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RECONSTRUCTION NUMBER OF TOPOLOGICAL SPACES WITH UNIQUE ISOLATED POINT

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Abstract. For a topological space X, the subspace X_x is called a *card* of X and the collection of all cards of X is called the *multideck* of X. The *reconstruction number* of X, denoted by $rn(X)$, is the minimum number of cards of X which can only belong to the multideck of X and not to the multideck of any other space Y, Y is not homeomorphic to X; these cards thus uniquely identifying X. It is shown that the reconstruction number is two or three for all finite topological spaces of order at least four with unique isolated point such that all open sets together form an ascending chain.

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1. Introduction

A vertex-deleted subgraph or *card* $G - v$ of a graph G is obtained by deleting the vertex v and all edges incident with v. The collection of all cards of G is called the *deck* of G. A graph H is a *reconstruction* of G if H has the same deck as G. A graph is said to be *reconstructible* if it is isomorphic to all its reconstructions. A parameter p defined on graphs is reconstructible if, for any graph G, it takes the same value on every reconstruction of G. The graph reconstruction conjecture, posed by Kelly and Ulam [10] in 1941, asserts that every graph G on n (≥ 3) vertices is reconstructible. More precisely, if G and H are finite graphs with at least three vertices such that $\mathscr{D}(H) = \mathscr{D}(G)$, then G and H are isomorphic. For a reconstructible graph G, Harary and Plantholt [7] defined the reconstruction number of a graph G, denoted by $rn(G)$, to be the minimum number of cards which can only belong to the deck of G and not to the deck of any other graph $H, H \ncong G$, these cards thus uniquely identifying G.

In 2016, Pitz and Suabedissen [9] have introduced the concept of reconstruction in topological spaces as follows. For a topological space X, the subspace X_x is called a *card* of X and it is denoted by X_x. The set $\mathscr{D}(X) = \{[X_x] \ : \ x \in X\}$ of subspaces of X is called the *deck* of X, where $[X_x]$ denotes the homeomorphism class of the card X_x. Given topological spaces X and Z, we say that Z is a *reconstruction* of X if their decks agree. A topological space X is said to be *reconstructible* if the only reconstructions of it are the spaces homeomorphic to X. Formally, a space X is reconstructible if $\mathscr{D}(X) = \mathscr{D}(Z)$ implies $X \cong Z$ and a property $\mathscr{P}$ of topological spaces is reconstructible if $\mathscr{D}(X) = \mathscr{D}(Z)$ implies "X has $\mathscr{P}$ if and only if Z has $\mathscr{P}$".

By *order of a set*, we mean the number of elements in the set. By *size of a space*, we mean the number of open sets in the space. Terms not defined here are taken as in [4]. A topological space X is said to have an *ascending chain* if all the open sets of X together form an ascending chain, or in other words, any two open sets in X are comparable. Gartside et al [5, 6, 9] have proved that the space of real numbers, the space of rational numbers, the space of irrational numbers, every compact Hausdroff space that has a card with a maximal finite compactification, and every Hausdroff continuum X with weight $\omega(X) < |X|$ are reconstructible. In their papers, they also proved certain properties of a space, namely all hereditary separation axioms and all cardinal invariants are reconstructible. Manvel et al [8] have done similar work in 1991 itself and they have reconstructed all finite sequences from their subsequences. Recently, Jini and Monikandan [2, 3] have proved that all finite topological spaces are reconstructible.

The *multideck* of a topological space X is the multiset $\mathscr{D}'(X) = \{X_x \ : \ x \in X\}$. In other words, the mutlideck not only knows which card occur, but also how often they occur. A space X is said to be *weakly reconstructible* if it is reconstructible from the multideck of X. Here we study the parameter, that is, the reconstruction number of topological spaces. Jini and Monikandan [1] have shown that all finite topological spaces are weakly reconstructible. For a weakly reconstructible space X, the *reconstruction number* of X, denoted by $rn(X)$, is defined to be the minimum number of cards which can only belong to the multideck of X and not to the multideck of any other space Y, Y is not homeomorphic to X; these cards thus uniquely identifying X. In this paper, it is proved that the reconstruction number of all finite topological spaces of order at least four with ascending chain and unique isolated point is two or three.

An *extension* of a card (X_x, τ_{X_x}) is a space (Y, τ_Y) $Y = (X - \{x\}) \cup \{y\}$, where y is an element not in $X - \{x\}$, and each U in τ_Y is either in τ_{X_x} or $U = V \cup \{y\}$ for some $V \in \tau_{X_x}$. The collection of all extensions of a card X_x is denoted by $\mathscr{E}(X_x)$.

For a space X, to prove $rn(X) = k$, we proceed as follows.

(i) First show, for any $k - 1$ cards in the multideck of X, that there exist two nonhomeomorphic spaces whose decks contain all these $k - 1$ cards (therefore $rn(X) \geq k$).

(ii) Next, we consider specific k cards of X.

(iii) Finally, show that every extension of at least one of these k cards is either homeomorphic to X or does not have all the other $k-1$ cards in its multideck (therefore $rn(X) \leq k$).

The next lemma asserts that no space can be determined (upto homeomorphism) from just one card $X - \{x\}$.

Lemma 1. *For any topological space* $X, rn(X) \geq 2$.

Proof. It suffices to show that only one card of X alone can not identify the topology on X uniquely (upto homeomorphism). In other words, it is enough to show that two non-homeomorphic spaces can be formed from any card of X. Consider any card X_x and the two collections $\tau_1 = \tau_{X_x} \cup \{X\}$ and $\tau_2 = \{U : U \in \tau_{X_x}\} \cup \{U \cup \{x\} : U \in \tau_{X_x}\}$. Clearly, the two spaces (X, τ_1) and (X, τ_2) are nonhomeomorphic, but they have X_x as its common card. Therefore, $rn(X) \geq 2$.

2. Topological spaces with ascending chain

By X, where $X = \{x_1, x_2, \ldots, x_n\}$, we mean a finite topological space of order $n(\geq 4)$ with ascending chain and unique isolated point, say x_1. By an *m-open set*, we mean an open set of order m. By U_j, we mean the open set of order j in X.

Lemma 2. *Let X be a finite space with acending chain and unique isolated point. Then X has an ascending chain if at least three cards of X have ascending chains.*

Proof. It suffices to show that if X does not have an ascending chain, then at most two cards of X have ascending chains. If X does not have an ascending chain, then there exist two open sets A and B such that none of them is contained in the other. Hence both A and B are non-empty and not equal to X. We proceed further by two cases.

Case 1. A or B is just the set $\{x_1\}$.

Since A and B are not comparable, it follows from the assumption in Case 1 that $A \cap B = \phi$. Now, the open sets of X must be of the form $\phi, \{x_1\}, U_1, \ldots, U_r, B, B \cup \{x_1\}, V_1, \ldots, V_s, X$ for some $r \geq 1$ and $s \geq 1$. Suppose that $U_1 = B \subset B \cup \{x_1\} \subset V_1 \subset ... \subset V_s \subset X$. Then the card X_{x_1} alone has an ascending chain. Otherwise, $U_1 \neq B$ and hence no card of X has an ascending chain. Thus, in this case, at most one card of X can have an ascending chain.

Case 2. Both A and B are not equal to $\{x_1\}$.

Now $|A| \geq 2$ and $|B| \geq 2$. If $A \cap B = \phi$, then no card of X has an ascending chain. So, assume that $A \cap B \neq \phi$. We have two subcases as below.

Case 2.1. $A \cap B = U \neq \{x_1\}$.

Assume that $x_1 \in U$ and that x_1 lies in all the open sets in X (as otherewise no card of X has an ascending chain). Then all the open sets of X are of the form $\phi, \{x_1\}, U_1, \ldots, U_r, A, B, V_1, \ldots, V_s, X$. If $U_i \subseteq U_{i+1}$ and $V_j \subseteq V_{j+1}$ for all i, j, $A \cap B = U_i, |A| = |B| = |U_i| + 1$ and $A \cup B = V_1$, then the two cards X_{x_r} and X_{x_s}, where $x_r \in A - U_i$ and $x_s \in B - U_i$, alone have an ascending chain. If $U_i \subseteq U_{i+1}$ and $V_j \subseteq V_{j+1}$ for all i, j, $A \cap B = U_i$, and $|A| > |U_i| + 1$ or $|B| > |U_i| + 1$, then no card has an ascending chain. Finally, if $\{U_1, U_2, \ldots, U_r\}$ or $\{V_1, V_2, \ldots, V_s\}$ does not form an ascending chain, then none of the cards has an ascending chain. Thus, in this case, at most two cards of X have an ascending chain.

Case 2.2. $A \cap B = \{x_1\}$.

Now we can assume that x_1 belongs to all the open sets in X (as otherwise no card of X has an ascending chain). Then all the open sets of X are of the form $\phi, \{x_1\}, U_1, \ldots, U_r, A, B, V_1, \ldots, V_s, X$. If $A = U_1, |A| = 2, |B| \geq 3$ and $V_j \subseteq V_{j+1}$ for all j, then the card X_{x_r}, where $x_r \in A - \{x_1\}$, alone has an ascending chain. If $A = U_1, |A| \geq 3, |B| \geq 3$, and $V_j \subseteq V_{j+1}$ for all j, then no card has an ascending chain. Also, if $\{U_1, U_2, \ldots, U_r\}$ or $\{V_1, V_2, \ldots, V_s\}$ does not form an ascending chain, then none of the cards has an ascending chain. Finally, if $A \neq U_1$, then no card has an ascending chain. Thus, in this case, at most one card of X has an ascending chain.

Lemma 3. ([2])

Let X be a finite topological space with ascending chain. Then X has an i-open set for each $i,\ i = 1, 2, \ldots, n$ if and only if the multideck of X contains exactly one card which has an ascending chain and it has an j-open set for each $j,\ j = 1, 2, \ldots, n-1$.

Lemma 4. ([2])

Let X be a finite topological space of size m with unique isolated point. Then X has an ascending chain and X has no open set of order i for some $i,\ 2 \leq i \leq m-1$ if and only if the multideck of X contains at least two cards and each card has an ascending chain.

Lemma 5. *Let X be a finite topological space of size m with ascending chain and unique isolated point. Then X has no k-open set for a unique $k, k \in \{2, 3, \ldots, n-1\}$ if and only if the multideck has only three mutually nonhomeomorphic cards of size $m-1, m-1$ and m respectively such that one card has no k-open set, one card has no $(k-1)$-open set and the other card has a j-open set for each $j,\ 1 \leq j \leq n-1$.*

Proof. By assumption, let $\tau_X = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_{k+1}, U_{k+2}, \ldots, U_n\}$. Then clearly, the cards $X_{x_r},\ x_r \in U_n - U_{k+1}$ does not have the k-open set and $|\tau_{X_{x_r}}| = m-1$, the cards $X_{x_s},\ x_s \in U_{k-1}$ does not have the $(k-1)$-open set and $|\tau_{X_{x_s}}| = m-1$

and the cards $X_{X_{x_t}}, x_t \in U_{k+1} - U_{k-1}$, has the j-open set for each $j,\ 1 \leq j \leq n-1$ and $|\tau_{X_{x_t}}| = m$. To prove the sufficiency, assume to the contrary, that τ_X was not equal to the given form. If X would have an i-open set for every $i,\ i \in \{1, 2, \ldots, n\}$, then by Lemma 4, any two cards would be homeomorphic, giving a contradiction. Therefore, X has no i-open sets for at least two distinct i's, $2 \leq i \leq n-1$. Then

$$\tau_X = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_{k+t}, \ldots, U_{l-1}, U_{l+s}, \ldots, X\},\ t, s \geq 1.$$

Consequently, the card X_{x_e}, where $x_e \in X - U_{l+s}$, has no open sets of order $k + t - 1, l + s - 1$; the card X_{x_f}, where $x_f \in U_{l+s} - U_{l-1}$ contains no open set of order $k+t-1$; the card X_{x_g}, where $x_g \in U_{l-1} - U_{k+t}$ has no open set of order $k+t-1$; the card X_{x_h}, where $x_h \in U_{k+t} - U_{k-1}$ contains no open set of order $l-1$; the card X_{x_k}, where $x_k \in U_{k-1}$ has no open sets of order $k-2$ or $l-2$. In other words, the multideck of X would not contain a card having a j-open set for each $j,\ 1 \leq j \leq n-1$, giving a contradiction to the hypothesis.

Corollary 6. *Let X be a finite topological space of size m with ascending chain and unique isolated point. If X has no i-open sets for at least two distinct i's, $2 \leq i \leq n-1$ if and only if each card of X has no i-open set for at least one i and the multideck has at least two nonhomeomorphic cards each of size m and $m-1$.*

Theorem 7. *Let X be a finite topological with ascending chain and unique isolated point. If X has an i-open set for every $i,\ i = 1, 2, \ldots, n$, then $rn(X) = 3$.*

Proof. By Lemma 1, we have $rn(X) \geq 2$. First we prove that $rn(X) > 2$. It suffices to prove that any two cards of X can not determine the topology of X uniquely (upto homeomorphism). That is, to prove, for any two cards of X, there exist two nonhomeomorphic spaces whose decks contain both the cards. By Lemma 3, we have any two cards in the multideck are homeomorphic. Let X_{x_r} and X_{x_s} be any two cards of X, where

$$\tau_{X_{x_r}} = \{U_1, U_2, \ldots, U_{n-1}\}.$$

Consider $\mathscr{E}(X_{x_r})$. Since each of these open sets in X_{x_r} has only two possibilities in the extension, at most $2n$ open sets can be formed in the extension, where $n = |\tau_{X_{x_r}}|$. Hence the possible sizes of the extensions of the card X_{x_r} are $n, n+1, \ldots, 2n$. Therefore, $\mathscr{E}(X_{x_r}) = \{H_1, H_2, \ldots, H_p\}$, where $p \geq 2$ and for $q = 1, 2, \ldots, p$, the size of H_q is $n, n+1, \ldots, 2n$, where $n = |\tau_{X_{x_r}}|$. Note that, there are more than one extensions of size $n, n+1, \ldots, 2n-1$ and the open sets in extensions of size n and $n+1$ are in the ascending chain and for the extensions of size greater than $n+1$, the open sets does not in the ascending chain. So we denote the extensions of size $n, n+1, \ldots, 2n-1$ by $H_{1(p)}, H_{2(p)}, \ldots, H_{n-1(p)}, H_{n(p)}$ where $p \geq 2$ and the extension of size $2n$ by H_{n+1}. Therefore, we rewrite the collection $\mathscr{E}(X_{x_r})$ as

$\mathscr{E}(X_{x_r}) = \{H_{1(p)}, H_{2(p)}, \ldots, H_{n-1(p)}, H_{n(p)}, H_{n+1}\},\ p \geq 2$. Consider the extensions, say $H_{1(r)}, H_{2(r)}$ of size n and $n+1$ respectively, where

$$\tau_{H_{1(r)}} = \{\phi, U_1, U_2, \ldots, U_{r-1}, U_r \cup \{x_r\}, U_{r+1} \cup \{x_r\}, \ldots, U_{n-1} \cup \{x_r\}\}$$

and

$$\tau_{H_{2(r)}} = \{\phi, U_1, U_2, \ldots, U_{r-1}, U_r, U_r \cup \{x_r\}, U_{r+1} \cup \{x_r\}, \ldots, U_{n-1} \cup \{x_r\}\}.$$

In extension $H_{1(r)}$, the two cards namely, x_r, x_e, where $x_e \in U_r - U_{r-1}$ has j-open set for each $j,\ j = 1, 2, \ldots, n-1$ and in extension $H_{2(r)}$, by Lemma 3, it has the two cards X_{x_r}, X_{x_s}. Thus the extensions $H_{1(r)}, H_{2(r)}$ have both the cards X_{x_r}, X_{x_s} in their decks and thus $rn(X) > 2$.

We now show that $rn(X) \leq 3$. That is, to prove, there exist three cards of X such that every extension of at least one of the three cards is either homeomorphic to X or does not have the other two cards together in its multideck. Consider arbitrary three cards X_{x_r}, X_{x_s} and X_{x_t} and $\mathscr{E}(X_{x_r})$. The extension, say $H_{2(r)}$ of size $n+1$ is clearly homeomorphic to X. Consider the extensions of size n. By Lemma 5, only two cards has j-open set for each $j,\ j \in \{1, 2, \ldots, n-1\}$ and hence one of the two cards X_{x_s}, X_{x_t} does not belong to its multideck. Since all extensions of size $n+1$ are homeomorphic, consider the extensions of size greater than $n+1$. Since these extensions does not have the ascending chain form, by Lemma 2, at most two cards can have the ascending chain and hence one of the two cards X_{x_s}, X_{x_t} does not belong to its multideck. Hence the only extension consisting all the above three cards is $H_{2(r)}$. Therefore every extension in $\mathscr{E}(X_{x_r})$ other than $H_{2(r)}$ does not have the other two cards in its multideck. Hence $rn(X) \leq 3$, which completes the proof.

Theorem 8. *Let X be a finite topological space of size m with ascending chain and unique isolated point. If X has no k-open set for a unique $k,\ 2 \leq k \leq n-1$, then $rn(X) = 3$.*

Proof. By view of Lemma 5, we can assume that the multideck of X has three nonhomeomorphic cards, say $X_{x_r}, X_{x_s}, X_{x_t}$ of size $m-1$, $m-1$, or m such that $X_{x_{x_r}}$ does not have an k-open set, X_{x_s} does not have an $(k-1)$-open set and X_{x_t} has an j-open set for each $j,\ 1 \leq j \leq n-1$, where

$$\tau_{X_{x_r}} = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_{k+1}, U_{k+2}, \ldots, X_{x_r}\}$$

$$\tau_{X_{x_s}} = \{\phi, U_1, U_2, \ldots, U_{k-2}, U_k, U_{k+1}, \ldots, X_{x_s}\} \ \ \textit{and}$$

$$\tau_{X_{x_t}} = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_k, U_{k+1}, \ldots, X_{x_t}\}.$$

Six cases arise as shown below.
Case 1. The two cards are X_{x_r} and X_{x_t}.

Clearly $\mathscr{E}(X_{x_t}) = \{H_{1(p)}, H_{2(p)}, \ldots, H_{m-1(p)}, H_{m(p)}, H_{m+1}\}$, where $p \geq 2$. Consider the extensions $H_{1(k)}, H_{1(k+1)}$ of size m, where

$$\tau_{H_{1(k)}} = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_k \cup \{x_t\}, U_{k+1} \cup \{x_t\}, \ldots, X_{x_t} \cup \{x_t\}\},$$

$$\tau_{H_{1(k+1)}} = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_k, U_{k+1} \cup \{x_t\}, U_{k+2} \cup \{x_t\}, \ldots, X_{x_t} \cup \{x_t\}\}.$$

By Lemma 5, the decks of these extensions contain both the cards X_{x_r} and X_{x_t}.
Case 2. The two cards are X_{x_s} and X_{x_t}.

Now consider $\mathscr{E}(X_{x_t})$ and the extensions $H_{1(k)}, H_{1(k-1)}$ of size m, where

$$\tau_{H_{k-1}} = \{\phi, U_1, U_2, \ldots, U_{k-2}, U_{k-1} \cup \{x_t\}, U_k \cup \{x_t\}, \ldots, X_{x_t} \cup \{x_t\}\}.$$

By Lemma 5, the decks of these extensions contain both the cards X_{x_r} and X_{x_t}.
Case 3. The two cards are X_{x_t} and X_{x_t}.

Now consider $\mathscr{E}(X_{x_t})$ and the extensions $H_{1(k)}, H_{2(k)}$ of size m and $m+1$ respectively, where

$$\tau_{H_{2(k)}} = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_k, U_k \cup \{x_t\}, U_{k+1}, \cup\{x_t\}, \ldots, X_{x_t} \cup \{x_t\}\}.$$

In the extension $H_{1(k)}$, the two cards, namely $x_{x_t},\ X_{x_e}, x_e \in U_k - U_{k-1}$ have j-open set for each $j = 1, 2, \ldots, n-1$. By Lemma 3, the extension $H_{2(k)}$ has the two cards $X_{x_t},\ X_{x_t}$ in its multideck.
Case 4. The two cards are X_{x_r} and X_{x_s}.

Clearly $\mathscr{E}(X_{x_r}) = \{I_{1(p)}, I_2, \ldots, I_{m-2(p)},\ I_{m-1(p)}, I_m\}$, where $p \geq 2$. Consider the extensions $I_{1(k-1)}, I_{2(k+1)}$, of size $m-1, m$ respectively, where

$$\tau_{I_{1(k-1)}} = \{\phi, U_1, U_2, \ldots, U_{k-2}, U_{k-1} \cup \{x_r\}, U_{k+1} \cup \{x_r\}, \ldots, X_{x_r} \cup \{x_r\}\},$$

$$\tau_{I_{2(k+1)}} = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_{k+1}, U_{k+1} \cup \{x_r\}, U_{k+2} \cup \{x_r\}, \ldots X_{x_r} \cup \{x_r\}\}.$$

In the extension $H_{1(k-1)}$, the two cards, namely $X_{x_r}, X_{x_e}\ x_e \in U_{k+1} - U_{k-1}$ do not have the k-open set and $k-1$-open set, respectively. In the extension $H_{2(k+1)}$, the two cards, namely $X_{x_r}, X_{x_f},\ x_f \in U_{k-1}$ do not have the k-open set and $(k-1)$-open set, respectively.
Case 5. The two cards are X_{x_r} and X_{x_r}.

Now consider $\mathscr{E}(X_{x_r})$ and the extensions $I_{1(k+1)}, I_{2(k+1)}$ of size $m-1, m$ respectively, where

$$\tau_{I_{1(k+1)}} = \{\phi, U_1, U_2, \ldots, U_{k-2}, U_{k-1}, U_{k+1} \cup \{x_r\}, U_{k+2} \cup \{x_r\}, \ldots, X_{x_r} \cup \{x_r\}\}.$$

In the extension $I_{1(k+1)}$, the two cards, namely $X_{x_r}, X_{x_e},\ x_e \in U_{k+1} - U_{k-1}$ do not have the k-open set and in the extension $I_{2(k+1)}$, the two cards, namely, $X_{x_r}, X_{x_f},\ x_f \in$

$U_{k+2} - U_{k+1}$ do not have the k-open set.
Case 6. The two cards are X_{x_s} and X_{x_s}.

Clearly $\mathscr{E}(X_{x_s}) = \{J_{1(p)}, J_{2(p)}, \ldots, J_{m-2(p)}, J_{m-1(p)}, J_m\}$, where $p \geq 2$. Consider the extensions $J_{1(k)}, J_{2(k+1)}$ of size $m-1, m$ respectively, where

$$\tau_{J_{1(k)}} = \{U_1, U_2, \ldots, U_{k-2}, U_k, U_{k+1} \cup \{x_s\}, U_{k+2} \cup \{x_s\}, \ldots, X_{x_s} \cup \{x_s\}\},$$

$$\tau_{J_{2(k+1)}} = \{\phi, U_1, U_2, \ldots, U_{k-3}, U_{k-2}, U_{k-2} \cup \{x_s\}, U_k \cup \{x_s\}, \ldots, X_{x_s} \cup \{x_s\}\}.$$

In the extension $J_{1(k)}$, the two cards, namely $X_{x_s}, X_{x_e},\ x_e \in U_{k+1} - U_k$ do not have the $(k-1)$-open set and in extension, $J_{2(k+1)}$, the two cards, namely, $X_{x_s}, X_{x_f},\ x_f \in U_{k-2}$ do not have the $(k-1)$-open set. Thus, in all the six cases, we have proved that $rn(X) > 2$.

Now we prove that $rn(X) \leq 3$. Consider the three nonhomeomorphic cards $X_{x_r}, X_{x_s},\ X_{x_t}$ and the collection $\mathscr{E}(X_{x_t})$. The extension, say $H_{1(k)}$ of size m is clearly homeomorphic to X. Consider the other extensions $H_{1(j)}$ of size m, where $j \neq k$. By Lemma 5, the cards do not have either an l-open set or an $(l-1)$-open set, $l \neq k$. Hence (at most) one of the two cards X_{x_r}, X_{x_s} does not belong to its multideck. Next, consider the extensions of size $m+1$. Since these extensions have i-open set for each $i,\ i \in \{1, 2, \ldots, n\}$, by Lemma 3, the two cards X_{x_r}, X_{x_s} do not belong to its multideck. Finally, consider the extensions of size greater than $m+1$. Since these extensions do not have the ascending chain, by Lemma 2, these extensions can have at most two cards with ascending chain and hence one of the two cards X_{x_r}, X_{x_s} does not belong to its multideck. Therefore every extension in $\mathscr{E}(X_{x_t})$ other than $H_{1(k)}$ does not have the other two cards in its multideck. Hence $rn(X) \leq 3$, which completes the proof.

In Theorem 8, the reconstruction number is determined for a space with ascending chain and with out an i-open set for a unique $i,\ 2 \leq i \leq n-1$. We shall now determine the reconstruction number for a space with ascending chain and with out i-open sets for at least two distinct i's, $\ 2 \leq i \leq n-1$. Let k be the smallest integer and l be an integer such that X has no open sets of order k and l. Then $1 < k < l < n$. Then τ_X can be equal to one of the following collection.

$$\{\phi, U_1, U_2, \ldots, U_{k-1}, U_{k+t}, \ldots, X\}, \quad t \geq 2 \qquad \cdots (C1)$$

$$\{\phi, U_1, U_2, \ldots, U_{k-1}, U_{k+1}, \ldots, U_{l-1}, U_{l+t}, \ldots, X\},\ t \geq 1 \text{ and } k+1 \neq l-1 \qquad \cdots (C2)$$

$$\{\phi, U_1, U_2, \ldots, U_{k-1}, U_{k+1} = U_{l-1}, U_{l+t}, \ldots, X\},\ t \geq 1 \qquad \cdots (C3)$$

Theorem 9. *Let X be a finite topological space of size m with ascending chain and unique isolated point. If X has no i-open sets for at least two distinct i's, $2 \leq i \leq n-1$ and τ_X is equal to the collection given in (C1), then $rn(X) = 2$.*

Proof. By Lemma 1, we have $rn(X) \geq 2$. Choose the two cards X_{x_1}, X_{x_r}, where x_1 is the isolated point and X_{x_r} is any card of size m. Since X does not have open sets of order $k, k+1, \ldots, k+t-1,\ t \geq 2$, the card X_{x_r} has the open set of order $(k+t-1)$ or not. Without loss of generality, assume the former case. Since X_{x_1} is the isolated point deleted card, it must have the $(k+t-1)$-open set. Thus

$$\tau_{X_{x_1}} = \{\phi, U_1, U_2, \ldots, U_{k-2}, U_{k+t-1}, \ldots, X_{x_1}\},$$

$$\tau_{X_{x_r}} = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_{k+t-1}, \ldots, X_{x_r}\}.$$

Consider $\mathscr{E}(X_{x_r})$, where $\mathscr{E}(X_{x_r}) = \{H_{1(p)}, H_{2(p)}, \ldots, H_{m-1(p)}, H_{m(p)}, H_{m+1}\}$, $p \geq 2$. The extension, say $H_{1(k+t)}$ of size m is clearly homeomorphic to X, where

$$\tau_{H_{1(k+t)}} = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_{k+t-1} \cup \{x_r\}, \ldots, X_{x_r} \cup \{x_r\}\}.$$

Consider the other extensions of size m and at first the extensions $H_{1(c)}$, where $c > k+t-1$ and

$$\tau_{H_{1(c)}} = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_{k+t-1}, \ldots, U_c \cup \{x_r\}, \ldots, X_{x_r} \cup \{x_r\}\}.$$

The cards of these extensions have an open set of order $(k-1)$ or $(k-2)$. The cards having $(k-1)$-open set are clearly not homeomorphic to the card X_{x_1}. Consider the cards with $(k-2)$-open set. Since these cards have the $(k+t-2)$-open set and the card X_{x_1} has no such open set, it follows that those cards are not homeomorphic to the card X_{x_1}. Consider next the extensions $H_{1(d)}$, where $d \leq k-1$ and

$$\tau_{H_{1(d)}} = \{\phi, U_1, U_2, \ldots, U_d \cup \{x_r\}, U_{d+1} \cup \{x_r\}, \ldots, U_{k-1} \cup \{x_r\},$$
$$U_{k+t-1} \cup \{x_r\}, \ldots, X_{x_r} \cup \{x_r\}\}.$$

The cards of these extensions have either k-open set or $(k-1)$-open set. Since the card X_{x_1} has no open set of order k as well as $(k-1)$, these cards are not homeomorphic to X_{x_1}. Consider the extensions of size $m+1$. The size of the cards of these extensions are either $m+1$ or m and hence the card X_{x_1} does not belong to its multideck, since the size of X_{x_1} is $m-1$. Finally, consider the extensions of size greater than $m+1$. Then these extensions have the open sets or order either '$k-1, k+t-1$ 'or '$k-1, k$ 'or '$k-1, k+t-1, k+t$. 'In the former case, the cards of these extensions have open sets of order '$k-1, k+t-1$ 'or '$k-1, k+t-2$ 'or '$k-2, k+t-2$. 'In the middile case, each card has the $(k-1)$-open set. For the latter case, the cards of these extensions have open sets of order '$k-1, k+t-1, k+t$ 'or '$k-1, k+t-1$ 'or '$k-1, k+t-2, k+t-1$ 'or $k-2, k+t-2, k+t-1$. 'Clearly, the card X_{x_1} does not belong to the multideck of these extensions. Therefore every extension in $\mathscr{E}(X_{x_r})$ other than $H_{1(k+t)}$ does not have the card X_{x_1} in its multideck. Hence $rn(X) \leq 2$, which completes the proof.

Theorem 10. *Let X be a finite topological space of size m with ascending chain and unique isolated point. If X has no i-open sets for at least two distinct i's, $2 \leq i \leq n-1$ and τ_X is equal to the collection given in (C2), then $rn(X) = 2$.*

Proof. By Lemma 1, we have $rn(X) \geq 2$. Choose two points $x_r \in U_{k+1} - U_{k-1}$ and $x_s \in U_{l+t} - U_{l-1}$. Then the cards X_{x_r} and X_{x_s} will have size m, where

$$\tau_{X_{x_r}} = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_k, U_{k+1} \ldots, U_{l-2}, U_{l+t-1}, \ldots, X_{x_r}\},$$

$$\tau_{X_s} = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_{k+1}, U_{k+2} \ldots, U_{l-1}, U_{l+t-1}, \ldots, X_{x_s}\}.$$

Consider $\mathscr{E}(X_{x_r}) = \{H_{1(p)}, H_{2(p)}, \ldots, H_{m-1(p)}, H_{m(p)}, H_{m+1}\}$, where $p \geq 2$. The extension $H_{1(k)}$ is clearly homeomorphic to X, where

$$\tau_{H_{1(k)}} = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_k \cup \{x_r\}, U_{k+1} \cup \{x_r\} \ldots, U_{l-2} \cup \{x_r\},$$
$$U_{l+t-1} \cup \{x_r\}, \ldots, X_{x_r} \cup \{x_r\}\}.$$

So consider the other extensions of size m and at first $H_{1(c)}$, where $c > k$ and

$$\tau_{H_{1(c)}} = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_k, U_{k+1} \ldots, U_c \cup \{x_r\}, \ldots U_{l-2} \cup \{x_r\},$$
$$U_{l+t-1} \cup \{x_r\}, \ldots, X_{x_r} \cup \{x_r\}\}.$$

Since these extensions have the open sets of order $k-1$ and k, the cards of these extensions have open sets of order either $k-1, k$ or $k-1$. The cards with k-open set is clearly not homeomorphic to X_{x_s}. So, consider the cards with the $(k-1)$-open set. These cards must be obtained by deleting the points in a k-open set from the extensions and so the size of these cards is $m-1$. Therefore the card X_{x_s} does not belong to its multideck. Consider the extensions $H_{1(d)}$, where $d < k$ and

$$\tau_{H_{1(d)}} = \{\phi, U_1, U_2, \ldots, U_d \cup \{x_r\}, U_{d+1} \cup \{x_r\}, \ldots, U_{k-1} \cup \{x_r\}, U_k \cup \{x_r\}, U_{k-1} \cup \{x_r\},$$
$$\ldots, U_{l-2} \cup \{x_r\}, U_{l+t-1} \cup \{x_r\}, \ldots, X_{x_r} \cup \{x_r\}\}.$$

Since these extensions have open sets of order $k-1, k$, and $k+1$, it follows that all the cards of these extensions have the open set of order k and hence the card X_{x_s} does not belong to its multideck. Consider now the extensions of size $m+1$. Since the card X_{x_r} has the open sets of order k, and $k+1$, the extensions of size $m+1$

must have the open sets of order $k, k+1$. Therefore all the cards of these extensions must have the k-open set and so the card X_{x_s} does not belong to its multideck. The similar arguments also hold for the extensions of size greater than $m+1$. Thus every extension in $\mathscr{E}(X_{x_r})$ other than $H_{1(k)}$ does not have the card X_{x_s} in its multideck. Hence $rn(X) \leq 2$, which completes the proof.

The only remaining case to determine the reconstruction number is when τ_X is equal to the collection given in (C3). That is,

$$\tau_X = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_{k+1} = U_{l-1}, U_{l+t}, \ldots, X\},\ t \geq 1.$$

Since $k+1 = l-1$, the order of the open set U_{l+t} must be at least $k+3$. We consider two more subcases depending upon the order of U_{l+t} as below.

$$\tau_X = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_{k+1}, U_{k+t}, \ldots, X\}, t \geq 4. \qquad \cdots(C3.1)$$

$$\tau_X = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_{k+1}, U_{k+3}, \ldots, X\}. \qquad \cdots(C3.2)$$

The latter case is again divided into three subcases depending upon how the order differs between two consecutive open sets occuring after U_{k+3} as follows.
The difference between any two consecutive open sets occuring after U_{k+3} is one. That is,

$$\tau_X = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_{k+1}, U_{k+3}, U_{k+4}, U_{k+5}, \ldots, X\}. \qquad \cdots(C3.2.1)$$

The difference between any two consecutive open sets occuring after U_{k+3} is two. That is,

$$\tau_X = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_{k+1}, U_{k+3}, U_{k+5}, U_{k+7}, \ldots, U_{n-2}, X\}. \qquad \cdots(C3.2.2)$$

The difference between at least two consecutive open sets occuring after U_{k+3} (can be anywhere after U_{k+3}) is at least two and τ_X is not equal to the collections given in (C3.2.1) and (C3.2.2). That is,

$$\tau_X = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_{k+1}, U_{k+3}, \ldots, U_l, U_{l+t}, \ldots, X\},\ t \geq 2. \qquad \cdots(C3.2.3)$$

Theorem 11. *Let X be a finite topological space of size m with ascending chain and unique isolated point. If X has no i-open sets for at least two distinct i's, $2 \leq i \leq n-1$ and τ_X is equal to the collection given in (C3.1), then $rn(X) = 2$.*

Proof. Choose two points $x_r \in U_{k+1} - U_{k-1}$ and $x_s \in U_{k+t} - U_{k+1}$. Then the two cards X_{x_r} and X_{x_s} have size m, where

$$\tau_{X_{x_r}} = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_k, U_{k+t-1}, \ldots, X_{x_r}\},$$

$$\tau_{X_{x_s}} = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_{k+1}, U_{k+t-1}, \ldots, X_{x_s}\}.$$

Consider the collection $\mathscr{E}(X_{x_r}) = \{H_{1(p)}, H_{2(p)}, \ldots, H_{m-1(p)}, H_{m(p)}, H_{m+1}\}, p \geq 2$. The extension $H_{1(k)}$ of size m is clearly homeomorphic to X, where

$$\tau_{H_{1(k)}} = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_k \cup \{x_r\}, U_{k+t-1} \cup \{x_r\}, \ldots, X_{x_r} \cup \{x_r\}\}.$$

So consider the other extensions of size m and at first the extensions $H_{1(c)},\ c > k$. These extensions have the open sets of order either '$k-1, k, k+t-1$'or '$k-1, k, k+t$'. In the former case, the cards of these extensions have open sets of order '$k-1, k, k+t-1$'or '$k-1, k, k+t-2$'or '$k-1, k+t-2$'or '$k-2, k-1, k+t-2$'. For the latter case, the cards of these extensions have open sets of order '$k-1, k, k+t$'or '$k-1, k, k+t-1$'or '$k-1, k+t-1$'or '$k-2, k-1, k+t-1$'. Cleary, the cards with k-open set is not homeomorphic to the card X_{x_s}. So consider the cards having no k-open set. Since $t \geq 4$, these cards do not have the open set of order $k+1$ and hence the card X_{x_s} does not belong to its multideck. Next, consider the extensions $H_{1(d)},\ d < k$. These extensions have the open sets of order $k, k+1, k+t$. Since these extensions have open sets of order $k, k+1$, all cards of these extensions have open set of order k and hence the card X_{x_s} does not belong to its multideck. Now consider the extensions of size $m+1$. Since the card X_{x_r} has the open sets of order $k, k+t-1$, the extensions have the open sets of order either '$k, k+t-1$'or '$k, k+t-1, k+t$'or '$k, k+1, k+t$'. Since $t \geq 4$, no cards of the extensions those have the open sets of order $k, k+t-1$ have no open set of order $k+1$ and all the cards of the extensions having the open sets of order $k, k+1, k+t$ have the k-open set and hence the card X_{x_s} does not belong to its multideck. Similar arguments hold for the extensions of size greater than $m+1$. Therefore every extension in $\mathscr{E}(X_{x_r})$ other than $H_{1(k)}$ does not have the card X_{x_s} in its multideck. Hence $rn(X) \leq 2$, which completes the proof.

The next two lemmas asserts that the spaces with τ_X given in (C3.2.1) or (C3.2.2) have only four mutually nonhomeomorphic cards so that these two spaces can be recognized.

Lemma 12. *Let X be a finite topological space of size m with ascending chain and unique isolated point. Then X has no i-open sets for at least two distinct i's, $2 \leq i \leq n-1$ and τ_X is equal to the collection given in (C3.2.1) if and only if the multideck of X has only four mutually nonhomeomorphic cards $X_{x_r}, X_{x_s}, X_{x_t}$ and X_{x_u}, where*

$$\tau_{X_{x_r}} = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_{k+1}, U_{k+2}, U_{k+3}, \ldots, X_{x_r}\},$$

$$\tau_{X_{x_s}} = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_k, U_{k+2}, U_{k+3}, \ldots, X_{x_s}\},$$

$$\tau_{X_{x_t}} = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_{k+1}, U_{k+3}, U_{k+4}, \ldots, X_{x_t}\},$$
$$\tau_{X_{x_u}} = \{\phi, U_1, U_2, \ldots, U_{k-2}, U_k, U_{k+2}, U_{k+3}, \ldots, X_{x_u}\},$$

$|\tau_{X_{x_r}}| = |\tau_{X_{x_s}}| = m$ *and* $|\tau_{X_{x_t}}| = |\tau_{X_{x_u}}| = m - 1.$

Proof. Necessity. Assume $\tau_X = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_{k+1}, U_{k+3}, U_{k+4}, U_{k+5}, \ldots, X\}$. Then we will get the above four nonhomeomorphic cards $X_{x_r}, X_{x_s}, X_{x_t}$ and X_{x_u} with desired cardinality by choosing the points x_r in $U_{k+3} - U_{k+1}$, x_s in $U_{k+1} - U_{k-1}$, x_t in $X - U_{k+3}$ and x_u in U_{k-1}.
Sufficiency. Assume, to the contrary, that τ_X is not equal to the collection given in (C3.2.1). Suppose that X has no i-open set for some i, $2 \le i \le k-1$, that is, $\tau_X = \{\phi, U_1, U_2, \ldots, U_{i-1}, U_{i+1}, \ldots, U_{k-1}, U_{k+1}, U_{k+3}, U_{k+4}, U_{k+5}, \ldots, X\}$. Then the cards of X have open sets of order '$i-1, i+1$'or '$i-1, i$'or '$i-2, i$'. The card having the open sets of order $i-2$ and i does not belong to the given multideck, a contradiction. Similarly, the same holds for X having no i-open set for some i, $k+3 \le i \le n-1$. *Suppose that X has no $(k+1)$-open set.* If X has no open sets of order k and $k+2$, then the multideck of X has only three mutually nonhomeomorphic cards X_{x_r}, X_{x_s} and X_{x_t}, where $x_r \in X - U_{k+3}$, $x_s \in U_{k+3} - U_{k-1}$, $x_t \in U_{k-1}$, giving a contradiction. If X has k-open set or $(k+2)$-open set but not both, then the multideck has only three mutually nonhomeomorphic cards X_{x_r}, X_{x_s} and X_{x_t}, where $x_r \in X - U_{k+3}$, $x_s \in U_{k+3} - U_k$, $x_t \in U_k$, $x_r \in X - U_{k+2}$, $x_s \in U_{k+2} - U_{k-1}$, $x_t \in U_{k-1}$ respectively, giving a contradiction. If X has both open sets of order k and $k+2$, then, by Lemma 5, the multideck has only three mutually nonhomeomorphic cards, a contradiction. *Assume now that X has the $(k+1)$-open set.* In addition, if X has the $(k+2)$-open set, then by Lemma 3, all cards are homeomorphic, a contradiction. Otherwise, by Lemma 5, the multideck has only three mutually nonhomeomorphic cards, again giving a contradiction.

Theorem 13. *Let X be a finite topological space of size m with ascending chain and unique isolated point. If X has no i-open sets for at least two distinct i's, $2 \le i \le n-1$ and τ_X is equal to the collection given in (C3.2.1), then $rn(X) = 3$.*

Proof. By Lemma 12, the multideck of X has only four mutually non-homeomorphic cards, namely X_{x_r}, X_s, X_t, X_u where

$$\tau_{X_{x_r}} = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_{k+1}, U_{k+2}, U_{k+3}, \ldots, X_{x_r}\},$$
$$\tau_{X_{x_s}} = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_k, U_{k+2}, U_{k+3}, \ldots, X_{x_s}\},$$
$$\tau_{X_{x_t}} = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_{k+1}, U_{k+3}, U_{k+4}, \ldots, X_{x_t}\},$$
$$\tau_{X_{x_u}} = \{\phi, U_1, U_2, \ldots, U_{k-2}, U_k, U_{k+2}, U_{k+3}, \ldots, X_{x_u}\},$$

$|\tau_{X_{x_r}}| = |\tau_{X_{x_s}}| = m$ and $|\tau_{X_{x_t}}| = |\tau_{X_{x_u}}| = m-1$. We shall first prove that $rn(X) > 2$ by considering ten cases as below.

Case 1. The two cards are X_{x_r} and X_{x_r}.

Consider $\mathscr{E}(X_{x_r}) = \{H_{1(p)}, H_{2(p)}, \ldots, H_{m-1(p)}, H_{m(p)}, H_{m+1}\}$, $p \geq 2$ and the extensions $H_{1(k+2)}$ and $H_{1(n-1)}$ of size m, where

$$\tau_{H_{1(k+2)}} = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_{k+1}, U_{k+2} \cup \{x_r\}, U_{k+3} \cup \{x_r\}, \ldots, X_{x_r} \cup \{x_r\}\},$$

$$\tau_{H_{1(n-1)}} = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_{k+1}, U_{k+2}, U_{k+3}, \ldots, X_{x_r} \cup \{x_r\}\}.$$

In $H_{1(k+2)}$, the card X_{x_e}, where $x_e \in U_{k+2} - U_{k+1}$, is homeomorphic to the card X_{x_r}. In $H_{1(n-1)}$, the card X_{x_f}, where $x_f \in X_{x_r} - U_{n-2}$, is homeomorphic to the card X_{x_r}.

Case 2. The two cards are X_{x_r} and X_{x_s}.

Consider $\mathscr{E}(X_{x_r})$ and the extensions $H_{1(k+2)}, H_{2(k-1)}$ of size m and $m+1$ respectively, where

$$\tau_{H_{2(k-1)}} = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_{k-1} \cup \{x_r\}, U_{k+1} \cup \{x_r\}, U_{k+2} \cup \{x_r\}, U_{k+3} \cup \{x_r\}, \ldots, X_{x_r} \cup \{x_r\}\}.$$

In $H_{2(k-1)}$, the card, X_e, $x_e \in U_{k+2} - U_{k+1}$ is homeomorphic to the card X_{x_s}. By Lemma 12, the extension $H_{1(k+2)}$ has both the cards X_{x_r} and X_{x_s}.

Case 3. The two cards are X_{x_r} and X_{x_t}.

Consider the extensions $H_{1(k+2)}$ and $H_{1(k+3)}$ of size m, where

$$\tau_{H_{1(k+3)}} = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_{k+1}, U_{k+2}, U_{k+3} \cup \{x_r\}, \ldots, X_{x_r} \cup \{x_r\}\}.$$

By Lemma 12, the extension $H_{1(k+2)}$ has both the cards X_{x_r} and X_{x_s}. In $H_{1(k+3)}$, the card X_{x_e}, $x_e \in U_{k+2} - U_{k+1}$ is homeomorphic to the card X_{x_t}.

Case 4. The two cards are X_{x_r} and X_{x_u}.

Consider $\mathscr{E}(X_{x_r})$ and the extensions $H_{1(k+2)}, H_{1(k-1)}$ of size m, where

$$H_{1(k-1)} = \{\phi, U_1, U_2, \ldots, U_{k-1} \cup \{x_r\}, U_{k+1} \cup \{x_r\}, U_{k+2} \cup \{x_r\}, U_{k+3} \cup \{x_r\}, \ldots, X_{x_r} \cup \{x_r\}\}.$$

By Lemma 12, the extension $H_{1(k+2)}$ has both the cards X_{x_r} and X_{x_u}. In $H_{1(k-1)}$, the card X_{x_e}, where $x_e \in U_{k+2} - U_{k+1}$ is homeomorphic to the card X_{x_u}.

Case 5. The two cards are X_{x_s} and X_{x_s}.

Consider $\mathscr{E}(X_{x_s}) = \{I_{1(p)}, I_{2(p)}, \ldots, I_{m-1(p)}, I_{m(p)}, I_{m+1}\}$, $p \geq 2$ and the extensions $I_{1(k)}, I_{1(n-1)}$ of size m, where

$$I_{1(k)} = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_k \cup \{x_s\}, U_{k+2} \cup \{x_s\}, U_{k+3} \cup \{x_s\}, \ldots, X_{x_s} \cup \{x_s\}\},$$

$$\tau_{I_{1(n-1)}} = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_k, U_{k+2}, U_{k+3}, \ldots, X_{x_s} \cup \{x_s\}\}.$$

In extension $I_{1(k)}$, the card $X_{x_e},\ x_e \in U_k - U_{k-1}$ is homeomorphic to the card X_{x_s}. In extension $I_{1(n-1)}$, the card $X_{x_e},\ x_e \in X_{x_s} - U_{n-2}$ is homeomorphic to the card X_{x_s}.

Case 6. The two cards are X_{x_s} and X_{x_t}.

Consider $\mathscr{E}(X_{x_s})$ and the extensions $I_{1(k)}, I_{1(k+3)}$ of size m, where

$$\tau_{I_{1(k+3)}} = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_k, U_{k+2}, U_{k+3} \cup \{x_s\}, \ldots, X_{x_s} \cup \{x_s\}\}.$$

By Lemma 12, the extension $I_{1(k)}$ has both the cards. In extension $I_{1(k+3)}$, the card $X_{x_e},\ x_e \in U_k - U_{k-1}$ is homeomorphic to the card X_{x_t}.

Case 7. The two cards are X_{x_s} and X_{x_u}.

Consider $\mathscr{E}(X_{x_s})$ and the extensions $I_{1(k)}, I_{1(k-1)}$ of size m, where

$$\tau_{I_{1(k-1)}} = \{\phi, U_1, U_2, \ldots, U_{k-1}\cup\{x_s\}, U_k\cup\{x_s\}, U_{k+2}\cup\{x_s\}, U_{k+3}\cup\{x_s\}, \ldots, X_{x_s}\cup\{x_s\}\}.$$

By Lemma 12, the extension $I_{1(k)}$ has both the cards. In $I_{1(k-1)}$, the card $X_{x_e},\ x_e \in U_k - U_{k-1}$ is homeomorphic to the card X_{x_u}.

Case 8. The two cards are X_{x_t} and X_{x_t}.

Consider $\mathscr{E}(X_{x_t}) = \{J_{1(p)}, J_{2(p)}, \ldots, J_{m-2(p)}, J_{m-1(p)}, J_m\}$, $p \geq 2$ and the extensions $J_{1(n-1)}, J_{1(k+1)}$ of size $m-1$, where

$$\tau_{J_{1(n-1)}} = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_{k+1}, U_{k+3}, U_{k+4}, \ldots, X_{x_t} \cup \{x_t\}\},$$

$$\tau_{J_{1(k+1)}} = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_{k+1}\cup\{x_t\}, U_{k+3}\cup\{x_t\}, U_{k+4}\cup\{x_t\}, \ldots, X_{x_t}\cup\{x_t\}\}.$$

In $J_{1(n-1)}$, the card $X_{x_e},\ x_e \in X_{x_t} - U_{n-2}$ is homeomorphic to the card X_{x_t}. In $J_{1(k+1)}$, the card $X_{x_f},\ x_f \in U_{k+1} - U_{k-1}$ is homeomorphic to the card X_{x_t}.

Case 9. The two cards are X_{x_t} and X_{x_u}.

Consider $\mathscr{E}(X_{x_t})$ and the extensions $J_{2(n-1)}, J_{1(k-1)}$ of size m and $m-1$ respectively, where

$$\tau_{J_{1(k-1)}} = \{\phi, U_1, U_2, \ldots, U_{k-1}\cup\{x_t\}, U_{k+1}\cup\{x_t\}, U_{k+3}\cup\{x_t\}, U_{k+4}\cup\{x_t\}, \ldots, X_{x_t}\cup\{x_t\}\},$$

$$\tau_{J(2(n-1))} = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_{k+1}, U_{k+3}, U_{k+4}, \ldots, X_{x_t}, X_{x_t} \cup \{x_t\}\}.$$

By Lemma 12, the extension $J_{2(n-1)}$ has both the cards. In $J_{1(k-1)}$, the card $X_{x_e},\ x_e \in U_{k+3} - U_{k+1}$ is homeomorphic to the card X_{x_u}.

Case 10. The two cards are X_{x_u} and X_{x_u}.

Consider $\mathscr{E}(X_{x_u}) = \{K_{1(p)}, K_{2(p)}, \ldots, K_{m-2(p)}, K_{m-1(p)}, K_m\},\ p \geq 2$ and the extensions $K_{1(n-1)}, K_{1(k)}$, of size $m-1$ where

$$\tau_{K_{1(n-1)}} = \{\phi, U_1, U_2, \ldots, U_{k-2}, U_k, U_{k+2}, U_{k+3}, \ldots, X_{x_u} \cup \{x_u\}\},$$

$\tau_{K_{1(k)}} = \{\phi, U_1, U_2, \ldots, U_{k-2}, U_k \cup \{x_u\}, U_{k+2} \cup \{x_u\}, U_{k+3} \cup \{x_u\}, \ldots, X_{x_u} \cup \{x_u\}\}$.
In $K_{1(n-1)}$, the card $X_{x_e},\ x_e \in X_{x_u} - U_{n-2}$ is homeomorphic to the card X_{x_u}. In $K_{1(k)}$, the card $X_{x_f},\ x_f \in U_k - U_{k-2}$ is homeomorphic to the card X_{x_u}. Thus, in all the ten cases, we have proved that $rn(X) > 2$.

Next we shall show that $rn(X) \leq 3$. Consider the three nonhomeomorphic cards $X_{x_r}, X_{x_s},\ X_{x_t}$ and the collection $\mathscr{E}(X_{x_r})$. The extension $H_{1(k+2)}$ of size m is clearly homeomorphic to X. Consider the other extensions of size m and at first the extensions $H_{1(c)},\ c > k+2$. These extensions have the open sets of order $k+1, k+2$ and hence all the cards of these extensions have the open set of order $k+1$ and hence the card X_{x_s} does not belong to its multideck. Consider the extensions $H_{1(d)},\ d < k+2$. These extensions have the open sets of order either '$k-1, k+2, k+3$'or '$k, k+2, k+3$'. In the former case, the cards of these extensions have open sets of order '$k-1, k+2, k+3$'or '$k-1, k+2$'or '$k-1, k+1, k+2$'or '$k-2, k+1, k+2$'. Since no card has the open set of order k, the card X_{x_s} does not belong to its multideck. For the latter case, the cards of these extensions have open sets of order '$k, k+2, k+3$'or '$k, k+2$'or '$k, k+1, k+2$'or '$k-1, k+1, k+2$'. Since each card has the open set of order $k+2$, it follows that the card X_{x_t} does not belong to its multideck. Next, consider the extensions of size $m+1$. Then the size of its cards are $m+1$ or m and hence the card X_{x_t} does not belong to its multideck, as size of X_{x_t} is $m-1$. Finally, consider the extensions of size greater than $m+1$. Since these extensions do not have the ascending chain, by Lemma 2, these extensions can have at most two cards with ascending chain and hence one of the two cards X_{x_s}, X_{x_t} does not belong to its multideck. Therefore every extension in $\mathscr{E}(X_{x_r})$ other than $H_{1(k+2)}$ does not have the other two cards in its multideck. Hence $rn(X) \leq 3$, which completes the proof.

Now we move on to the collection given in (C3.2.2). From the structure of the open sets of X, it is enough to prove the result for the open sets of order $1, 2, \ldots, k-1, k+1, k+3, k+5$.

Lemma 14. *Let X be a finite topological space of size m with ascending chain and unique isolated point. Then X has no i-open sets for at least two distinct i's, $2 \leq i \leq n-1$ and τ_X is equal to the collection given in (C3.2.2) if and only if the multideck of X has only four mutually nonhomeomorphic cards $X_{x_r}, X_{x_s}, X_{x_t}$ and X_{x_u}, where*

$$\tau_{X_{x_r}} = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_{k+1}, U_{k+3}, U_{k+4}\},$$

$$\tau_{X_{x_s}} = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_{k+1}, U_{k+2}, U_{k+4}\},$$

$$\tau_{X_{x_t}} = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_k, U_{k+2}, U_{k+4}\},$$

$$\tau_{X_{x_u}} = \{\phi, U_1, U_2, \ldots, U_{k-2}, U_k, U_{k+2}, U_{k+4}\},$$

$|\tau_{X_{x_r}}| = |\tau_{X_{x_s}}| = |\tau_{X_{x_t}}| = m$ *and* $|\tau_{X_{x_u}}| = m - 1$.

Proof. Necessity: Assume that $\tau_X = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_{k+1}, U_{k+3}, U_{k+5}\}$. Then we will get the above four nonhomeomorphic cards $X_{x_r}, X_{x_s}, X_{x_t}$ and X_{x_u} with desired cardinality by choosing the points x_r in $U_{k+5} - U_{k+3}$, x_s in $U_{k+3} - U_{k+1}$, x_t in $U_{k+1} - U_{k-1}$, and x_u in U_{k-1}.
Sufficiency: Assume, to the contrary, that τ_X was not equal to the collection given in (C3.2.2). Suppose that X has no i-open set for some i, $2 \leq i \leq k-1$. The cards of X have open sets of order '$i-1, i+1$'or '$i-1, i$'or '$i-2$, i'. The cards having open sets of order $i-2$ and i do not belong to the given multideck, a contradiction. *Suppose that X has no $(k+1)$-open set.* If X has no open sets of order $k, k+2, k+3$ and $k+4$, then the multideck has only two nonhomeomorphic cards X_{x_r} and X_{x_s}, where $x_r \in U_{k+5} - U_{k-1}$, $x_s \in U_{k-1}$, a contradiction. If X has the i-open set, where $i = k, k+2, k+3$, or $k+4$ and X has no j-open set, where $j \in \{k, k+2, k+3, k+4\} - \{i\}$, then the multideck has only two or three mutually nonhomeomorphic cards, a contradiction. If X has the k-open set and the $(k+2)$-open set, then the card X_{x_r}, where $x_r \in U_{k+2} - U_k$, does not belong to the given multideck. Similarly, the same hold when X has at most three open sets of different order from $\{k, k+2, k+3, k+4\}$. If X has open sets of all order from $\{k, k+2, k+3, k+4\}$, then by Lemma 5, the multideck has only three mutually nonhomeomorphic cards, a contradiction. Similarly, the same hold for the case that X has no $(k+3)$-open set. *Now assume that X has the $(k+1)$-open set.* If X has no open sets of order $k, k+2, k+3$ and $k+4$, then the multideck has only three mutually nonhomeomorphic cards X_{x_r}, X_{x_s} and X_{x_t}, where $x_r \in U_{k+5} - U_{k+1}$, $x_s \in U_{k+1} - U_{k-1}$, $x_t \in U_{k-1}$, a contradiction. If X has the i-open set, where $i = k, k+2, k+4$ and X has no j-open set, where $j \in \{k, k+2, k+4\} - \{i\}$, then the card X_{x_r}, where $x_r \in U_{k+5}$, $x_r \in U_{k+1}$, $x_r \in U_{k+5}$ does not belong to the given multideck for each i respectively. If X has the k-open set and the $(k+2)$-open set, then the card X_{x_r}, where $x_r \in U_{k+5}$, does not belong to the given multideck. Similarly, the same hold when X has at most three open sets of different order from $\{k, k+2, k+3, k+4\}$. If X has open sets of all order from $\{k, k+2, k+3, k+4\}$, then by Lemma 3, all cards are homeomorphic, a contradiction. Similarly, the same hold for the case that X has the $(k+3)$-open set.

Theorem 15. *Let X be a finite topological space of size m with ascending chain and unique isolated point. If X has no i-open sets for at least two distinct i's, $2 \leq i \leq n-1$ and τ_X is equal to the collection given in (C3.2.2), then $rn(X) = 3$.*

Proof. By Lemma 14, the multideck of X has only four mutually non-homeomorphic cards, namely X_{x_r}, X_s, X_t, X_u where $|\tau_{X_{x_r}}| = |\tau_{X_{x_s}}| = |\tau_{X_{x_t}}| = m$, $|\tau_{X_{x_u}}| = m-1$,

$$\tau_{X_{x_r}} = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_{k+1}, U_{k+3}, U_{k+4}\},$$

$$\tau_{X_{x_s}} = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_{k+1}, U_{k+2}, U_{k+4}\},$$
$$\tau_{X_{x_t}} = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_k, U_{k+2}, U_{k+4}\}, \text{ and}$$
$$\tau_{X_{x_u}} = \{\phi, U_1, U_2, \ldots, U_{k-2}, U_k, U_{k+2}, U_{k+4}\}.$$

We shall first prove that $rn(X) > 2$ by considering ten cases as below.
Case 1. The two cards are X_{x_r} and X_{x_r}.

Consider $\mathscr{E}(X_{x_r}) = \{H_{1(p)}, H_{2(p)}, \ldots, H_{m-1(p)}, H_{m(p)}, H_{m+1}\}$, $p \geq 2$ and the extensions $H_{1(k+4)}, H_{1(k+3)}$ of size m, where

$$\tau_{H_{1(k+4)}} = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_{k+1}, U_{k+3}, U_{k+4} \cup \{x_r\}\},$$
$$\tau_{H_{1(k+3)}} = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_{k+1}, U_{k+3} \cup \{x_r\}, U_{k+4} \cup \{x_r\}\}.$$

In $H_{1(k+4)}$, the card $X_{x_e},\ x_e \in U_{k+4} - U_{k+3}$ is homeomorphic to the card X_{x_r}. In $H_{1(k+3)}$, the card $X_{x_f},\ x_f \in U_{k+3} - U_{k+1}$ is homeomorphic to the card X_{x_r}.
Case 2. The two cards are X_{x_r} and X_{x_s}.

Consider $\mathscr{E}(X_{x_r})$ and the extensions $H_{1(k+4)}, H_{2(k+1)}$ of size m and $m+1$ respectively, where

$$\tau_{H_{2(k+1)}} = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_{k+1}, U_{k+1} \cup \{x_r\}, U_{k+3} \cup \{x_r\}, U_{k+4} \cup \{x_r\}\}.$$

By Lemma 14, the extension $H_{1(k+4)}$ has both the cards. In the extension $H_{2(k+1)}$, the card x_e, where $x_e \in U_{k+4} - U_{k+3}$, is homeomorphic to the card X_{x_s}.
Case 3. The two cards are X_{x_r} and X_{x_t}.

Consider $\mathscr{E}(X_{x_r})$ and the extensions $H_{1(k+4)}, H_{2(k-1)}$ of size m and $m+1$ respectively, where

$$\tau_{H_{2(k-1)}} = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_{k-1} \cup \{x_r\}, U_{k+1} \cup \{x_r\}, U_{k+3} \cup \{x_r\}, U_{k+4} \cup \{x_r\}\}.$$

By Lemma 14, the extension $H_{1(k+4)}$ has both the cards. In the extension $H_{2(k-1)}$, the card x_e, where $x_e \in U_{k+4} - U_{k+3}$, is homeomorphic to the card X_{x_t}.
Case 4. The two cards are X_{x_r} and X_{x_u}.

Consider $\mathscr{E}(X_{x_r})$ and the extensions $H_{1(k+4)}, H_{1(k-1)}$ of size m, where

$$\tau_{H_{1(k-1)}} = \{\phi, U_1, U_2, \ldots, U_{k-1} \cup \{x_r\}, U_{k+1} \cup \{x_r\}, U_{k+3} \cup \{x_r\}, U_{k+4} \cup \{x_r\}\}.$$

By Lemma 14, the extension $H_{1(k+4)}$ has both the cards. In the extension $H_{1(k-1)}$, the card x_e, where $x_e \in U_{k+4} - U_{k+3}$, is homeomorphic to the card X_{x_u}.
Case 5. The two cards are X_{x_s} and X_{x_s}.

Consider $\mathscr{E}(X_{x_s}) = \{I_{1(p)}, I_{2(p)}, \ldots, I_{m-1(p)}, I_{m(p)}, I_{m+1}\}$, where $p \geq 2$. Consider the extensions $I_{1(k+2)}, I_{1(k+4)}$ of size m, where

$$\tau_{I_{1(k+2)}} = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_{k+1}, U_{k+2} \cup \{x_s\}, U_{k+4} \cup \{x_s\}\},$$

$$\tau_{I_{1(k+4)}} = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_{k+1}, U_{k+2}, U_{k+4} \cup \{x_s\}\}.$$

In $I_{1(k+2)}$, the card X_{x_e}, where $x_e \in U_{k+2} - U_{k+1}$, is homeomorphic to the card X_{x_s}. In $I_{1(k+4)}$, the card X_{x_f}, where $x_f \in U_{k+4} - U_{k+2}$, is homeomorphic to the card X_{x_s}.
Case 6. The two cards are X_{x_s} and X_{x_t}.

Consider the extensions $I_{1(k+2)}, I_{2(k-1)}$ of size m and $m+1$ respectively, where

$$\tau_{H_{2(k-1)}} = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_{k-1} \cup \{x_s\}, U_{k+1} \cup \{x_s\}, U_{k+2} \cup \{x_s\}, U_{k+4} \cup \{x_s\}\}.$$

By Lemma 14, the extension $I_{1(k+2)}$ has both the cards and in extension $I_{2(k-1)}$, the card x_e, where $x_e \in U_{k+2} - U_{k+1}$, is homeomorphic to the card X_{x_t}.
Case 7. The two cards are X_{x_s} and X_{x_u}.

Consider $\mathscr{E}(X_{x_s})$ and the extensions $I_{1(k+2)}, I_{1(k-1)}$ of size m, where

$$\tau_{I_{1(k-1)}} = \{\phi, U_1, U_2, \ldots, U_{k-1} \cup \{x_s\}, U_{k+1} \cup \{x_s\}, U_{k+2} \cup \{x_s\}, U_{k+4} \cup \{x_s\}\}.$$

By Lemma 14, the extension $I_{1(k+2)}$ has both the cards. In $I_{1(k-1)}$, the card x_e, where $x_e \in U_{k+2} - U_{k+1}$, is homeomorphic to the card X_{x_u}.
Case 8. The two cards are X_{x_t} and X_{x_t}.

Consider $\mathscr{E}(X_{x_t}) = \{J_{1(p)}, J_{2(p)}, \ldots, J_{m-1(p)}, J_{m(p)}, J_{m+1}\}$, $p \geq 2$ and the extensions $J_{1(k)}, J_{1(k+4)}$ of size m, where

$$\tau_{J_{1(k)}} = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_k \cup \{x_t\}, U_{k+2} \cup \{x_t\}, U_{k+4} \cup \{x_t\}\},$$

$$\tau_{J_{1(k+4)}} = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_k, U_{k+2}, U_{k+4} \cup \{x_t\}\}.$$

In $J_{1(k)}$, the card X_{x_e}, where $x_e \in U_k - U_{k-1}$, is homeomorphic to the card X_{x_t}. In $J_{1(k+4)}$, the card X_{x_f}, where $x_f \in U_{k+4} - U_{k+2}$, is homeomorphic to the card X_{x_t}.
Case 9. The two cards are X_{x_t} and X_{x_u}.

Consider $\mathscr{E}(X_{x_t})$ and the extensions $J_{1(k)}, J_{1(k-1)}$ of size m, where

$$\tau_{J_{1(k-1)}} = \{\phi, U_1, U_2, \ldots, U_{k-1} \cup \{x_t\}, U_k \cup \{x_t\}, U_{k+2} \cup \{x_t\}, U_{k+4} \cup \{x_t\}\}.$$

By Lemma 14, the extension $J_{1(k)}$ has both the cards. In $J_{1(k-1)}$, the card x_e, where $x_e \in U_k - U_{k-1}$, is homeomorphic to the card X_{x_u}.
Case 10. The two cards are X_{x_u} and X_{x_u}.

Consider $\mathscr{E}(X_{x_u}) = \{K_{1(p)}, K_{2(p)}, \ldots, K_{m-2(p)}, K_{m-1(p)}, K_m\}$, $p \geq 2$ and the extensions $K_{1(k+4)}, K_{1(k+2)}$ of size $m-1$, where

$$\tau_{K_{1(k+4)}} = \{\phi, U_1, U_2, \ldots, U_{k-2}, U_k, U_{k+2}, U_{k+4} \cup \{x_u\}\},$$

$$\tau_{K_{1(k+2)}} = \{\phi, U_1, U_2, \ldots, U_{k-2}, U_k, U_{k+2} \cup \{x_u\}, U_{k+4} \cup \{x_u\}\}.$$

In $K_{1(k+4)}$, the card X_{x_e}, where $x_e \in U_{k+4} - U_{k+2}$, is homeomorphic to the card X_{x_u}. In $K_{1(k+2)}$, the card X_{x_f}, where $x_f \in U_{k+2} - U_k$, is homeomorphic to the card X_{x_u}. Thus, in all the ten cases, we have proved that $rn(X) > 2$.

Now we shall show that $rn(X) \leq 3$. Consider the three nonhomeomorphic cards $X_{x_r}, X_{x_s}, X_{x_u}$ and the collection $\mathscr{E}(X_{x_r})$. The extension $H_{1(k+4)}$ of size m is clearly homeomorphic to X. Consider the other extensions of size m. These extensions have the open sets of order either '$k+1, k+4, k+5$'or '$k+2, k+4, k+5$'. For the former case, the cards of the extensions have open sets of order '$k+1, k+4$'or '$k+1, k+3, k+4$'or '$k, k+3, k+4$'. For the latter case, the cards of the extensions have open sets of order '$k+2, k+4$'or '$k+2, k+3, k+4$'or '$k+1, k+3, k+5$'. In both cases, the cards X_{x_s} and X_{x_u} do not belong to the multideck of the extensions. Next, consider the extensions of size $m+1$. The size of the cards are either $m+1$ or m and hence the card X_{x_u} does not belong to its multideck, as the size of X_{x_u} is $m-1$. Finally, consider the extensions of size greater than $m+1$. Since these extensions do not have the ascending chain, by Lemma 2, these extensions can have at most two cards with ascending chain and hence one of the two cards X_{x_s}, X_{x_u} does not belong to its multideck. Therefore every extension in $\mathscr{E}(X_{x_r})$ other than H_{k+4} does not have the other two cards in its multideck. Hence $rn(X) \leq 3$, which completes the proof.

Theorem 16. *Let X be a finite topological space of size m with ascending chain and unique isolated point. If X has no i-open sets for at least two distinct i's, $2 \leq i \leq n-1$ and τ_X is equal to the collection given in (C3.2.3), then $rn(X) = 2$.*

Proof. By Lemma 1, we have $rn(X) \geq 2$. Choose two points $x_r \in U_{k+1} - U_{k-1}$ and $x_s \in U_{k+3} - U_{k+1}$. Then the cards X_{x_r} and X_{x_s} will have size m, where

$$\tau_{X_{x_r}} = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_k, U_{k+2}, \ldots, U_{l-1}, U_{l+t-1}, \ldots, X_{x_r}\},$$

$$\tau_{X_{x_s}} = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_{k+1}, U_{k+2}, \ldots, U_{l-1}, U_{l+t-1}, \ldots, X_{x_s}\}.$$

Consider the collection $\mathscr{E}(X_{x_r}) = \{H_{1(p)}, H_{2(p)}, \ldots, H_{m-1(p)}, H_{m(p)}, H_{m+1}\}, p \geq 2$. The extension $H_{1(k)}$ of size m is clearly homeomorphic to X, where

$$\tau_{H_{1(k)}} = \{\phi, U_1, U_2, \ldots, U_{k-1}, U_k \cup \{x_r\}, U_{k+2} \cup \{x_r\}, \ldots, U_{l-1} \cup \{x_r\}, U_{l+t-1} \cup \{x_r\}, \ldots, X_{x_r} \cup \{x_r\}\}.$$

So consider the other extensions of size m and at first the extensions $H_{1(c)},\ c > k$. These extensions have the open sets of order either '$l-1, l+t-1$'or '$l-1, l+t$'or '$l, l+t$'. In the former case, the cards of these extensions have open sets of order '$l-1, l+t-1$'or '$l-1, l+t-2$'or '$l-2, l+t-2$'. Cleary, the cards having

$(l+t-2)$-open set is not homeomorphic to the card X_{x_s}. So consider the cards having both $(l-1)$-open set and $(l+t-1)$-open set. These cards must be obtained by deleting the points which is not in the $(l+t-1)$-open set from the extensions and so these cards have the k-open set. Therefore the card X_{x_s} does not belong to its multideck. In the middle case, the cards of these extensions have open sets of order '$l-1, l+t$'or '$l-1, l+t-1$'or '$l-2, l+t-1$'. Cleary, the cards having $(l-2)$-open set and $(l+t)$-open set are not homeomorphic to the card X_{x_s}. So consider the cards having both $(l-1)$-open set and $(l+t-1)$-open set. These cards must be obtained by deleting the points $(l+t)$-open set which is not in the $(l-1)$-open set from the extensions and so these cards have the k-open set. Therefore the card X_{x_s} does not belong to its multideck. For the latter case, the cards of these extensions have open sets of order '$l, l+t$'or '$l, l+t-1$'or '$l-1, l+t-1$'. Cleary, the cards with l-open set are not homeomorphic to the card X_{x_s}. So consider the cards having both $(l-1)$-open set and $(l+t-1)$-open set. These cards must be obtained by deleting the points in the l-open set from the extensions. If the deleted point belongs to $U_l - U_k$, then these cards have the k-open set. Otherwise, the cards have no k-open set but size of the card is $m-1$ and hence the card X_{x_s} does not belong to its multideck. Next, consider the extensions $H_{1(d)},\ d < k$. These extensions have the open sets of order $k, k+1$. Since these extensions have open sets of order $k, k+1$, all cards of these extensions have open set of order k and hence the card X_{x_s} does not belong to its multideck. Now consider the extensions of size $m+1$. Since the card X_{x_r} has the open sets of order $k-1, k$, the extensions have the open sets of order either '$k-1, k$'or '$k-1, k, k+1$'. For the former case, the cards of these extensions have open sets of order '$k-1, k$'or '$k-1$'. Cleary, the cards with k-open set are not homeomorphic to the card X_{x_s}. So consider the cards having $k-1$-open set. These cards must be obtained by deleting the points in the k-open set from the extensions and so size of the card is $m-1$ and hence the card X_{x_s} does not belong to its multideck. For the latter case, all the cards of the extensions have the k-open set and hence the card X_{x_s} does not belong to its multideck. Similar arguments hold for the extensions of size greater than $m+1$. Therefore every extension in $\mathscr{E}(X_{x_r})$ other than $H_{1(k)}$ does not have the card X_{x_s} in its multideck. Hence $rn(X) \leq 2$, which completes the proof.

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