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SOLVING NONLINEAR INTEGRO-DIFFERENTIAL EQUATIONS BY USING NUMERICAL TECHNIQUES

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Abstract. In this paper, nonlinear initial value problems for Volterra integro-differential equations are solved by Modified Decomposition Method (MDM) and Modified Homotopy Perturbation Method (MHPM). The solutions of the problems are derived by infinite convergent series which are easily computable and then graphical representation shows that both methods are most effective and convenient. In order to show the efficiency of the presented techniques, we compare our results obtained with the exact results. Finally, some examples are included to demonstrate the validity and applicability of the proposed techniques.

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1. Introduction

In recent years, there has been a growing interest in the linear and nonlinear Volterra integro-differential equations which are a combination of differential and integral equations. The nonlinear Volterra integro-differential equations play an important role in many branches of nonlinear functional analysis and their applications in the theory of engineering, mechanics, physics, electrostatics, biology, chemistry and economics [4]. In this paper, we consider the Volterra integro-differential equations of the type:

$$Z^{(j)}(x) = f(x) + \gamma \int_a^x K(x,t)G(Z(t))dt \tag{1}$$

with the initial conditions

$$Z^{(r)}(a) = b_r, \quad r = 0,1,2,\cdots,(j-1), \tag{2}$$

where $Z^{(j)}(x)$ is the j^{th} derivative of the unknown function $Z(x)$ that will be determined, $K(x,t)$ is the kernel of the equation, $f(x)$ is an analytic function, G is nonlinear function of Z and a, b, γ, and b_r are real finite constants. Recently, many authors focus on the development of numerical and analytical techniques for integro-differential equations. For instance, we can remember the following works. Abbasbandy and Elyas [1] studied some applications on variational iteration method for solving system of nonlinear volterra integro-differential equations, Hamoud and Ghadle [5] applied the hybrid methods for solving nonlinear Volterra-Fredholm integro-differential equations, Alao et al. [2] used Adomian decomposition and variational iteration methods for solving integro-differential equations, Yang and Hou [21] applied the Laplace decomposition method to solve the fractional integro-differential equations, Mittal and Nigam [19] applied the Adomian decomposition method to approximate solutions for fractional integro-differential equations, and Behzadi et al. [4] solved some class of nonlinear Volterra-Fredholm integro-differential equations by homotopy analysis method. Moreover, several authors have applied the Adomian decomposition method and the variational iteration method to find the approximate solutions of various types of integro-differential equations [5, 6, 7, 9, 10, 11, 12, 17, 19, 21].

The main objective of the present paper is to study the behavior of the solution that can be formally determined by semi-analytical approximated methods as the modified decomposition method and modified homotopy perturbation method.

2. Description of the Methods

Some powerful methods have been focusing on the development of more advanced and efficient methods for integro-differential equations such as the MDM [6, 8] and MHPM [1, 2, 3, 13, 14, 15, 16, 18]. We will describe all these methods in this section:

2.1. Description of the MDM

Assuming $f(x)$ has a series expansion, finds its series expansion and then applies the Laplace transformation $\mathcal{L}$ on both sides of Eq.(1)

$$\mathcal{L}[Z^{(n)}(x)] = \mathcal{L}\Big[f(x) + \gamma \int_a^x K(x,t)G(Z(t))dt\Big].$$

Using the differentiation property of the Laplace transform, we have

$$\begin{aligned} & s^n\mathcal{L}[Z(x)] - D^{(n-1)}Z(0) - sD^{(n-2)}Z(0) - \cdots - s^{(n-1)}Z(0) \\ & = \mathcal{L}\Big[f(x) + \gamma \int_a^x K(x,t)G(Z(t))dt\Big]. \end{aligned} \tag{3}$$

Further simplification of Eq.(3) resulted into

$$\begin{aligned}\mathcal{L}[Z(x)] &= \frac{1}{s^n}\Big[\mathcal{L}[f(x)] + D^{(n-1)}Z(0) + sD^{(n-2)}Z(0) + \cdots + s^{(n-1)}Z(0)\Big] \\ &+ \frac{1}{s^n}\mathcal{L}\Big[\gamma\int_a^x K(x,t)G(Z(t))dt\Big]. \end{aligned} \tag{4}$$

Now we apply MDM

$$G(Z(x)) = \sum_{n=0}^{\infty} A_n, \tag{5}$$

where A_n; $n \geq 0$ are the Adomian polynomials determined formally as follows:

$$A_n = \frac{1}{n!}\Big[\frac{d^n}{d\mu^n}G(\sum_{i=0}^{\infty}\mu^i Z_i)\Big]\Big|_{\mu=0}. \tag{6}$$

The Adomian polynomials were introduced in [**?**, 20, 21] as:

$$\begin{aligned} A_0 &= G(Z_0); \\ A_1 &= Z_1 G^{'}(Z_0); \\ A_2 &= Z_2 G^{'}(Z_0) + \frac{1}{2!}Z_1^2 G^{''}(Z_0); \\ A_3 &= Z_3 G^{'}(Z_0) + Z_1 Z_2 G^{''}(Z_0) + \frac{1}{3!}Z_1^3 G^{'''}(Z_0), ... \end{aligned}$$

The standard decomposition technique represents the solution of Z as the following series:

$$Z = \sum_{i=0}^{\infty} Z_i. \tag{7}$$

By substituting (5) and (7) in Eq.(4) we have

$$\begin{aligned}\sum_{i=0}^{\infty} Z_i(x) &= \frac{1}{s^n}\Big[\mathcal{L}[f(x)] + D^{(n-1)}Z(0) + sD^{(n-2)}Z(0) + \cdots + s^{(n-1)}Z(0)\Big] \\ &+ \frac{1}{s^n}\mathcal{L}\Big[\gamma\int_a^x K(x,t)\sum_{i=0}^{\infty} A_i dt\Big]. \end{aligned} \tag{8}$$

The components $Z_0, Z_1, Z_2, \cdots$ are usually determined by using the condition in (2).

2.2. Description of the MHPM

This method is applied to solve a large class of linear and nonlinear problems with approximations converging rapidly to exact solutions. This section is devoted to reviewing MHPM for solving nonlinear integro-differential equation. To explain MHPM, we consider the above integro-differential equation as

$$L[u] = Z^{(j)}(x) - f(x) - \gamma \int_a^x K(x,t)G(Z(t))dt \tag{9}$$

with solution $Z(x)$. As a possible remedy, we can define homotopy $H(u,p)$ by

$$H(u,0) = F(u), \qquad H(u,1) = L(u),$$

where $F(u)$ is a functional operator with known solution v_0, which can be obtained easily. In MHPM, we define

$$v_0(x) = a + bx + cx^2 + dx^3,$$

which is dependent on the order of differentiation. Typically, we may choose a convex homotopy by

$$H(u,p) = (1-p)F(u) + pL(u) = 0. \tag{10}$$

and continuously trace an implicitly defined curve from a starting point $H(v_0, 0)$ to a solution function $H(Z, 1)$. The embedding parameter p monotonously increases from zero to unit as trivial problem $F(u) = 0$ is continuously deformed to the original problem $L(u) = 0$. The embedding parameter $p \in (0, 1]$ can be considered as an expanding parameter [20].

The HPM uses the homotopy parameter p as an expanding parameter to obtain

$$u = v_0 + pv_1 + p^2 v_2 + \cdots \tag{11}$$

When $p \longrightarrow 1$, Eq.(11) corresponds to Eq.(10) and becomes the approximate solution of Eq.(9), i.e.,

$$Z = \lim_{p \longrightarrow 1} u = v_0 + v_1 + v_2 + \cdots \tag{12}$$

Series Eq.(12) is convergent for most cases, and the rate of convergence depends on $L(u)$.

3. Numerical Results

In this section, we present the numerical techniques based on MDM and MHPM to solve Volterra integro-differential equations.

Example 1. Consider the following nonlinear Volterra integro-differential equation:

$$Z^{(4)}(x) = e^{-3x} + e^{-x} - 1 + 3\int_0^x Z^3(s)ds,$$

with the conditions

$$Z(0) = Z''(0) = 1, \quad Z'(0) = Z'''(0) = -1,$$

and the exact solution is

$$Z(x) = e^{-x}.$$

Table 1: Numerical Results of the Example 1.

x	Exact solution	MDM	MHPM
0	1	1	1
0.04	0.9607894392	0.960789545	0.9608106692
0.08	0.9231163464	0.923118053	0.9232854120
0.12	0.8869204367	0.886929077	0.8874885866
0.16	0.8521437890	0.852171094	0.8534850921
0.20	0.8187307531	0.818797419	0.8213405980
0.24	0.7866278611	0.786766100	0.7911217470
0.28	0.7557837415	0.756039847	0.7628963355
0.32	0.7261490371	0.726585945	0.7367334755
0.36	0.6976763261	0.698376168	0.7127037390

Example 2. Consider the following Volterra integro-differential equation:

$$Z^{(4)}(x) = e^{x} - \frac{1}{2}e^{2x} + \frac{1}{2} - \int_0^x Z(s)Z''(s)ds,$$

with the conditions

$$Z(0) = Z'(0) = Z''(0) = 1, \quad Z'''(0) = k,$$

and the exact solution is

$$Z(x) = e^{x}.$$

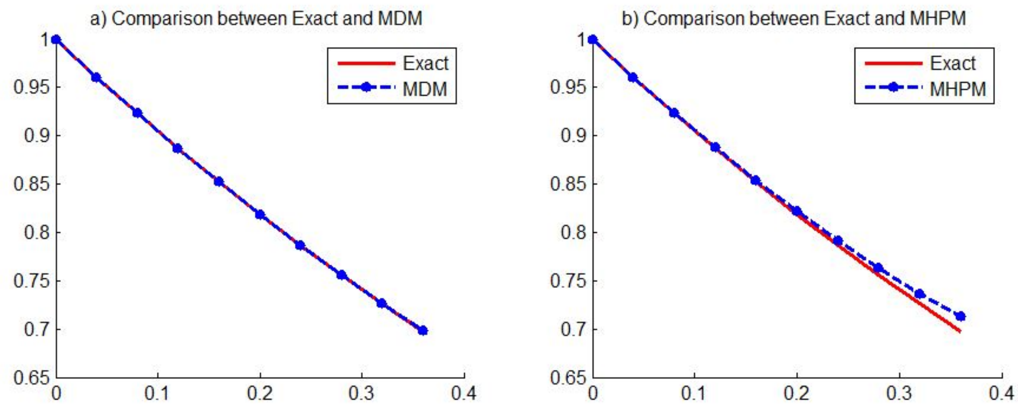

Figure 1: Numerical Results of the Example 1.

Table 2: Numerical Results of the Example 2.

x	Exact solution	MDM	MHPM
0	1	1	1
0.1	1.105170918	1.105170968	1.105230748
0.2	1.221402758	1.221403165	1.221843785
0.3	1.349858808	1.349860175	1.351209687
0.4	1.491824698	1.491827922	1.494673169
0.5	1.648721271	1.648727480	1.653535684
0.6	1.822118800	1.822129142	1.829034189
0.7	2.013752707	2.013767812	2.022315475
0.8	2.225540928	2.225559662	2.234405354
0.9	2.459603111	2.459619956	2.466171899
1.0	2.718281828	2.718281828	2.718281826

4. Comparison Among the Methods

The comparison among of the methods, it can be seen from the results of the above examples:

- The methods are powerful, efficient and give approximations of higher accuracy. Also, they can produce closed-form solutions if they exist.

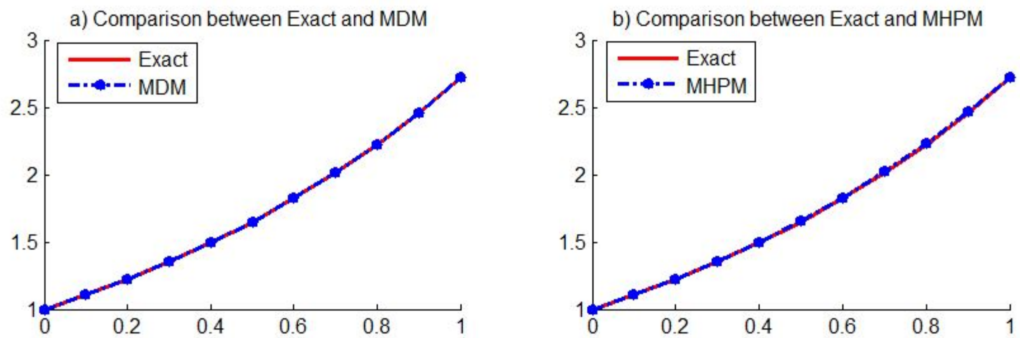

Figure 2: Numerical Results of the Example 2.

- Although the results obtained by these methods when applied to nonlinear Volterra integro-differential equations are the same approximately. MDM is seen to be much easier and more convenient than the MHPM.
- Tables 1 and 2 displayed the comparison of MDM, MHPM with the exact solutions. The error of the results obtained from the tables show that MDM gives a better result than MHPM. It was also discovered from the figures that the MDM converged to the exact more rapidly than the MHPM.

5. Conclusion

We present a comparative study between the MDM and MHPM for solving nonlinear Volterra integro-differential equations. From the computational viewpoint, the MDM is more efficient, convenient and easy to use. The methods are very powerful and efficient in finding analytical as well as numerical solutions for wide classes of linear and nonlinear Volterra integro-differential equations. The numerical results establish the precision and efficiency of the proposed techniques.

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