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A REMARK ON CO-IDEALS IN IMPLICATIVE SEMIGROUPS WITH APARTNESS

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Abstract. In 2001, Jun and Kim introduced the concept of ideals in implicative semigroups. Implicative semigroups with apartness were introduced and analyzed in 2016-19 by the author in his four published articles. In the last of these (D. A. Romano. *On co-ideals in implicative semiroups with apartness*. Turk. J. Math. Comput. Sci., 11(2)(2019), 101-106), the author analyzes the concept of co-ideals in such semigroups. In this paper, as a continuation of the aforementioned articles, the author shows that the concept of co-ideals is consistent with the concept of ideals, introduced by Jun and Kim. The author demonstrates this by proving that a strong complement of a co-ideal in an implicative semigroup with apartness is an ideal in terms of Jun and Kim.

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1. Introduction

The notions of implicative semigroup were introduced by Chan and Shum [6]. For the general development of implicative semilattice theory, the ordered ideals and filters play an important role. It has been shown by Nemitz [11]. Motivated by this, Chan and Shum [6] established some elementary properties and constructed quotient structure of implicative semigroups via ordered filters. Jun and Kim [9] discussed ordered ideals of implicative semigroups.

In paper [12], in setting of Bishop's constructive mathematics, following the ideas of Chan and Shum and other authors mentioned above, the author introduced the notion of implicative semigroups with tight apartness and gave some fundamental characterization of these semigroups. In [12, 13, 16, 17] and in this article, using sets with apartness and co-order relations introduced by the author, instead of partial order the author analyzed the implicative semigroups with apartness. An interested reader can find in the articles [14, 15] a little more information on the relations of co-quasiorder and co-order and their applications in algebraic structures built using sets with apartness as carriers. In this research, the author studied side effects induced by existence of apartness and co-orders. Additionally, in [12] the author introduced the notion of co-filters in an implicative semigroup. Further, in [13] he analyzed a connection between co-filters and strongly extensional homomorphisms of implicative semigroup with apartness. In article [17], the concept of co-ideals in implicit semigroups with apartness is introduced and analyzed.

In this article, as a continuation of his mentioned articles [12, 13, 16, 17], the author discuss about concept of co-ideals in such semigroups. The idea that this paper focuses on is the generally accepted principled-philosophical commitment in the Bishop's constructive framework that a concept of A, introduced by an apartness relation and strongly extensional predicates, associated with a classical concept of B if strong compliment $A^{\triangleleft}$ of A satisfies the conditions by which that concept B is determined. In this case, we show that the concept of co-ideals in implicative semigroups with apartness, introduced by the author in article [17], is associated with the concept of ideals of implicative semigroups introduced in article [9] by Jun and Kim.

2. Preliminaries

2.1. The Bishop's constructive framework

This report is in Bishop's constructive algebra in a sense of papers [5, 7, 8, 14, 15] and books [1, 2, 3, 4], [18](Chapter 8: Algebra). For example, in the articles [5, 7, 8] the focus is on semigroups with apartness. An interested reader can find much more information in our article [15] on algebraic structures determined on sets with apartness.

Let $(S, =, \neq)$ be a constructive set (i.e. it is a relational system with the relation "$\neq$"). The diversity relation "$\neq$" ([2]) is a binary relation on S, which satisfies the following properties:

$$\neg(x \neq x),\ x \neq y \Longrightarrow y \neq x,\ x \neq y \wedge y = z \Longrightarrow x \neq z\ .$$

If it satisfies the following condition

$$(\forall x, z \in S)(x \neq z \Longrightarrow (\forall y \in S)(x \neq y \vee y \neq z)),$$

then, it is called apartness (A. Heyting). In this paper, is assume that the basic apartness is tight, i.e. that it satisfies the following

$$(\forall x, y \in S)(\neg(x \neq y) \Longrightarrow x = y).$$

For subset X of S, we say that it is a strongly extensional subset of S if and only if $(\forall x \in X)(\forall y \in S)(x \neq y \vee y \in S)$ holds. For subsets X and Y of S, it is said that the subset X is set-set apart from the subset Y, and it is denoted by $X \bowtie Y$, if and only if $(\forall x \in X)(\forall y \in Y)(x \neq y)$. It's labeled like this $x \lhd Y$, instead of $\{x\} \bowtie Y$, and, of course, $x \neq y$ instead of $\{x\} \bowtie \{y\}$. With $X^{\lhd} = \{x \in S : x \lhd X\}$ is denoted strong complement of X.

For a function $f : (S, =, \neq) \longrightarrow (T, =, \neq)$ is says that it is strongly extensional if and only if

$$(\forall a, b \in S)(f(a) \neq f(b) \Longrightarrow a \neq b).$$

For relation $\alpha \subseteq S \times S$, it is says that it is an co-order relation on semigroup S, if it is consistent, co-transitive and linear

$$\alpha \subseteq \neq,\ \alpha \subseteq \alpha * \alpha,\ \neq \subseteq \alpha \cup \alpha^{-1},$$

where α has to be compatible with the semigroup operation in the following way

$$(\forall x, y, z \in S)(((xz, yz) \in \alpha \vee (zx, zy) \in \alpha) \Longrightarrow (x, y) \in \alpha).$$

In addition to this term, the term 'anti-order relation' is used (See, for example: [12, 13]). In this article both terms are used. The α is said to be a co-quasiorder if it is consistent and co-transitive relation.

Speaking by the language of the classical algebra, the relation α is left and right cancellative. Here, $*$ is the filed product between relations defined by the following way: If α and β are relations on set S, then filed product $\beta * \alpha$ of relation α and β is the relation given by $\{(x, z) \in X \times X : (\forall y \in X)((x, y) \in \alpha \vee (y, z) \in \beta)\}$.

For undefined notions and notations the reader can referred to the following papers [5, 7, 8, 12, 13, 14, 15].

2.2. Implicative semigroup with apartness

In this subsection, some definitions and the necessary results will be repeated. When it comes to a *negatively anti-ordered* semigroup (briefly, n.a-o. semigroup) ([12, 13]), then it is meant a set S with a co-order α and a binary internal operation '$\cdot$' (sometimes written by xy instead of $x \cdot y$) such that for all $x, y, z \in S$ the following holds:

(1) $(xy)z = x(yz)$,
(2) $(xz, yz) \in \alpha$ or $(zx, zy) \in \alpha$ implies $(x, y) \in \alpha$, and
(3) $(xy, x) \lhd \alpha$ and $(xy, y) \lhd \alpha$.

In that case for anti-order α we will say that it is a *negative anti-order relation* on semigroup. The operation '$\cdot$' is extensional and strongly extensional function from $S \times S$ into S, i.e. it has to be

$$(x, y) = (x', y') \Longrightarrow xy = x'y'$$
$$(xy \neq x'y \vee yx \neq yx') \Longrightarrow x \neq x'$$

for any elements x, x', y, y' of S.

A n.a-o. semigroup $(S, =, \neq, \cdot, \alpha)$ is said to be *implicative* if there is an additional binary operation $\otimes : S \times S \longrightarrow S$ such that the following is true

(4) $(z, x \otimes y) \in \alpha \Longleftrightarrow (zx, y) \in \alpha$ for any elements x, y, z of S.

In addition, let us recall that the internal binary operation '$\otimes$' must satisfy the following implications:

$$(a, b) = (u, v) \Longrightarrow a \otimes b = u \otimes v,$$
$$a \otimes b \neq u \otimes v \Longrightarrow (a, b) \neq (u, v).$$

The operation $\otimes$ is called *implication*. From now on, an implicative n.a-o. semigroup is simply called an *implicative semigroup*.

An implicative semigroup is to be *commutative* if it satisfies the following condition

$$(\forall x \in S)(\forall y \in S)(x \cdot y = y \cdot x).$$

Let α be a relation on S. For an element a of S we put $a\alpha = \{x \in S : (a, x) \in \alpha\}$ and $\alpha a = \{x \in S : (x, a) \in \alpha\}$. In the following proposition we give some properties of negative anti-order relation on semigroup.

Theorem 1. ([12], Theorem 3.1) *If $\alpha \subseteq S \times S$ is an anti-order relation on a semigroup S, then the following statements are equivalent:*

(i) *α is a negative co-order relation;*

(ii) *αb for any b in S has the following properties:*

$xy \in \alpha b \Longrightarrow x \in \alpha b \wedge y \in \alpha b,$

$x \in \alpha b \Longrightarrow (x, y) \in \alpha \vee y \in \alpha b;$

(iii) $(\forall a, b \in S)(\alpha a \cup \alpha b \subseteq \alpha(ab))$*;*

(iv) *$a\alpha$ is an ideal of S for any a in S;*

(v) $(\forall a, b \in S)((ab)\alpha \subseteq a\alpha \cap b\alpha)$.

In any implicative semigroup S there exist a special element of S, the biggest element in $(S, \alpha^{\triangleleft})$, which is the left neutral element in $(S, \cdot)$.

Some elementary properties of semigroup with apartness are given in the following proposition ([12], Theorem 3.3, Theorem 3.4, Corollary 3.2 and Corollary 3.3).

Theorem 2. (a) $(\forall x \in S)(x \otimes x = 1)$*;*

(b) $(\forall x \in S)(\forall y \in S)((x, y) \in \alpha \Longleftrightarrow 1 \neq x \otimes y)$*;*

(c) $(\forall x \in S)(1 = x \otimes 1)$ *and* $(\forall x \in S)(x = 1 \otimes x)$.

2.3. The concept of co-ideals

In this subsection the author introduce the concept of co-ideals of an implicative semigroup with apartness:

Definition 1 ([17] Definition 3.1)**.** *A subset K of S is called* co-ideal *if the following holds:*
(K1) $(\forall x, y \in S)(x \cdot y \in K \Longrightarrow y \in K)$ *and*
(K2) $(\forall x, y, z \in S)(x \otimes z \in K \Longrightarrow ((x \cdot y, z) \in \alpha \vee y \in K))$.
We say for co-ideal K of S that it is proper *co-ideal if $K \subset S$ is valid.*

The condition (K2) is equivalent to
(K2') $(\forall x, y, z \in S)(x \otimes z \in K \Longrightarrow ((x, y \otimes z) \in \alpha \vee y \in K))$
according to (4).

3. The Main Results

It is easy to check that the following holds:

Proposition 1. *If K is a co-ideal in an implicative semigroup S, then*
(K3) $(\forall y, z \in S)(z \in K \Longrightarrow ((y, z) \in \alpha \vee y \in K))$.

Proof. If we put x = 1 in (K2), we get

$$1 \otimes z \in K \Longrightarrow ((1 \cdot y, z) \in \alpha \vee y \in K)).$$

Given that $1 \otimes z = z$ according to (c) of Theorem 2 and $1 \cdot y = y$ according to Corollary 3.1 in [12], we obtain (K3). □

Proposition 2. *If S is a commutative implicative semigroup with apartness, then condition (K3) implies the condition (K2).*

Proof. Let K be a subset of S and let $x, y, z \in S$ be arbitrary such that $x \otimes z \in K$. Then $(y, x \otimes z) \in \alpha \vee y \in K$ by (K3). Thus $(yx, z) \in \alpha \vee y \in K$ by (4). Since S is comutative, we have $(xy, z) \in \alpha \vee y \in K$. So we have proved the condition (K2). □

It is easily to prove the following

Proposition 3. *Let K be a co-ideal in an implicative semigroup S with apartness. Then*
(K4) $(\forall x, y \in S)(x \otimes y \in K \Longrightarrow y \in K)$.

Proof. If we put $z = y$ in (K2), we get

$$x \otimes y \in K \implies ((x \cdot y, y) \in \alpha \vee y \in K).$$

Since the first option is impossible by (3), must to be $y \in K$. □

The following proposition gives a little more information about the properties of co-ideals

Proposition 4. *Let K be a co-ideal in an implicative semigroup S with apartness. Then*
(K5) $(\forall y, z \in S)((y \otimes z) \otimes z \in K \implies (y \in K \wedge z \in K))$

Proof. If we put $x = y \otimes z$ in (K2), we get

$$(y \otimes z) \otimes z \in K \implies ((y \otimes z, y \otimes z) \in \alpha \vee y \in K).$$

So, we have $y \in K$ because the first option is impossible. On the other hand, from $(x \otimes z) \otimes z \in$ follows $z \in K$ by Proposition 3. Therefore, (K5) is proved. □

The claim of the following lemma is demonstrable in the classical case ([6], Theorem 1.4 (7)). This claim holds also if S is an implicative semigroup with apartness with evidence slightly different from the original evidence.

Lemma 3. *Let S be an implicative semigroup with apartness. Then*
(5) $(\forall x, y, z \in S)(x \otimes (y \otimes z) = (x \cdot y) \otimes z)$.

Proof. Let $x, y, z \in S$ be arbitrary elements. Then, from $x \otimes (y \otimes z) = x \otimes (y \otimes z)$ follows $(x \otimes (y \otimes z), x \otimes (y \otimes z)) \lhd \alpha$. In [12], Theorem 3.2, it is shown that (4) implies
(4 ') $(\forall x, y, z \in S)((x, y \otimes z) \lhd \alpha \iff (x \cdot y, z) \lhd \alpha$.
So, we have $((x \otimes (y \otimes z)) \cdot x, y \otimes z) \lhd \alpha$ and by (4') again $((x \otimes (y \otimes z)) \cdot x \cdot y, z) \lhd \alpha$ with respect (1). Thus $((x \otimes (y \otimes z)), (x \cdot y) \otimes z) \lhd \alpha$. On the other hand, from $(x \cdot y) \otimes z = (x \cdot y) \otimes z$ follows $((x \cdot y) \otimes z, (x \cdot y) \otimes z) \lhd \alpha$. Thus $(((x \cdot y) \otimes z) \cdot x \cdot y, z) \lhd \alpha$ with respect (1). Then $(((x \cdot y) \otimes z) \cdot x, y \otimes z) \lhd \alpha$ by (4') and again by (4') we have $(((x \cdot y) \otimes z), x \otimes (y \otimes z)) \lhd \alpha$. Thus, equality (5) is proved. □

Proposition 5. *Let K be a co-ideal in an implicative semigroup S with apartness. Then*

$$(\forall x, y, z \in S)((x \otimes (y \otimes z)) \otimes z \in K \implies x \cdot y \in K).$$

Proof. From $(x \otimes (y \otimes z)) \otimes z \in K$ follows

$$((x \otimes (y \otimes z)), u \otimes z) \in \alpha \vee u \in K$$

for any $u \in S$. If we choose $u = x \cdot y$, we get

$$((x \otimes (y \otimes z)), (x \cdot y) \otimes z) \in \alpha \vee x \cdot y \in K.$$

The first option is impossible by Lemma 3. So, have to be $x \cdot y \in K$. $\square$

Remark 1. *From this Proposition immediately follows the assertion of the Proposition 4 if we put* $x = 1$.

In the article [9], Definition 3.1, the concept of ideals in implicit semigroups is defined as a subset J in S that satisfies the following two conditions
(J1) $(\forall x, y \in S)(y \in J \implies x \otimes y \in J)$ and
(J2) $(\forall x, yz \in S)(x \in J \wedge y \in J \implies (x \otimes (y \otimes z)) \otimes z \in J)$.
Let us show that our determination of co-ideal K in an implicative semigroup S with apartness by Definition 1 is correct, i.e. let us show that the concepts of ideals and co-ideals are associated. In order to achieve this, we will show that $K^{\lhd}$ is an ideal in the sense of the foregoing description.

Theorem 4. *Let K be a co-ideal of an implicative semigroup S with apartness. Then $K^{\lhd}$ satisfies the conditions (J1) and (J2).*

Proof. Let $x, y, u \in S$ be arbitrary elements such that $y \in K^{\lhd}$ and $u \in K$. Then $u \neq x \otimes y \vee x \otimes y \in K$ by strongly extensionality of K. From option $x \otimes y \in K$ follows $y \in K$ by (K4). It is impossible in accordance with hypothesis $y \lhd K$. So, have to be $x \otimes y \neq u \in K$. Therefore, $x \otimes y \in K^{\lhd}$. So, we have proved that set $K^{\lhd}$ satisfies condition (J1).

Let $x, y, z, u \in S$ be arbitrary elements such that $x \in K^{\lhd}$, $y \in K^{\lhd}$ and $u \in K$. Then

$$u \neq (x \otimes (y \otimes z)) \otimes z \vee (x \otimes (y \otimes z)) \otimes z \in K$$

by strongly extensionality of K in S. From $(x \otimes (y \otimes z)) \otimes z \in K$ follows $x \cdot y \in K$ by Proposition 5. Thus $y \in K$ by (K1). This contradicts the hypothesis $y \lhd K$. So, it has to be $(x \otimes (y \otimes z)) \otimes z \neq u \in K$. Finally, we have $(x \otimes (y \otimes z)) \otimes z \in K^{\lhd}$. We have proved that the set $K^{\lhd}$ satisfies the condition (J2). $\square$

4. Final observation

The concept of co-ideals in implicit semigroups with apartness in Bishop's framework ([1, 2, 3, 4, 10, 18]) was introduced and analyzed in our article [17]. This paper was preceded by the papers [12, 13, 16] in which the concepts of co-ideals and co-filters of such semigroups were analyzed. In this article, we shown that our concept of co-ideals is consistent with the concept of ideals introduced in 2001 by Jun and Kim in article [9]. The connection between the concept of ideals (in classical algebra) and the concept of co-ideals (in Bishop's constructive mathematics) in implicative semigroups with apartness is one of the specificity of Bishop's constructive algebra. This article shows that the concept of co-ideals is well-defined since it is associated with the concept of the classically defined concept of ideals in these semigroups. The main result of this paper is Theorem 4 in which this connection is proved. Of course, a few of necessary propositions previously proved preceded the proof of this theorem.

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