E F_{h,n-m}(-x)]. \tag{2.14}$$

Proof. From (2.1), we have

$$\frac{-t}{1 + h(x)t - t^2} \frac{2}{e^{-t} + 1} = \sum_{n=0}^{\infty} {}_E F_{h,n}(x) \frac{(-t)^n}{n!},$$

which on replacing x by $-x$ yields

$$\frac{t}{1 - h(x)t - t^2} \frac{2}{e^{-t} + 1} = \sum_{n=0}^{\infty} (-1)^n {}_E F_{h,n}(-x) \frac{t^n}{n!}$$

$$e^t \sum_{n=0}^{\infty} {}_E F_{h,n}(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} (-1)^n {}_E F_{h,n}(-x) \frac{t^n}{n!}. \tag{2.15}$$

Now expanding e^t and comparing the coefficients of t^n, we get the result (2.13).

Next we add (2.1) to (2.15)

$$(1 + e^t) \sum_{n=0}^{\infty} {}_E F_{h,n}(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} {}_E F_{h,n}(x) \frac{t^n}{n!} + \sum_{n=0}^{\infty} (-1)^n {}_E F_{h,n}(-x) \frac{t^n}{n!},$$

so that

$$\sum_{n=0}^{\infty} {}_E F_{h,n}(x) \frac{t^n}{n!} = \frac{1}{1 + e^t} \sum_{n=0}^{\infty} [{}_E F_{h,n}(x) \frac{t^n}{n!} + (-1)^n {}_E F_{h,n}(-x) \frac{t^n}{n!}].$$

Now using the definition of Euler polynomials and comparing the coefficients of t^n, we get the result (2.14).

Binet's formulas are well known in the theory of Fibonacci numbers. These formulas can also be carried out for the $h(x)$-Fibonacci polynomials. Let $\alpha(x)$ and $\beta(x)$ be the roots of the characteristic equation

$$\nu^2 - h(x)\nu - 1 = 0 \tag{2.16}$$

Then

$$\alpha(x) = \frac{h(x) + \sqrt{h^2(x)+4}}{2}, \beta(x) = \frac{h(x) - \sqrt{h^2(x)+4}}{2}. \tag{2.17}$$

Note that $\alpha(x) + \beta(x) = h(x)$, $\alpha(x)\beta(x) = 1$ and $\alpha(x) - \beta(x) = \sqrt{h^2(x)+4}$.

Theorem 2.6. For $n \geq 1$, we have

$$_EF_{h,n}(x) = \frac{2^{1-n+m}n!}{m!} \sum_{m=0}^{n} \sum_{i=0}^{[\frac{n-m-1}{2}]} \binom{n-m}{2i+1} E_m h^{n-m-2i-1}(x)(h^2(x)+4)^i. \tag{2.18}$$

Proof. By (2.15) and (2.16), we have

$$\alpha^n(x) - \beta^n(x) = 2^{-n}\left[\left(h(x) + \sqrt{h^2(x)+4}\right)^n - \left(h(x) - \sqrt{h^2(x)+4}\right)^n\right]$$

$$= 2^{-n}\left[\sum_{i=0}^{n}\binom{n}{i} h^{n-i}(x)\left(\sqrt{h^2(x)+4}\right)^i - \sum_{i=0}^{n}\binom{n}{i} h^{n-i}(x)\left(-\sqrt{h^2(x)+4}\right)^i\right]$$

$$= 2^{-n+1}\sum_{i=0}^{[\frac{n-1}{2}]}\binom{n}{2i+1} h^{n-2i-1}(x)\left(\sqrt{h^2(x)+4}\right)^{2i+1}. \tag{2.19}$$

Now

$$\sum_{n=0}^{\infty} {}_EF_{h,n}(x)\frac{t^n}{n!} = \left(\frac{\alpha^n(x)-\beta^n(x)}{\alpha(x)-\beta(x)}\right)\frac{2}{e^t+1} \tag{2.20}$$

and so by substituting from (2.20), we have

$$\sum_{n=0}^{\infty} {}_EF_{h,n}(x)\frac{t^n}{n!} = \frac{2^{1-n+m}}{m!}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{i=0}^{[\frac{n-m}{2}]}\binom{n-m}{2i+1} E_m h^{n-m-2i-1}(x)(h^2(x)+4)^i t^n.$$

Comparing the coefficients of t^n, we get the result (2.18).

3. The $h(x)$-Lucas-Euler numbers and polynomials

The $h(x)$-Lucas polynomials introduced by Nalli and Haukkanen [9, p.3183(3.6)] are

$$\frac{2-h(x)t}{1-h(x)t-t^2}=\sum_{n=0}^{\infty}L_{h,n}(x)t^n. \tag{3.1}$$

For $h(x)=x$, we obtain the Lucas polynomials and for $h(x)=1$, we obtain the usual Lucas numbers.

Let $h(x)$ be a polynomial with real coefficients.We define h(x)-Lucas-Euler polynomials ${}_EL_{h,n}(x)$ by the generating function

$$\frac{2-h(x)t}{1-h(x)t-t^2}\frac{2}{e^t+1}=\sum_{n=0}^{\infty}{}_EL_{h,n}(x)\frac{t^n}{n!}. \tag{3.2}$$

We may now rewrite (3.2) as

$$\sum_{n=0}^{\infty}{}_EL_{h,n}(x)\frac{t^n}{n!}=\sum_{n=0}^{\infty}L_{h,n}(x)t^n\sum_{m=0}^{\infty}E_m\frac{t^m}{m!}.$$

Replace n by $n-m$ in R.H.S and comparing the coefficients of t^n to get the following representation for h(x)-Lucas-Euler polynomials

$${}_EL_{h,n}(x)=n!\sum_{m=0}^{n}L_{h,n-m}(x)\frac{E_m}{m!}. \tag{3.3}$$

Theorem 3.1. For $n\geq 1$, we have

$$2E_n\frac{1}{n!}={}_EF_{h,n}(x)\frac{1}{n!}-h(x)\left[h(x){}_EF_{h,n-1}(x)\frac{1}{(n-1)!}+{}_EL_{h,n-1}(x)\frac{1}{(n-1)!}\right]$$
$$-\left[h(x){}_EF_{h,n-2}(x)\frac{1}{(n-2)!}+{}_EL_{h,n-2}(x)\frac{1}{(n-2)!}\right]. \tag{3.4}$$

$$F_{h,n}(x)=\frac{1}{2}\sum_{m=0}^{\infty}\binom{n}{m}{}_EL_{h,n-m}(x)+\frac{1}{2}{}_EL_{h,n}(x)\frac{1}{n!}. \tag{3.5}$$

Proof. By using equation (3.2), we can write

$$\frac{2}{1-h(x)t-t^2}\frac{2}{e^t+1}=h(x)\sum_{n=0}^{\infty}{}_EF_{h,n}(x)\frac{t^n}{n!}+\sum_{n=0}^{\infty}{}_EL_{h,n}(x)\frac{t^n}{n!}$$

$$\frac{2}{1-h(x)t-t^2}\sum_{n=0}^{\infty}E_n\frac{t^n}{n!}=h(x)\sum_{n=0}^{\infty}{}_EF_{h,n}(x)\frac{t^n}{n!}+\sum_{n=0}^{\infty}{}_EL_{h,n}(x)\frac{t^n}{n!}$$

$$2\sum_{n=0}^{\infty}E_n\frac{t^n}{n!}=(1-h(x)t-t^2)\left[h(x)\sum_{n=0}^{\infty}{}_EF_{h,n}(x)\frac{t^n}{n!}+\sum_{n=0}^{\infty}{}_EL_{h,n}(x)\frac{t^n}{n!}\right]$$

$$2\sum_{n=0}^{\infty}E_n\frac{t^n}{n!}=\sum_{n=0}^{\infty}{}_EF_{h,n}(x)\frac{t^n}{n!}+\sum_{n=0}^{\infty}{}_EL_{h,n}(x)\frac{t^n}{n!}-h(x)t\left[h(x)\sum_{n=0}^{\infty}{}_EF_{h,n}(x)\frac{t^n}{n!}+\sum_{n=0}^{\infty}{}_EL_{h,n}(x)\frac{t^n}{n!}\right]$$
$$-t^2\left[h(x)\sum_{n=0}^{\infty}{}_EF_{h,n}(x)\frac{t^n}{n!}+\sum_{n=0}^{\infty}{}_EL_{h,n}(x)\frac{t^n}{n!}\right].$$

Comparing the coefficients of t^n, we get the result (3.4).

Again we rewrite the equation (3.2) as

$$2\frac{2-h(x)t}{1-h(x)t-t^2}=(e^t+1)\sum_{n=0}^{\infty}{}_EL_{h,n}(x)\frac{t^n}{n!}$$

$$2\sum_{n=0}^{\infty}F_{h,n}(x)t^n=\sum_{m=0}^{\infty}\frac{t^m}{m!}\sum_{n=0}^{\infty}{}_EL_{h,n}(x)\frac{t^n}{n!}+\sum_{n=0}^{\infty}{}_EL_{h,n}(x)\frac{t^n}{n!}.$$

Comparing the coefficients of t^n, we get the result (3.5).

Theorem 3.2. For $n\geq 1$, we have

$${}_EL_{h,n}(x)=\sum_{m=0}^{n}\sum_{i=0}^{[\frac{n-m}{2}]}\begin{pmatrix} n-m-i \\ i \end{pmatrix}\frac{n-m}{(n-m-i)!(m)!}h^{n-m-2i}(x)E_m. \tag{3.6}$$

Proof. Let us write

$$\frac{2-h(x)t}{1-h(x)t-t^2}\frac{2}{e^t+1}=\frac{2}{e^t+1}(2-h(x)t)\sum_{n=0}^{\infty}(h(x)t+t^2)^n$$

$$=\frac{2}{e^t+1}\sum_{n=0}^{\infty}\sum_{i=0}^{[\frac{n}{2}]}\begin{pmatrix} n \\ i \end{pmatrix}(h(x)t)^{n-i}(t^2)^i$$

$$=\frac{2}{e^t+1}\sum_{n=0}^{\infty}\sum_{i=0}^{[\frac{n}{2}]}\frac{n}{n-i}\begin{pmatrix} n-i \\ i \end{pmatrix}h^{n-2i}(x)t^n$$

$$= \sum_{m=0}^{\infty} E_m \frac{t^m}{m!} \sum_{n=0}^{\infty} \sum_{i=0}^{[\frac{n}{2}]} \frac{n}{n-i} \left(\begin{array}{c} n-i \\ i \end{array} \right) h^{n-2i}(x) t^n.$$

Replacing n by $n-m$ and comparing the coefficients of t^n, we get the result (3.6).

Remark 3.1. For $m = 0$ in equation (3.6), the result reduces to known result of Nalli and Haukkanen [9, p.3183(3.11)].

Corollary 3.1. For $n \geq 1$, we have

$$L_{h,n}(x) = \sum_{i=0}^{[\frac{n}{2}]} \left(\begin{array}{c} n-i \\ i \end{array} \right) \frac{n}{n-i} h^{n-2i}(x).$$

Theorem 3.3. For $n \geq 1$, we have

$$_E L_{h,n}(x) = \frac{m!}{2^{n-m-1}} \sum_{m=0}^{n} \sum_{i=0}^{[\frac{n-m}{2}]} \left(\begin{array}{c} n-m \\ 2i \end{array} \right) h^{n-m-2i}(x)(h^2(x)+4)^i E_m. \qquad (3.7)$$

Proof. Let

$$\alpha^n(x) + \beta^n(x) = 2^{-n} \left[\left(h(x) + \sqrt{h^2(x)+4} \right)^n + \left(h(x) - \sqrt{h^2(x)+4} \right)^n \right]$$

$$= 2^{-n} \left[\sum_{i=0}^{n} \left(\begin{array}{c} n \\ i \end{array} \right) h^{n-i}(x) \left(\sqrt{h^2(x)+4} \right)^i + \sum_{i=0}^{n} \left(\begin{array}{c} n \\ i \end{array} \right) h^{n-i}(x) \left(\sqrt{h^2(x)+4} \right)^i \right]$$

$$= \frac{1}{2^{-n+1}} \sum_{i=0}^{[\frac{n}{2}]} \left(\begin{array}{c} n \\ 2i \end{array} \right) h^{n-2i}(x) \left(\sqrt{h^2(x)+4} \right)^i.$$

Now

$$\sum_{n=0}^{\infty} {}_H L_{h,n}(x,y,z) t^n = (\alpha^n(x) + \beta^n(x)) e^{yt+zt^2}$$

$$\sum_{n=0}^{\infty} {}_H L_{h,n}(x,y,z) t^n = \frac{1}{2^{1-n+m}} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{i=0}^{[\frac{n-m}{2}]} \left(\begin{array}{c} n-m \\ 2i \end{array} \right) E_m h^{n-m-2i}(x)(h^2(x)+4)^i t^n.$$

Comparing the coefficients of t^n, we get the result (3.7).

Remark 3.2. For $m = 0$ in Equation (3.7), the result reduces to known result of Nalli and Haukkanen [9, p.3183(3.12)].

Corollary 3.2. For $n \geq 1$, we have

$$L_{h,n}(x) = \frac{1}{2^{1-n}} \sum_{i=0}^{[\frac{n}{2}]} \binom{n}{2i} h^{n-2i}(x)(h^2(x)+4)^i.$$

4. Symmetric identities for $h(x)$-Fibonacci-Euler polynomials

In our previous articles (Pathan [10], Pathan and Khan [11, 12] and Khan [6]), it was shown that symmetric identities for Hermite based generalized polynomials unified many properties and identities of Hermite-Bernoulli and Hermite-Euler polynomials. In this section, we give general symmetric identities for the generalized $h(x)$-Fibonacci Euler polynomials ${}_EF_{h,n}(x)$ by applying the generating functions (2.1) and (1.14).

Theorem 4.1. Let $n \geq 0$. Then the following identity holds true:

$$\sum_{m=0}^{n} \binom{n}{m} a^{n-m} b^m {}_EF_{h,n-m}(x) {}_EF_{h,m}(x)$$

$$= \sum_{m=0}^{n} \binom{n}{m} b^{n-m} a^m {}_EF_{h,n-m}(x) {}_EF_{h,m}(x). \tag{4.1}$$

Proof. Let

$$g(t) = \left(\frac{abt^2}{(1 - ah(x)t - a^2t^2)(1 - bh(x)t - b^2t^2)} \right) \left(\frac{4}{(e^{at}+1)(e^{bt}+1)} \right).$$

Then the expression for $g(t)$ is symmetric in a and b and we can expand $g(t)$ into series in two ways to obtain

$$g(t) = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} a^{n-m} b^m {}_EF_{h,n-m}(x) {}_EF_{h,m}(x) \right) \frac{t^n}{n!}.$$

On the similar lines we can show that

$$g(t) = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} b^{n-m} a^m {}_EF_{h,n-m}(x) {}_EF_{h,m}(x) \right) \frac{t^n}{n!}.$$

Comparing the coefficients of t^n on the right hand sides of the last two equations, we arrive at the desired result.

Remark 4.1. By setting $b = 1$ in Theorem (4.1), we immediately get the following corollary

Corollary 4.1. The following identity holds true:

$$\sum_{m=0}^{n} \binom{n}{m} a^{n-m}{}_EF_{h,n-m}(x)_EF_{h,m}(x)$$

$$= \sum_{m=0}^{n} \binom{n}{m} a^{m}{}_EF_{h,n-m}(x)_EF_{h,m}(x).$$

Theorem 4.2. For each pair of integers a and b and all integers and $n \geq 1$, the following identity holds true:

$$\sum_{k=0}^{n} \binom{n}{k} a^{n-k}b^k{}_EF_{h,n-k}(x)\sum_{l=0}^{k} \binom{k}{l} M_l(a-1)_EF_{h,k-l}(x)$$

$$= \sum_{k=0}^{n} \binom{n}{k} b^{n-k}a^k{}_EF_{h,n-k}(x)\sum_{l=0}^{k} \binom{k}{l} M_l(b-1)_EF_{h,k-l}(x). \tag{4.2}$$

Proof. Let

$$g(t) = \left(\frac{abt^2}{(1-ah(x)t-at^2)(1-bh(x)t-bt^2)}\right)\frac{(1+e^{abt})}{(e^{bt}+1)}\left(\frac{4}{(e^{at}+1)(e^{bt}+1)}\right)$$

$$= \sum_{n=0}^{\infty} {}_EF_{h,n}(x)\frac{(at)^n}{n!}\sum_{l=0}^{\infty} M_l(a-1)\frac{(bt)^l}{l!}\sum_{k=0}^{\infty} {}_EF_{h,k}(x)\frac{(bt)^k}{k!} \tag{4.3}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \binom{n}{k} a^{n-k}b^k{}_EF_{h,n-k}(x)\sum_{l=0}^{k} \binom{k}{l} M_l(a-1)_EF_{h,k-l}(x)\right)\frac{t^n}{n!}.$$

On the other hand

$$g(t) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \binom{n}{k} b^{n-k}a^k{}_EF_{h,n-k}(x)\sum_{l=0}^{k} \binom{k}{l} M_l(b-1)_EF_{h,k-l}(x)\right)\frac{t^n}{n!}.$$

By comparing the coefficients of t^n on the right hand sides of the last two equations, we arrive at the desired result.

5. Symmetric identities for $h(x)$-Lucas Euler polynomials

In this section, we give general symmetric identities for the generalized $h(x)$-Lucas Euler polynomials ${}_EL_{h,n}(x)$ by applying the generating functions (3.2) and (1.14). For some known symmetric identities for Hermite based generalized polynomials, we refer (Pathan [10], Pathan and Khan [11, 12] and Khan [6]).

Theorem 5.1. Let $n \geq 0$. Then the following identity holds true:

$$\sum_{m=0}^{n} \binom{n}{k} a^{n-m} b^{m} {}_EL_{h,n-m}(x) {}_EL_{h,m}(x)$$

$$= \sum_{m=0}^{n} \binom{n}{m} b^{n-m} a^{m} {}_EL_{h,n-m}(x) {}_EL_{h,m}(x). \tag{5.1}$$

Proof. Let

$$g(t) = \left(\frac{(2-ah(x)t)(2-bh(x)t)}{(1-ah(x)t-a^2t^2)(1-bh(x)t-b^2t^2)} \right) \left(\frac{4}{(e^{at}+1)(e^{bt}+1)} \right).$$

Then the expression for $g(t)$ is symmetric in a and b and we can expand $g(t)$ into series in two ways to obtain

$$g(t) = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} a^{n-m} b^{m} {}_EL_{h,n-m}(x) {}_EL_{h,m}(x) \right) \frac{t^n}{n!}.$$

On the similar lines we can show that

$$g(t) = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} b^{n-m} a^{m} {}_EL_{h,n-m}(x) {}_EL_{h,m}(x) \right) \frac{t^n}{n!}.$$

Comparing the coefficients of t^n on the right hand sides of the last two equations, we arrive at the desired result.

Remark 5.1. By setting $b = 1$ in Theorem (5.1), we immediately get the following corollary

Corollary 5.1. The following identity holds true:

$$\sum_{m=0}^{n} \binom{n}{m} a^{n-m} {}_EL_{h,n-m}(x) {}_EL_{h,m}(x)$$

$$= \sum_{m=0}^{n} \binom{n}{m} a^m {}_E L_{h,n-m}(x) {}_E L_{h,m}(x).$$

Theorem 5.2. For each pair of integers a and b and all integers and $n \geq 1$, the following identity holds true:

$$\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k {}_E L_{h,n-k}(x) \sum_{l=0}^{k} \binom{k}{l} M_l(a-1) {}_E L_{h,k-l}(x)$$

$$= \sum_{k=0}^{n} \binom{n}{k} b^{n-k} a^k {}_E L_{h,n-k}(x) \sum_{l=0}^{k} \binom{k}{l} M_l(b-1) {}_E L_{h,k-l}(x). \qquad (5.2)$$

Proof. Let

$$g(t) = \left(\frac{(2-ah(x)t)(2-bh(x)t)}{(1-ah(x)t-a^2t^2)(1-bh(x)t-b^2t^2)} \right) \frac{(1+e^{abt})}{(e^{bt}+1)} \left(\frac{4}{(e^{at}+1)(e^{bt}+1)} \right)$$

$$= \sum_{n=0}^{\infty} {}_E F_{h,n}(x) \frac{(at)^n}{n!} \sum_{l=0}^{\infty} M_l(a-1) \frac{(bt)^l}{l!} \sum_{k=0}^{\infty} {}_E F_{h,k}(x) \frac{(bt)^k}{k!} \qquad (5.3)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k {}_E L_{h,n-k}(x) \sum_{l=0}^{k} \binom{k}{l} M_l(a-1) {}_E L_{h,k-l}(x) \right) \frac{t^n}{n!}.$$

On the other hand, we have

$$g(t) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} b^{n-k} a^k {}_E L_{h,n-k}(x) \sum_{l=0}^{k} \binom{k}{l} M_l(b-1) {}_E L_{h,k-l}(x) \right) \frac{t^n}{n!}.$$

By comparing the coefficients of t^n on the right hand sides of the last two equations, we arrive at the desired result.

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