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UNIFIED REPRESENTATION OF CERTAIN HARMONIC UNIVALENT FUNCTIONS STARLIKE AND CONVEX WITH RESPECT TO SYMMETRIC POINTS

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Abstract. Let H denote the class of functions which are harmonic and univalent in the open unit disc $D = \{z : |z| < 1\}$. In this paper, we define and investigate a family of complex-valued harmonic functions that are sense-preserving and univalent in D. We obtain growth result, extreme points, convolution, convex combinations and the closure property under certain integral operator for this family of functions.

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Introduction

A continuous complex-valued function $f = u + iv$ is defined in a simply connected complex domain E is said to be harmonic in E if both u and v are real harmonic in E. In any simply connected complex domain, we can write

$$f(z) = h(z) + \overline{g(z)}, \tag{1}$$

where h and g are analytic in E. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in E is that $|h'(z)| > |g'(z)|$ in E (see clunie and sheil-Small [3]). Let $D = \{z : |z| < 1\}$ be the open unit disc and H denote the class of functions of the form (1) which are harmonic, univalent and sense-preserving in D for $f(0) = f_z(0) - 1 = 0$. Each $f \in H$ can be expressed as $f = h + \overline{g}$ where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad (|b_1| < 1), \tag{2}$$

are analytic in D.

Note that, if the co-analytic part of f is zero, the class H reduces to the class of normalized analytic functions.

Also, let $\overline{H}$ be the subclass of H consisting of functions $f = h + \overline{g}$ such that the functions h and g take the form

$$h(z) = z - \sum_{n=2}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad (a_n \geq 0,\ b_n \geq 0,\ |b_1| < 1). \tag{3}$$

In 2006, Darus et al. [4] defined the class $HS_s^*(\beta)$ of harmonic starlike function with respect to symmetric points of order β as follows:

Definition 1. *[4] Let $f \in H$. Then $f \in HS_s^*(\beta)$ is said to be harmonic starlike of order β with respect to symmetric points if and only if, for $0 \leq \beta < 1$, $z' = \frac{\partial}{\partial\theta}\left(z = re^{i\theta}\right)$, $f'(z) = \frac{\partial}{\partial\theta}\left(f(z) = f\left(re^{i\theta}\right)\right)$, $0 \leq r < 1$ and $0 \leq \theta < 2\pi$,*

$$\operatorname{Re}\left\{\frac{2\left(zh'(z) + z\overline{g'(z)}\right)}{z'\left[(h(z) - h(-z)) + \left(\overline{g(z) - g(-z)}\right)\right]}\right\} \geq \beta, \tag{4}$$

and $\overline{H}S_s^*(\beta) = HS_s^*(\beta) \cap \overline{H}$.

Lemma 1. *[4] Let $f = h + \overline{g}$ with h and g of the form (2). If*

$$\sum_{n=1}^{\infty}\left[\left(\frac{2n - \beta(1 - (-1)^n)}{2(1-\beta)}\right)|a_n| + \left(\frac{2n + \beta(1 - (-1)^n)}{2(1-\beta)}\right)|b_n|\right] \leq 2, \quad (0 \leq \beta < 1). \tag{5}$$

then f is harmonic, sense-preserving, univalent in D and $f \in HS_s^*(\beta)$. The condition (5) is also necessary if $f \in \overline{H}S_s^*(\beta)$

In 2011, Janteng and Halim [7, *with* $b = 1$] defined the class $HS_c^*(\beta)$ of harmonic starlike function with respect to conjugate points of order β as follows

Definition 2. *Let $f \in H$. Then, for $0 \leq \beta < 1$,*

$$z' = \frac{\partial}{\partial\theta}(z = re^{i\theta}),$$

$$f'(z) = \frac{\partial}{\partial\theta}(f(z) = f(re^{i\theta})),$$

$$0 \leq r < 10 \leq \theta < 2\pi,$$

$f \in HS_c^*(\beta)$ is said to be harmonic starlike functions of order β with respect to conjugate points, if and only if,

$$Re\left\{\frac{2\left(zh'(z)+z\overline{g'(z)}\right)}{z'\left[\left(h(z)+\overline{h(\overline{z})}\right)+\left(\overline{g(z)+\overline{g(\overline{z})}}\right)\right]}\right\}\geq\beta, \tag{6}$$

and $\overline{H}S_c^*(\beta)=HS_c^*(\beta)\cap\overline{H}$.

Lemma 2. $[7, with b=1]$ *Let $f=h+\overline{g}$ with h and g of the form (2). If*

$$\sum_{n=1}^{\infty}\left[\frac{n-\beta}{1-\beta}|a_n|+\frac{n+\beta}{1-\beta}|b_n|\right]\leq 2,\ \ 0\leq\beta<1. \tag{7}$$

then f is harmonic, sense-preserving, univalent in D and $f\in HS_c^*(\beta)$. The condition (7) is also necessary if $f\in\overline{H}S_c^*(\beta)$.

In 2008, Janteng et al. [8] defined the class $HS_{sc}^*(\beta)$ of harmonic starlike functions of order β with respect to symmetric conjugate points as follows:

Definition 3. [8] *Let $f\in H$. Then, for $0\leq\beta<1$, $z'=\frac{\partial}{\partial\theta}(z=re^{i\theta})$, $f'(z)=\frac{\partial}{\partial\theta}(f(z)=f(re^{i\theta})), 0\leq r<1$ and $0\leq\theta<2\pi$, a function $f\in HS_{sc}^*(\beta)$ is said to be harmonic starlike of order β with respect to symmetric conjugate points, if and only if,*

$$Re\left\{\frac{2\left(zh'(z)+z\overline{g'(z)}\right)}{z'\left[\left(h(z)-\overline{h(\overline{-z})}\right)+\left(\overline{g(z)-\overline{g(\overline{-z})}}\right)\right]}\right\}\geq\beta, \tag{8}$$

and $\overline{H}S_{sc}^*(\beta)=HS_{sc}^*(\beta)\cap\overline{H}$.

Lemma 3. [8] *Let $f=h+\overline{g}$ with h and g of the form (2). If*

$$\sum_{n=1}^{\infty}\left[\left(\frac{2n-\beta\left(1-(-1)^n\right)}{2(1-\beta)}\right)|a_n|+\left(\frac{2n+\beta\left(1-(-1)^n\right)}{2(1-\beta)}\right)|b_n|\right]\leq 2,(0\leq\beta<1). \tag{9}$$

then f is harmonic, sense-preserving, univalent in D and $f\in HS_{sc}^*(\beta)$. The condition (9) is also necessary if $f\in\overline{H}S_{sc}^*(\beta)$.

Now, we shall recall some definitions and lemmas about the harmonic univalent functions which are convex with respect to other points of order β.

Definition 4. [6] *Let $f \in H$. Then $f \in HC_s(\beta)$ is said to be harmonic convex of order β with respect to symmetric points, if and only if, for $0 \leq \beta < 1$,*

$$Re\left\{\frac{2\left[z^2h''(z)+zh'(z)+\overline{z^2g''(z)+zg'(z)}\right]}{zh'(z)-\overline{zg'(z)}+zh'(-z)-\overline{zg'(-z)}}\right\}\geq\beta, \tag{10}$$

and $\overline{H}C_s(\beta)=HC_s(\beta)\cap\overline{H}$.

The following lemma proved by Janteng and Halim [6].

Lemma 4. [6] *Let $f = h + \overline{g}$ with h and g of the form (2). If*

$$\sum_{n=1}^{\infty} n\left[\frac{(2n-\beta(1-(-1)^n))}{2(1-\beta)}|a_n|+\frac{(2n+\beta(1-(-1)^n))}{2(1-\beta)}|b_n|\right]\leq 2,\ \ (0\leq\beta<1). \tag{11}$$

then f is harmonic, sense-preserving, univalent in D and $f \in HC_s(\beta)$. The condition (11) is also necessary if $f \in \overline{H}C_s(\beta)$.

Definition 5. *[5] Let $f \in H$. Then $f \in HC_c(\beta)$ is said to be harmonic convex of order β with respect to conjugate points, if and only if, for $0 \leq \beta < 1$,*

$$Re\left\{\frac{2\left[z^2h''(z)+zh'(z)+\overline{z^2g''(z)+zg'(z)}\right]}{zh'(z)-\overline{zg'(z)}+\overline{zh'(\overline{z})}-\overline{z\overline{g'(\overline{z})}}}\right\}\geq\beta, \tag{12}$$

and $\overline{H}C_c(\beta)=HC_c(\beta)\cap\overline{H}$.

Lemma 5. *[5] Let $f = h + \overline{g}$ with h and g of the form (2). If*

$$\sum_{n=1}^{\infty} n\left[\frac{(n-\beta)}{1-\beta}|a_n|+\frac{(n+\beta)}{1-\beta}|b_n|\right]\leq 2,\ \ (0\leq\beta<1), \tag{13}$$

then f is harmonic, sense-preserving, univalent in D and $f \in HC_c(\beta)$. The condition (13) is also necessary if $f \in \overline{H}C_c(\beta)$.

Similarly, we can define the class $HC_{sc}(\beta)$ as follows

Definition 6. *Let $f \in H$. Then $f \in HC_{sc}(\beta)$ is said to be harmonic convex of order β with respect to symmetric conjugate points, if and only if, for $0 \leq \beta < 1$,*

$$Re\left\{\frac{2\left[z^2h''(z)+zh'(z)+\overline{z^2g''(z)+zg'(z)}\right]}{zh'(z)-\overline{zg'(z)}+\overline{zh'(-\overline{z})}-\overline{z\overline{g'(-\overline{z})}}}\right\}\geq\beta, \tag{14}$$

and $\overline{H}C_{sc}(\beta) = HC_{sc}(\beta) \cap \overline{H}$.

Lemma 6. *Let* $f = h + \overline{g}$ *with* h *and* g *of the form* (2)*. If*

$$\sum_{n=1}^{\infty} n \left[\frac{(2n - \beta(1-(-1)^n))}{2(1-\beta)} |a_n| + \frac{(2n + \beta(1-(-1)^n))}{2(1-\beta)} |b_n| \right] \le 2, \quad (0 \le \beta < 1), \tag{15}$$

then f *is harmonic, sense-preserving, univalent in* D *and* $f \in HC_{sc}(\beta)$*. The condition* (15) *is also necessary if* $f \in \overline{H}C_{sc}(\beta)$*.*

In view of Lemma 2 and Lemma 8, we will define and study the following an interesting unification class $\overline{H}G_s^*(\beta,\gamma)$ as follows:-

Definition 7. *Let* $f = h + \overline{g}$ *with* h *and* g *of the form* (3)*. A function* $f \in \overline{H}$ *is said to be in the class* $\overline{H}G_s^*(\beta,\gamma)$ *if it satisfies the following condition,*

$$\sum_{n=1}^{\infty} n^{\gamma} \left[\frac{(2n - \beta(1-(-1)^n))}{2(1-\beta)} |a_n| + \frac{(2n + \beta((1-(-1)^n)))}{2(1-\beta)} |b_n| \right] \le 2, \tag{16}$$

where $0 \le \beta < 1$ *and* $\gamma \ge 0$.

(i) Putting $\gamma = 0$ in Definition 13, we obtain the class $\overline{H}S_s^*(\beta)$ which defined by Darus et al.[4];

(ii) Putting $\gamma = 1$ in Definition 13, we obtain the class $\overline{H}C_s^*(\beta)$ which defined by Janteng and Halim [6];

(iii) Putting $\gamma = m$ in Definition 13, we obtain the class $SH_s(m,\beta)$ which defined by AL-Khal and AL Kharsani [1].

In this paper, we obtain some geometric properties for the class $\overline{H}G_s^*(\beta,\gamma)$ such as distortion theorems, extreme points, convolution, convex combinations and the closure property under certain integral operator for this family of functions.

1. Main Results

Unless otherwise mentioned, we assume in the reminder of this paper that $0 \le \beta < 1$, $\gamma \ge 0$.

Theorem 7. *. Let $f = h + \overline{g}$ with h and g of the form (3) and $f(z) \in \overline{H}G_s^*(\beta, \gamma)$. Then $f(z)$ is sense-preserving harmonic univalent in D.*

Proof. If $z_1 \neq z_2$,

$$\begin{aligned}\left|\frac{f(z_2)-f(z_1)}{h(z_2)-h(z_1)}\right| &\geq 1-\left|\frac{g(z_2)-g(z_1)}{h(z_2)-h(z_1)}\right| \\ &= 1-\left|\frac{\sum_{n=1}^{\infty} b_n\left(z_2^n-z_1^n\right)}{(z_2-z_1)-\sum_{n=2}^{\infty} a_n\left(z_2^n-z_1^n\right)}\right| \geq 1-\frac{\sum_{n=1}^{\infty} n\left|b_n\right|}{1-\sum_{n=2}^{\infty} n\left|a_n\right|}\end{aligned}$$

by using (16), we have

$$\left|\frac{f(z_2)-f(z_1)}{h(z_2)-h(z_1)}\right| \geq 1-\frac{\sum_{n=1}^{\infty} n^\gamma\left(\frac{2n+\beta((1-(-1)^n))}{2(1-\beta)}\right)\left|b_n\right|}{1-\sum_{n=2}^{\infty} n^\gamma\left(\frac{2n-\beta(1-(-1)^n)}{2(1-\beta)}\right)\left|a_n\right|}>0$$

which proves the univalence. Also f is sense-preserving in D since

$$\begin{aligned}\left|h'(z)\right| &\geq 1-\sum_{n=2}^{\infty} n\left|a_n\right|\left|z^{n-1}\right|>1-\sum_{n=2}^{\infty} n\left|a_n\right| \\ &\geq 1-\sum_{n=2}^{\infty} n^\gamma\left(\frac{2n-\beta(1-(-1)^n)}{2(1-\beta)}\right)\left|a_n\right| \\ &\geq \sum_{n=1}^{\infty} n^\gamma\left(\frac{2n+\beta(1-(-1)^n)}{2(1-\beta)}\right)\left|b_n\right| \\ &\geq \sum_{n=1}^{\infty} n\left|b_n\right|\left|z^{n-1}\right| \geq \left|g'(z)\right|.\end{aligned}$$

The harmonic univalent functions

$$f(z)=z-\sum_{n=2}^{\infty} \frac{2(1-\beta)}{n^\gamma\left(2n-\beta\left(1-(-1)^n\right)\right)} x_n z^n+\sum_{n=1}^{\infty} \frac{2(1-\beta)}{n^\gamma\left(2n+\beta\left(1-(-1)^n\right)\right)} \overline{y_n z^n}. \tag{17}$$

where

$$\sum_{n=2}^{\infty}\left|x_n\right|+\sum_{n=1}^{\infty}\left|y_n\right|=1,$$

shows that the coefficient bound given by (16) is sharp. The function of the form

(17) is in the class $\overline{H}G_s^*(\beta,\gamma)$ because

$$\begin{aligned} &\sum_{n=1}^{\infty}\frac{n^{\gamma}}{2}\left[\left(\frac{2n-\beta\left(1-(-1)^n\right)}{(1-\beta)}\right)|a_n|+\left(\frac{2n+\beta\left((1-(-1)^n)\right)}{(1-\beta)}\right)|b_n|\right] \\ = \;& 1+\sum_{n=2}^{\infty}|x_n|+\sum_{n=1}^{\infty}|y_n|=2. \end{aligned}$$

This completes the proof of Theorem 7.

Theorem 8. *Let the function* $f=h+\overline{g}$ *with* h *and* g *of the form* (3) *and* $f(z)\in\overline{H}G_s^*(\beta,\gamma)$, *then*

$$|f(z)|\geq(1-|b_1|)\,r-\frac{1}{2^{\gamma}}\left[\frac{1-\beta}{2}-\frac{1+\beta}{2}|b_1|\right]r^2,\;\;|z|=r<1, \tag{18}$$

and

$$|f(z)|\leq(1+|b_1|)\,r+\frac{1}{2^{\gamma}}\left[\frac{1-\beta}{2}-\frac{1+\beta}{2}|b_1|\right]r^2,\;\;|z|=r<1. \tag{19}$$

The equalities in (18) *and* (19) *are attained for the functions* f *given by*

$$f(z)=(1-b_1)\,\overline{z}-\frac{1}{2^{\gamma}}\left[\frac{1-\beta}{2}-\frac{1+\beta}{2}|b_1|\right]\overline{z}^2, \tag{20}$$

and

$$f(z)=(1+b_1)\,\overline{z}+\frac{1}{2^{\gamma}}\left[\frac{1-\beta}{2}-\frac{1+\beta}{2}|b_1|\right]\overline{z}^2, \tag{21}$$

where $|b_1|\leq\frac{1-\beta}{1+\beta}$.

Proof. Let $f(z) \in \overline{H}G_s^*(\beta,\gamma)$, then we have

$$\begin{aligned}
|f(z)| &\geq (1-|b_1|)\,r - \sum_{n=2}^{\infty}(|a_n|+|b_n|)\,r^n \\
&\geq (1-|b_1|)\,r - r^2\sum_{n=2}^{\infty}(|a_n|+|b_n|) \\
&= (1-|b_1|)\,r - \frac{(1-\beta)}{2^{\gamma+1}}r^2\sum_{n=2}^{\infty}2^{\gamma}\left(\frac{2}{1-\beta}|a_n| + \frac{2}{1-\beta}|b_n|\right) \\
&\geq (1-|b_1|)\,r - \frac{(1-\beta)}{2^{\gamma+1}}r^2. \\
&\quad \cdot\sum_{n=2}^{\infty}\frac{n^{\gamma}}{2}\left[\left(\frac{2n-\beta(1-(-1)^n)}{(1-\beta)}\right)|a_n| + \left(\frac{2n+\beta((1-(-1)^n))}{(1-\beta)}\right)|b_n|\right] \\
&\geq (1-|b_1|)\,r - \frac{1-\beta}{2^{\gamma+1}}\left[1-\frac{1+\beta}{1-\beta}|b_1|\right]r^2. \\
&\geq (1-|b_1|)\,r - \frac{1}{2^{\gamma}}\left[\frac{1-\beta}{2}-\frac{1+\beta}{2}|b_1|\right]r^2.
\end{aligned}$$

which proves the assertion (18) of Theorem 8. The proof of assertion (19) is similar, thus, we omit it.

The following covering result follows from the left hand inequality of Theorem 8

Corollary 9. *Let $f(z) \in \overline{H}G_s^*(\beta,\gamma)$, then*

$$\left\{w : |w| < \left(1-\frac{1-\beta}{2^{\gamma+1}}\right) - \left(1-\frac{1+\beta}{2^{\gamma+1}}\right)|b_1|\right\} \subset f(D)$$

where

$$|b_1| \leq \frac{1-\beta}{1+\beta}$$

Theorem 10. *Let the function $f = h + \overline{g}$ with h and g of the form (3). Then $f(z) \in clco\ \overline{H}G_s^*(\beta,\gamma)$, if and only if*

$$f(z) = \sum_{n=1}^{\infty}\left[X_n h_n(z) + Y_n g_n(z)\right], \tag{22}$$

where

$$h_1(z) = z, \tag{23}$$

$$h_n(z) = z - \frac{2(1-\beta)}{n^{\gamma}(2n-\beta(1-(-1)^n))}z^n, \quad (n=2,3,...) \tag{24}$$

and

$$g_n(z) = z + \frac{2(1-\beta)}{n^\gamma(2n+\beta(1-(-1)^n))}\overline{z}^n \quad (n = 1, 2, ...), \tag{25}$$

where

$$\sum_{n=1}^{\infty}(X_n + Y_n) = 1, X_n \geq 0 \text{ and } Y_n \geq 0.$$

In particular, the extreme points of the class $\overline{H}G_s^*(\beta,\gamma)$ *are* $\{h_n\}$ $(n \geq 2)$ *and* $\{g_n\}$ $(n \geq 1)$, *respectively.*

Proof. For a function $f(z)$ of the form (22), we have

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty}[X_n h_n(z) + Y_n g_n(z)] \\ &= \sum_{n=1}^{\infty}\left[\begin{array}{c} X_n\left(z - \frac{2(1-\beta)}{n^\gamma(2n-\beta(1-(-1)^n))}z^n\right) \\ +Y_n\left(z + \frac{2(1-\beta)}{n^\gamma(2n+\beta(1-(-1)^n))}\overline{z}^n\right)\end{array}\right] \\ &= z - \sum_{n=2}^{\infty}\frac{2(1-\beta)}{n^\gamma(2n-\beta(1-(-1)^n))}X_n z^n \\ &\quad + \sum_{n=1}^{\infty}\frac{2(1-\beta)}{n^\gamma(2n+\beta(1-(-1)^n))}Y_n \overline{z}^n. \end{aligned}$$

But

$$\begin{aligned} &\sum_{n=2}^{\infty}\left(\frac{n^\gamma(2n-\beta(1-(-1)^n))}{2(1-\beta)}.\frac{2(1-\beta)}{n^\gamma(2n-\beta(1-(-1)^n))}X_n\right) \\ &+\sum_{n=1}^{\infty}\left(\frac{n^\gamma(2n+\beta(1-(-1)^n))}{2(1-\beta)}.\frac{2(1-\beta)}{n^\gamma(2n+\beta(1-(-1)^n))}Y_n\right) \\ =\ &\sum_{n=2}^{\infty}X_n + \sum_{n=1}^{\infty}Y_n = 1 - X_1 \leq 1. \end{aligned}$$

Thus $f(z) \in clco\ \overline{H}G_s^*(\beta,\gamma)$.

Conversely, assume $f(z) \in clco\ \overline{H}G_s^*(\beta,\gamma)$. Set

$$X_n = \frac{n^\gamma(2n-\beta(1-(-1)^n))}{2(1-\beta)}|a_n| \quad (n = 2, 3, ...),$$

$$Y_n = \frac{n^\gamma(2n+\beta(1-(-1)^n))}{2(1-\beta)}|b_n| \quad (n = 1, 2, ...).$$

Then, by using (16) , we have

$$0 \leq X_n \leq 1 \ \ (n = 2, 3, ...) \text{ and } 0 \leq Y_n \leq 1 \ \ (n = 1, 2, ...) .$$

Define

$$X_1 = 1 - \sum_{n=2}^{\infty} X_n - \sum_{n=1}^{\infty} Y_n.$$

Thus, we obtain

$$f(z) = \sum_{n=1}^{\infty} [X_n h_n(z) + Y_n g_n(z)] .$$

This completes the proof of Theorem 10 .

Now, we discuss the convolution properties and convex combination. Let the functions $f_m(z)$ $(m = 1, 2)$ be defined by

$$f_m(z) = z - \sum_{n=2}^{\infty} |a_{n,m}| z^n + \sum_{n=1}^{\infty} |b_{n,m}| \overline{z}^n \ \ (m = 1, 2) \tag{26}$$

are in the class $\overline{H}G_s^*(\beta, \gamma)$ the convolution of $f_m(z)$ $(m = 1, 2)$ is defined as,

$$(f_1 * f_2)(z) = z - \sum_{n=2}^{\infty} |a_{n,1}| |a_{n,2}| z^n + \sum_{n=1}^{\infty} |b_{n,1}| |b_{n,2}| \overline{z}^n.$$

We first show that the class $\overline{H}G_s^*(\beta, \gamma)$ is closed under convolution.

Theorem 11. *For* $0 \leq \delta \leq \beta < 1$, *let the functions* $f_1 \in \overline{H}G_s^*(\beta, \gamma)$ *and* $f_2 \in \overline{H}G_s^*(\delta, \gamma)$ *Then*

$$(f_1 * f_2)(z) \in \overline{H}G_s^*(\beta, \gamma) \subset \overline{H}G_s^*(\delta, \gamma) . \tag{27}$$

Proof. Let the functions $f_m(z)$ $(m = 1, 2)$ are given by (26) where f_1 be in the class $\overline{H}G_s^*(\beta, \gamma)$ and f_2 be in the class $\overline{H}G_s^*(\delta, \gamma)$. We wish to show that the coefficients of $(f_1 * f_2)(z)$ satisfy the required condition given in (16) . For $f_2 \in \overline{H}G_s^*(\delta, \gamma)$, we note that $|a_{n,2}| < 1$ and $|b_{n,2}| < 1$. Now for the convolution functions $(f_1 * f_2)(z)$, we obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{n^{\gamma}}{2}\left[\left(\frac{2n-\delta\left(1-(-1)^{n}\right)}{(1-\delta)}\right)|a_{n,1}|\,|a_{n,2}|+\left(\frac{2n+\delta\left((1-(-1)^{n})\right)}{(1-\delta)}\right)|b_{n,1}|\,|b_{n,2}|\right]\\
\leq\ &\sum_{n=1}^{\infty}\frac{n^{\gamma}}{2}\left[\left(\frac{2n-\delta\left(1-(-1)^{n}\right)}{(1-\delta)}\right)|a_{n,1}|+\left(\frac{2n+\delta\left((1-(-1)^{n})\right)}{(1-\delta)}\right)|b_{n,1}|\right]\\
\leq\ &\sum_{n=1}^{\infty}\frac{n^{\gamma}}{2}\left[\left(\frac{2n-\beta\left(1-(-1)^{n}\right)}{(1-\beta)}\right)|a_{n,1}|+\left(\frac{2n+\beta\left((1-(-1)^{n})\right)}{(1-\beta)}\right)|b_{n,1}|\right]\\
\leq\ &2,
\end{aligned}$$

since $0\leq\delta\leq\beta<1$ and $f_1\in\overline{H}G_s^{*}(\beta,\gamma)$. Thus $(f_1*f_2)(z)\in\overline{H}G_s^{*}(\beta,\gamma)\subset\overline{H}G_s^{*}(\delta,\gamma)$. This completes the proof of Theorem 4.

Theorem 12. *The class* $\overline{H}G_s^{*}(\beta,\gamma)$ *is closed under convex combinations.*

Proof. For $i=1,2,...,$ let $f_i\in\overline{H}G_s^{*}(\beta,\gamma)$ where

$$f_i(z)=z-\sum_{n=2}^{\infty}|a_{n,i}|\,z^{n}+\sum_{n=1}^{\infty}|b_{n,i}|\,\overline{z}^{n}.$$

then, for $\sum_{i=1}^{\infty}t_i=1$, $0\leq t_i<1$, the convex combination of f_i can be written as

$$\sum_{i=1}^{\infty}t_if_i(z)=z-\sum_{n=2}^{\infty}\left(\sum_{i=1}^{\infty}t_i\,|a_{n,i}|\right)z^{n}+\sum_{n=1}^{\infty}\left(\sum_{i=1}^{\infty}t_i\,|b_{n,i}|\right)\overline{z}^{n}. \tag{28}$$

By (16), we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{n^{\gamma}}{2}\left[\left(\frac{2n-\beta\left(1-(-1)^{n}\right)}{(1-\beta)}\right)\left(\sum_{i=1}^{\infty}t_i\,|a_{n,i}|\right)+\left(\frac{2n+\beta\left(1-(-1)^{n}\right)}{(1-\beta)}\right)\left(\sum_{i=1}^{\infty}t_i\,|b_{n,i}|\right)\right]\\
=\ &\sum_{i=1}^{\infty}t_i\left(\sum_{n=1}^{\infty}\frac{n^{\gamma}}{2}\left[\left(\frac{2n-\beta\left(1-(-1)^{n}\right)}{(1-\beta)}\right)|a_{n,i}|+\left(\frac{2n+\beta\left(1-(-1)^{n}\right)}{(1-\beta)}\right)|b_{n,i}|\right]\right)\\
\leq\ &2\sum_{i=1}^{\infty}t_i=2.
\end{aligned}$$

This completes the proof of Theorem 12 .

Finally, we will examine a closure property of the class $\overline{H}G_s^{*}(\beta,\gamma)$ under the modified generalized Bernardi-Libera-Livingston integral operator; see [2, 9, 10] for harmonic univalent functions $f(z)$ given by (3) as follows; see [11]

$$\begin{aligned} L_c f(z) &= L_c(h)+\overline{L_c(g)}, \qquad (c>-1) \\ &= \frac{c+1}{z^c}\int_0^z t^{c-1}\left(t-\sum_{n=2}^{\infty}a_n t^n\right)dt+\overline{\frac{c+1}{z^c}\int_0^z t^{c-1}\left(\sum_{n=1}^{\infty}b_n t^n\right)dt} \\ &= z-\sum_{n=2}^{\infty}\frac{c+1}{c+n}a_n z^n+\overline{\sum_{n=1}^{\infty}\frac{c+1}{c+n}b_n z^n}. \end{aligned} \tag{29}$$

Theorem 13. *Let $f(z)\in\overline{H}G_s^*(\beta,\gamma)$. Then $L_c f(z)$ defined by (29), is in the class $\overline{H}G_s^*(\beta,\gamma)$.*

Proof. If $f(z)$ given by (3), and

$$L_c f(z)=z-\sum_{n=2}^{\infty}A_n z^n+\overline{\sum_{n=1}^{\infty}B_n z^n}.$$

Form (29), it follows that

$$A_n=\frac{c+1}{c+n}a_n \text{ and } B_n=\frac{c+1}{c+n}b_n.$$

Therefore, we have

$$\begin{aligned} &\sum_{n=1}^{\infty}\frac{n^\gamma}{2}\left[\left(\frac{2n-\beta(1-(-1)^n)}{(1-\beta)}\right)\left(\frac{c+1}{c+n}\right)|a_n|+\left(\frac{2n+\beta(1-(-1)^n)}{(1-\beta)}\right)\sum_{i=1}^{\infty}\left(\frac{c+1}{c+n}\right)|b_n|\right] \\ \leq\ &\sum_{n=1}^{\infty}\frac{n^\gamma}{2}\left[\left(\frac{2n-\beta(1-(-1)^n)}{(1-\beta)}\right)|a_n|+\left(\frac{2n+\beta(1-(-1)^n)}{(1-\beta)}\right)|b_n|\right]\leq 2. \end{aligned}$$

Since $f(z)\in\overline{H}G_s^*(\beta,\gamma)$, by Theorem 7, then $L_c f(z)\in\overline{H}G_s^*(\beta,\gamma)$. This completes the proof of Theorem 13.

Theorem 14. *Let c be real number such that $c>-1$. If $L_c f(z)\in\overline{H}G_s^*(\beta,\gamma)$, then the function $f(z)$ define by (29) is univalent in $|z|<R^*$, where $R^*=\min\{r_1,r_2\}$,*

$$r_1=\inf_n\left\{\frac{n^{\gamma-1}(2n-\beta(1-(-1)^n))(1+c)}{2(1-\beta)(n+c)}\right\}^{\frac{1}{n-1}},$$

$$r_2=\inf_n\left\{\frac{n^{\gamma-1}(2n+\beta((1-(-1)^n)))(1+c)}{2(1-\beta)(c-n)}\right\}^{\frac{1}{n-1}}.$$

The result is sharp.

Proof. It follows from (29) that

$$\begin{aligned}
f(z) &= \frac{z^{1-c}\left(z^{c}L_{c}f(z)\right)'}{c+1}, \qquad (c>-1)\\
&= \frac{cL_{c}f(z)+z\left(L_{c}f(z)\right)'}{c+1}\\
&= z-\sum_{n=2}^{\infty}\frac{n+c}{1+c}a_{n}z^{n}+\sum_{n=1}^{\infty}\overline{\frac{c-n}{1+c}b_{n}\overline{z}^{n}}\\
&= H(z)+\overline{G(z)}.
\end{aligned}$$

Since, we need to prove

$$\left|f'(z)-1\right|<1, \text{ in } |z|<R^{*}=\min\left\{r_{1},r_{2}\right\}.$$

Now

$$\begin{aligned}
\left|f'(z)-1\right| &= \left|H'(z)-\overline{G'(z)}-1\right|\\
&= \left|\sum_{n=2}^{\infty}\frac{n(n+c)}{1+c}a_{n}z^{n-1}+\sum_{n=1}^{\infty}\frac{n(c-n)}{1+c}\overline{b_{n}z^{n-1}}\right|\\
&\leq \sum_{n=2}^{\infty}\left|\frac{n(n+c)}{1+c}a_{n}z^{n-1}\right|+\sum_{n=1}^{\infty}\left|\frac{n(c-n)}{1+c}\overline{b_{n}z^{n-1}}\right|\leq 1. \qquad (30)
\end{aligned}$$

From inequality (16), (30) will be true if

$$\begin{aligned}
&\sum_{n=2}^{\infty}\left|\frac{n(n+c)}{1+c}a_{n}z^{n-1}\right|+\sum_{n=1}^{\infty}\left|\frac{n(c-n)}{1+c}\overline{b_{n}z^{n-1}}\right|\\
\leq\ &\sum_{n=2}^{\infty}n^{\gamma}\frac{\left(2n-\beta\left(1-(-1)^{n}\right)\right)}{2(1-\beta)}\left|a_{n}\right|+\sum_{n=1}^{\infty}n^{\gamma}\frac{\left(2n+\beta\left(\left(1-(-1)^{n}\right)\right)\right)}{2(1-\beta)}\left|b_{n}\right|,
\end{aligned}$$

or

$$\begin{aligned}
\left|z^{n-1}\right| &= \frac{n^{\gamma}\left(2n-\beta\left(1-(-1)^{n}\right)\right)(1+c)}{2(1-\beta)n(n+c)},\\
r_{1} &= \inf_{n}\left\{\frac{n^{\gamma-1}\left(2n-\beta\left(1-(-1)^{n}\right)\right)(1+c)}{2(1-\beta)(n+c)}\right\}^{\frac{1}{n-1}}, \qquad (n\geq 2)
\end{aligned}$$

and

$$\begin{aligned}
\left|\overline{z^{n-1}}\right| &= \frac{n^{\gamma}\left(2n+\beta\left(\left(1-(-1)^{n}\right)\right)\right)(1+c)}{2(1-\beta)n(c-n)},\\
r_{2} &= \inf_{n}\left\{\frac{n^{\gamma-1}\left(2n+\beta\left(\left(1-(-1)^{n}\right)\right)\right)(1+c)}{2(1-\beta)(c-n)}\right\}^{\frac{1}{n-1}}, \qquad (n\geq 1).
\end{aligned}$$

Therefore, $f(z)$ is univalent in $|z| < R^* = \min\{r_1, r_2\}$.

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