

ACTA UNIV. SAPIENTIAE, MATHEMATICA, 1, 1 (2009) 21-34

Anti-Ramsey numbers of spanning double stars

Balázs Montágh University of Western Hungary email: montagh@ttmk.nyme.hu

Abstract. In this paper we shall prove and sharpen a conjecture of A. Bialostocki on anti-Ramsey colorings of the complete graph K_n . Assume the edges of a K_n are colored by t colors. The basic question is how many colors ensure that K_n has a spanning subtree of diameter at most d in which each edge has a different color. The surprising fact is that the answers are the same for every $d \geq 3$. Moreover, we can set the maximal degree of the spanning tree at least n - 4 without altering the answer. This implies that in these cases there is an extremal anti-Ramsey coloring using only one color more than once. Recently Jiang showed that this is not the case for d = 2. We also prove a new extremal property of Moore graphs of diameter 2 (e.g. the Petersen graph), that yields a bit shorter proof of a weaker version of our main theorem.

1 Introduction

Erdős, Simonovits and Sós [10] initiated the investigation of anti-Ramsey problems for graphs. Call an edge-colored graph *totally multicolored* (TMC, for short) if any two edges have different colors. Given a family \mathcal{L} of graphs, what is the maximum t for which there exist t-colorings of the edges of K_n , where every color is used at least once, without a TMC subgraph that belongs to \mathcal{L} ? This maximum will be denoted by $R^*(n, \mathcal{L})$. When \mathcal{L} consists a single graph G, we shall use $R^*(n, G)$ for $R^*(n, \{G\})$.

These questions are related to extremal graph problems. Define $ex(n, \mathcal{L})$ as the maximal integer m of edges such that there is a graph with n vertices

AMS 2000 subject classifications: 05C55

Key words and phrases: anti-Ramsey theory, double stars, Moore graphs

and \mathfrak{m} edges which does not contain a subgraph isomorphic to a graph in \mathcal{L} . Denote $\operatorname{ex}(\mathfrak{n}, \{G\})$ by $\operatorname{ex}(\mathfrak{n}, G)$.

It is easy to see that $\mathbb{R}^*(\mathfrak{n}, \mathcal{L}) \leq \operatorname{ex}(\mathfrak{n}, \mathcal{L})$; indeed, if we have a coloring which uses t colors and no TMC subgraph belonging to \mathcal{L} is obtained, then, by choosing for every color an edge with this color arbitrarily we have a graph with t edges which does not contain subgraphs belonging to \mathcal{L} . On the other hand, let $\mathcal{L}^* = \{L - e : L \in \mathcal{L}, e \in E(L)\}$, and let G be a subgraph of K_n containing no member of \mathcal{L}^* . Then, as it was observed in [10], every coloring of K_n for which all edges not belonging to G has the same color contains no TMC member of \mathcal{L} . In particular, since G can be chosen to have $\operatorname{ex}(\mathfrak{n}, \mathcal{L}^*)$ edges,

$$\mathbf{R}^*(\mathbf{n}, \mathcal{L}) \ge \exp(\mathbf{n}, \mathcal{L}^*) + \mathbf{1}.$$
 (1)

The above inequality is sharp iff there is an extremal anti-Ramsey coloring using only one color more than once. Erdős, Simonovits and Sós showed also that sometimes this is the case, and sometimes it is not. (1) is sharp if \mathcal{L} consists of one clique with at least 4 vertices and **n** is large enough. That is,

$$\mathbf{R}^*(\mathbf{n}, \mathbf{K}_k) = \exp(\mathbf{n}, \mathbf{K}_k - \mathbf{e}) + \mathbf{1}$$
(2)

for sufficiently large n and $k \ge 4$. (Their proofs gave $R^*(n, K_k)$ explicitly, since they proved (2) by showing

$$R^*(n, K_k) \le ex(n, K_{k-1}) + 1.$$
(3)

for sufficiently large n and $k \ge 4$. (3), combining with (1) and an earlier theorem of Dirac [7] stating Recently Montellano-Ballesteros and Neumann-Lara [18] and Schiermeyer [19] proved (2) for every possible n.) On the other hand, it was also shown in [10] that (1) is not sharp if \mathcal{L} consists one cycle of length k and $n \ge 2k - 1$.

Calculation of the exact anti-Ramsey numbers has proven to be very hard. They have been found only for few \mathcal{L} so far, and, for most of these \mathcal{L} s, $|\mathcal{L}| = 1$ (see [10], [18] and [19] for cliques, [10], [1] and [14] for short cycles, [20] and [17] for paths, [13] for stars, [15] for brooms, that is, for trees formed by identifying the center of a star and an endpoint of a path, and [19] for matchings).

The case $|\mathcal{L}| \geq 2$ was examined first by Bialostocki and Voxman [5] and by Jiang and West [15]. They considered the family of all trees with k edges, denoted by \mathcal{T}_k . $R^*(n, \mathcal{T}_k)$ makes sense iff $k \leq n-1$. Bialostocki and Voxman [5] gave $R^*(n, \mathcal{T}_{n-1})$ for every n; independently Jiang and West [15] gave $R^*(n, \mathcal{T}_k)$ for every n and k. (1) proved to be sharp for all of them. In particular, \mathcal{T}_{n-1}^* obviously consists of forests of order n with exactly two components, hence for $n \geq 3$ a graph of order n is \mathcal{T}_{n-1}^* -free iff it has at least three components; therefore it is easy to see that

$$ex(n, \mathcal{T}_{n-1}^*) = \binom{n-2}{2}$$
(4)

for $n \geq 3$. Bialostocki and Voxman [5] and Jiang and West [15] proved the next theorem:

Theorem 1 For $n \geq 3$,

$$\mathsf{R}^*(\mathfrak{n},\mathcal{T}_{\mathfrak{n}-1}) = \binom{\mathfrak{n}-2}{2} + 1.$$

(As we said, Jiang and West proved a much more general theorem.) Bialostocki [3] conjectured a generalization in another direction.

Let \mathcal{T}_{n-1}^d be the family of the trees of size n-1 and diameter at most d. Bialostocki conjectured that

$$\mathbf{R}^{*}(\mathbf{n}, \mathcal{T}_{\mathbf{n}-1}^{4}) = \mathbf{R}^{*}(\mathbf{n}, \mathcal{T}_{\mathbf{n}-1}).$$
(5)

In other words, if the number of colors is greater than $\binom{n-2}{2} + 1$ then there is a TMC spanning tree of diameter at most 4. His motivation was a result of him and his collaborators about corresponding Ramsey-type questions. It is well-known that if a graph is disconnected, i.e., if its diameter is infinite, then the diameter of its complement is at most 2. Bialostocki, Dierker and Voxman [4] proved that the same is true for complements of graphs of diameter greater than 4.

Our main aim in this paper is to show that much more is true. First of all,

$$R^*(n, \mathcal{T}^3_{n-1}) = R^*(n, \mathcal{T}_{n-1}).$$
(6)

That is, if the number of colors is greater than $\binom{n-2}{2} + 1$ then there is a TMC spanning tree with an edge dominating the whole tree, i.e., every vertex is adjacent to an endpoint of this edge. We shall call such a tree a *double star*, and the edge dominating the whole tree the *central edge*. (Note that a star is also a double star, and all its edges are central.) It is obvious that in (6) the diameter 3 cannot be replaced by 2, since $T_{n-1}^2 = \{K_{1,n-1}\}, T_{n-1}^{2^*} = \{K_{1,n-2}\}$

B. Montágh

and therefore $R^*(n, K_{1,n-2}) \ge ex(n, K_{1,n-2}) + 1 = \binom{n}{2} - n + 1 > R^*(n, \mathcal{T}_{n-1})$ for $n \ge 4$. Jiang [13] showed the considerably more interesting fact, that

$$R^*(n, \mathcal{T}_k^2) - ex(n, \mathcal{T}_k^{2^*}) > 1,$$
(7)

for $k \ge n/2 + 2$. In particular,

$$R^*(n, T_{n-1}^2) - ex(n, T_{n-1}^{2^*}) > 1$$

for $n \ge 6$. That is, for these cases the extremal anti-Ramsey colorings use at least two colors more than once. (7) disproves a conjecture of Manoussakis, Spyratos, Tuza and Voigt [16] who expected equality in (7) for every k and n. (7) is implied from the trivial facts that $T_k^2 = \{K_{1,k}\}$, for $k \ge 2$, $T_k^{2^*} = \{K_{1,k-1}\}$ and

$$\operatorname{ex}(\mathfrak{n},\mathsf{K}_{1,k-1}) = \left\lfloor \frac{\mathfrak{n}(k-2)}{2} \right\rfloor,\,$$

and the next theorem of Jiang [13]:

Theorem 2 For every $n > k \ge 2$,

$$0 \leq R^*(n, K_{1,k}) - \left(\left\lfloor \frac{n(k-2)}{2} \right\rfloor + \left\lfloor \frac{n}{n-k+2} \right\rfloor \right) \leq 1,$$

and the lower bound is sharp unless all of n, k and |2n/(n-k+2)| are odd.

(Jiang conjectures that the lower bound is the truth also in the remaining case.)

However, $\mathbb{R}^*(\mathfrak{n}, \mathcal{L}) = \mathbb{R}^*(\mathfrak{n}, \mathcal{T}_{n-1})$ holds even for a family \mathcal{L} much smaller than \mathcal{T}_{n-1}^3 . Let \mathcal{DS}_{n-1}^m be the families of the double stars with \mathfrak{n} vertices whose maximal degree is at least $\mathfrak{n} - \mathfrak{m}$, that is, their second largest degrees are at most \mathfrak{m} . Obviously, for $\mathfrak{n} \geq 2$ the second largest degree of a double star of size $\mathfrak{n} - \mathfrak{l}$ can be any positive integer between 1 and $\mathfrak{n}/2$, and all of these possible second largest degrees determines the double star uniquely, therefore $|\mathcal{T}_{n-1}^3| = \lfloor \mathfrak{n}/2 \rfloor$ for $\mathfrak{n} \geq 2$, and $|\mathcal{DS}_{\mathfrak{n}-1}^m| = \mathfrak{m}$ for $\mathfrak{n} \geq 2\mathfrak{m}$. In comparison, it is easy to see that $|\mathcal{T}_{\mathfrak{n}-1}^4| > \mathsf{P}(\lfloor (\mathfrak{n} - 1)/2 \rfloor)$, where P is the partition function, that is, $\mathsf{P}(k)$ is the number of writing the integer k as a sum of positive integers without regard to order. Hardy and Ramanujan [11] showed that, for some absolute constants A and B , $e^{\mathsf{A}\sqrt{k}} < \mathsf{P}(\mathsf{k}) < e^{\mathsf{B}\sqrt{\mathsf{k}}}$.

We shall show that

$$R^{*}(n, \mathcal{T}_{n-1}) = R^{*}(n, \mathcal{D}S_{n-1}^{4}).$$
(8)

Thus the conjectured equality (5) is true even if we replace the superpolinomially large family \mathcal{T}_{n-1}^4 by the family \mathcal{DS}_{n-1}^4 consisting four graphs.

Jiang's theorem above shows that $\mathcal{D}S^4_{n-1}$ cannot be replaced by $\mathcal{D}S^3_{n-1}$. Since all of the three members of $\mathcal{D}S^3_{n-1}$ contain $K_{1,n-3}$, $R^*(n, \mathcal{D}S^3_{n-1}) \geq R^*(n, K_{1,n-3})$. Therefore Theorem 2 implies that, for $n \geq 25$,

$$\begin{split} \mathsf{R}^*(\mathfrak{n},\mathcal{D}\mathsf{S}^3_{\mathfrak{n}-1}) \geq \frac{\mathfrak{n}(\mathfrak{n}-5)}{2} + \left\lfloor \frac{\mathfrak{n}}{5} \right\rfloor &= \binom{\mathfrak{n}-2}{2} + \left\lfloor \frac{\mathfrak{n}}{5} \right\rfloor - 3 \\ > & \binom{\mathfrak{n}-2}{2} + 1 = \mathsf{R}^*(\mathfrak{n},\mathcal{T}_{\mathfrak{n}-1}). \end{split}$$

In the next section we prove our main result.

In the last section we present an alternative, simpler proof of statement (6), by proving another theorem that perhaps is of its own interest. A graph is called Moore graph if it has diameter d and girth 2d+1 for some integer d. The trivial examples are complete graphs and odd cycles. Hoffman and Singleton proved that: every regular nontrivial Moore graph of diameter 2 has degree 3, 7 or 57; the unique 3-regular Moore graph of diameter 2 is the Petersen graph; there is exactly one 7-regular Moore graph (now it is called Hoffman-Singleton graph); and there is no nontrivial regular Moore graph of diameter 3. Erdős and Rényi [8] (see also [9]) found the next extremal property of regular Moore graphs of diameter 2.

Theorem 3 Let G be a graph of order n, diameter d and maximal degree Δ . Then

$$|\mathsf{E}(\mathsf{G})| \ge \frac{n(n-1)(\Delta-2)}{2((\Delta-1)^d-1)},$$

and equality holds iff G is a regular Moore graph.

Later Singleton [21] showed that, perhaps surprisingly, every Moore graph is regular. Finally, Bannai and Ito [2] and, independently, Damerell [6] proved that every nontrivial Moore graph has diameter 2. It yields that equality can hold in Theorem 3 only if d = 2 and $\Delta \in \{2; 3; 7; 57\}$. It is still unknown if there are 57-regular Moore graphs. It is easy to see that if they exist then their order is $1 + 57 + 57 \cdot 56 = 3250$.

We will show the next extremal property of the Moore graphs of diameter 2. If the graph G has diameter 2 and minimal degree δ then its size is at least $((\delta + 1)/2)n - (\delta^2 + 1)/2$, and equality holds iff G is a Moore graph. In case of d = 2, this fact can be considered as a dual of Theorem 3.

2 Proof of the main theorem

The main result of this chapter is the following.

Theorem 4 For $n \geq 3$,

$$\mathsf{R}^*(\mathfrak{n},\mathcal{D}\mathsf{S}^4_{\mathfrak{n}-1}) = \binom{\mathfrak{n}-2}{2} + 1.$$

Theorem 4 and Theorem 1 immediately imply (8) for $n \ge 3$, and it is obvious for n = 2, since then (and even for $n \le 4$) $\mathcal{T}_{n-1} = \mathcal{D}S_{n-1}^4$.

As usual, we shall denote by $N_G(x)$ the set of the vertices adjacent to xin G, and let $N_G[x]$ be the set $N_G(x) \cup \{x\}$. For any subgraph G of K_n , we define V(G) as the whole $V(K_n)$ even if, for some $v \in V(K_n)$, there is no edge of G incident to v. We say that a set $D \subset V(K_n)$ is a dominating set in a subgraph G of K_n if every vertex of \overline{D} is adjacent to at least one vertex of D. We shall denote the minimal degree of a graph G by $\delta(G)$ and the subgraph of G formed by the edges joining the disjoint subsets $A, B \subseteq V(G)$ by G[A, B].

Before starting the proof of Theorem 4, we state and prove three lemmas. All of them are very simple, but stating them as lemmas will make the proof of Theorem 4 easier to read.

The first lemma does not concern colorings, but rather only subgraphs of K_n whose complements do not contain double stars with high maximal degree.

Lemma 1 Let \mathfrak{m} , \mathfrak{n} be integers with $\mathfrak{m} \leq \mathfrak{n}/2$. Let G be a subgraph of $K_{\mathfrak{n}}$ for which \overline{G} does not contain a spanning double star with maximal degree at least $\mathfrak{n} - \mathfrak{m}$, and let \mathfrak{u} be a vertex of G of degree at most $\mathfrak{m} - 1$. Then $N_G(\mathfrak{u})$ is a dominating set in G.

We will use this lemma for $\mathfrak{m} = 4$ only, but its proof is essentially the same for every \mathfrak{m} .

Proof. Suppose that $N_G(u)$ is not a dominating set in G, that is, there is a vertex v whose distance from u in G is at least 3. Then $uv \in E(\overline{G})$, and $N_G(u) \cap N_G(v) = \emptyset$. Hence uv is a central edge of a spanning double star contained in \overline{G} in which the degree of u is $|\overline{N_G}[u]| \ge n - m$.

The next two lemmas concern edge-colored K_n s and their some particular subgraphs we define as follows. A subgraph of an edge-colored K_n is called *representing* iff it has exactly one edge of every color appearing on K_n . We shall call a subgraph H *representing-complement* (RC, for short) iff \overline{H} is representing. Furthermore, for a given edge-coloring of K_n , we shall call a subgraph H special representing-complement (SRC, for short) iff $\delta(H)$ is minimal over all RC subgraphs.

Lemma 2 Let a subgraph H of an edge-colored K_n be SRC, and let v be a vertex whose degree in H is $\delta(H)$. Then for any edge e of H incident to v there is an edge of \overline{H} incident to v colored with the color of e.

Proof. Since H is RC, there is an edge f of \overline{H} colored with the color of e. Since \overline{H} is representing, $\overline{H} - f + e$ is also representing, hence H - e + f is RC. If f was not incident to v, then the degree of v in H - e + f would be $\delta(H) - 1$, contradicting the fact that H is SRC.

Lemma 3 In every SRC subgraph H of an edge-colored K_n , the vertices whose degree in H is $\delta(H)$ forms an independent set.

Proof. Let u, v be two vertices whose degrees in H is $\delta(H)$. By Lemma 2 in \overline{H} there are edges f_1 , f_2 incident to u, v, respectively, colored with the color of uv. Since H is RC, only one edge of \overline{H} can be colored with the color of uv, hence $f_1 = f_2$. But then f_1 is incident to u and v both, that is, $f_1 = uv$, contradicting the facts $uv \in E(H)$ and $f_1 \in E(\overline{H})$.

$$\label{eq:proof.} \begin{split} \mathbf{Proof.} [\mathrm{Proof} \mbox{ of Theorem 4}] \mbox{ Obviously } R^*(n,\mathcal{D}S^4_{n-1}) \geq R^*(n,\mathcal{T}_{n-1}), \mbox{ and, for } n \geq 3, \end{split}$$

$$\mathsf{R}^*(\mathfrak{n},\mathcal{T}_{\mathfrak{n}-1}) \ge \binom{\mathfrak{n}-2}{2} + 1 \tag{9}$$

is a part of Theorem 1. (As we saw in the introduction, (9) is the easy direction of Theorem 1, since it immediately follows from (1) and (4).) Therefore, all we need to prove is

$$\mathsf{R}^*(\mathfrak{n},\mathcal{D}\mathsf{S}^4_{\mathfrak{n}-1}) \leq \binom{\mathfrak{n}-2}{2} + 1.$$

For the sake of brevity, we shall call a subgraph *good* if it is RC and for some vertex of degree at most 3, the set of its neighbors is not dominating. By Lemma 1 it suffices to prove that, for every coloring of $E(K_n)$ that uses $\binom{n-2}{2} + 2$ colors, there is a good subgraph.

Consider an arbitrary coloring of the edges of K_n that uses $\binom{n-2}{2} + 2$ colors. Let H be an arbitrary SRC subgraph. Then its size is $\binom{n}{2} - \binom{n-2}{2} - 2 = 2n-5$. If $\delta(H) = 0$ then there is a vertex u with $N_H(u) = \emptyset$, hence H is good. If $\delta(H) = 1$ then let u, v be vertices such that v is the only neighbor of u in H. By Lemma 2 there is a vertex w different from v such that the colors of uv and uw are the same, hence H - uv + uw is also RC. If neither H nor H - uv + uw is good, then v is adjacent to every vertex in H, and w is adjacent to every vertex in H – uv + uw, that is, to every vertex other than u in H. Hence the degrees of v and w in H are n - 1 and n - 2, respectively, so H has at least 2n - 4 edges, a contradiction.

Suppose that $\delta(H) = 2$. In the remainder of this proof, the graph, whose adjacency relation is considered, is always H. Let u be a vertex of degree 2 and with neighbors v_1, v_2 . First we assume that uv_1 and uv_2 have the same color, say, red. By Lemma 2 there is a vertex v_3 different from v_1 and v_2 such that uv_3 is also red. For each $1 \le i \le 3$, $H - uv_i + uv_3$ is RC. Thus if none of them is good, then for any permutation (i, j, k) of (1, 2, 3), $\{v_i, v_j\}$ is a dominating set. Therefore every element of $V(K_n) \setminus \{u, v_1, v_2, v_3\}$ has at least 2 neighbors among the v_js , and every v_i has at least 1 neighbor among the other two v_js . Hence there are at least 2(n-4) edges between $\{v_1, v_2, v_3\}$ and $V(K_n) \setminus \{u, v_1, v_2, v_3\}$, and at least 2 edges inside of $\{v_1, v_2, v_3\}$. Since the degree of u is 2, we have again the contradiction $|E(H)| \ge 2(n-4) + 2 + 2 = 2n - 4$.

We may therefore assume that uv_1 and uv_2 have different colors. By Lemma 2 there are $w_1, w_2 \in V(K_n) \setminus \{u, v_1, v_2\}$ such that uw_i have the same color as uv_i . Let z_1, \ldots, z_{n-5} be the remaining vertices. If none of the RC graphs H, $H - uv_1 + uw_1$, $H - uv_2 + uw_2$, $H - uv_1 + uw_1 - uv_2 + uw_2$ is good, then any pair containing one vertex among v_1, w_1 and one vertex among v_2, w_2 form a dominating set. Hence, for every $1 \le i \le n-5$, z_i is adjacent to either v_1 and w_1 or to v_2 and w_2 . Therefore there are at least 2(n-5) edges that are incident with the set $\{z_1, \ldots, z_{n-5}\}$.

Assume that $v_1w_1 \notin E(H)$. If H is not good, then the set $\{v_1, v_2\}$ is dominating, hence $v_2w_1 \in E(H)$. Similarly, if $H - uv_1 + uw_1$, $H - uv_2 + uw_2$ or $H - uv_1 + uw_1 - uv_2 + uw_2$ is not good then, in order, v_2v_1 , w_2w_1 or w_2v_1 is an edge of H. But, since the degree of u is 2, these imply that if none of the four mentioned subgraphs are good, then there are at least 2(n-5)+4+2 = 2n-4 edges in H. Thus $v_1w_1 \in E(H)$. Similarly, $v_2w_2 \in E(H)$, and therefore there is an edge e of H such that the 2n - 6 edges of H - e are the following: uv_1 , uv_2 , v_1w_1 , v_2w_2 , and for every $1 \le i \le n-5$, either the pair v_1z_i , w_1z_i or the pair v_1z_i , w_1z_i .

If $n \ge 8$ then there is an integer l such that $1 \le l \le n-5$ and e is not adjacent to z_l . Hence the only neighbors of z_l are either v_1 and w_1 or v_2 and w_2 . Thus if $N(z_l)$ is dominating then there are at least 2 edges between $\{v_1, w_l\}$ and $\{v_2, w_2\}$. However, as both are outside of H-e, this is impossible.



Figure 1: Case $\delta(H) = 2$ if uv_1 and uv_2 has different colors.

Therefore H is good.

On the other hand, by Lemma 3 there are no two adjacent vertices of degree 2. Hence there are at least two vertices of degree greater than 2, therefore the sum of the degrees is at least 2n + 2. Since the size of H is 2n - 5, we have $4n - 10 \ge 2n + 2$ and $n \ge 6$. Moreover, n > 6 since otherwise H would have two vertices of degree 3 and they would be adjacent to all the other four vertices, a contradiction.

Therefore the only remaining case is when n = 7 and $e = z_1 z_2$. Then $N(w_1)$ is not dominating since v_2 and w_2 are not adjacent to any vertices in it. Clearly, the degree of w_1 is at most 3. (In fact, it is 2, since otherwise the degree of w_2 is 1 and so $\delta(H) \neq 2$.) Hence H is good.



Figure 2: Subcase n = 7.

Finally, let $\delta(H) = 3$ and let \mathfrak{u} be a vertex of degree 3, with neighbors ν_1, ν_2, ν_3 . As in the previous argument, we can assume that there are (not necessarily different) vertices $w_1, w_2, w_3 \in V(K_n) \setminus \{\mathfrak{u}, \nu_1, \nu_2, \nu_3\}$ such that $\mathfrak{u}w_i$ have the same color as $\mathfrak{u}\nu_i$. Set $W = \{\nu_1, \nu_2, \nu_3, w_1, w_2, w_3\}$ and let Z be the set of the remaining vertices, that is, $Z = V(K_n) \setminus (W \cup \{\mathfrak{u}\})$. As in the previous cases, every vertex of Z has at least 2 neighbors in W and every vertex of W has at least 1 neighbor in W. As we shall show, these facts lead to a lower bound on the sum of all degrees, that implies an upper bound on

n. The degree of u is 3. The sum of the degrees of the vertices in W is $|E(H[W, \{u\}])| + 2|E(H[W])| + |E(H[W, Z])|$, that is at least 3 + |W| + 2|Z|. The sum of the degrees of the vertices in Z is at least 3|Z|. Hence the sum of all degrees is at least 6 + |W| + 5|Z| = 6 + |W| + 5(n - |W| - 1) = 1 + 5n - 4|W|, which is at least 5n - 23. Since this sum is exactly 2|E(H)| = 4n - 10, we have $5n - 23 \le 4n - 10$, that is, $n \le 13$.

On the other hand, 2|E(H)| = 4n - 10 and $\delta(H) = 3$ imply that there are at least 10 vertices of degree 3. By Lemma 3, they form an independent set. Hence the graph has at least 30 edges. It implies that $2n - 5 \ge 30$, that is, $n \ge 18$, a contradiction.

3 An extremal property of the nontrivial Moore graphs

Theorem 1 and Theorem 4 imply immediately the following theorem.

Theorem 5 For $n \geq 3$,

$$\mathsf{R}^*(\mathsf{n},\mathcal{T}^3_{\mathsf{n}-1}) = \binom{\mathsf{n}-2}{2} + 1.$$

In this section we shall prove an extremal property of the Moore graphs of diameter 2. As we saw in the introduction, the family of these graphs consists of C_5 , the Petersen graph, the Hoffman-Singleton graph and perhaps some 57-regular graphs of order 3250. The fact that the Petersen graph possesses this property leads to a proof of Theorem 5 that is a bit shorter than our proof for Theorem 4.

We will prove the following theorem.

Theorem 6 Let G be a graph of order n, diameter 2 and minimal degree δ . Then

$$|\mathsf{E}(\mathsf{G})| \geq \frac{(\delta+1)n - \delta^2 - 1}{2},$$

and equality holds iff G is a Moore graph.

First, we show how can one simplify the proof of Theorem 5 by using Theorem 6.

Proof. [Proof of Theorem 5.] We repeat the proof of Theorem 4, changing the end only, which handled the case $\delta(H) = 3$. As we saw in the proof of

Lemma 1, if there are two vertices \mathbf{u} , \mathbf{v} of a graph G such that their distance is at least 3, than $\overline{\mathbf{G}}$ contains a spanning double star (with central edge \mathbf{uv}). In other words, if a graph does not contain a spanning double star, then the diameter of its complement is at most 2. Therefore, if a coloring of $\mathsf{E}(\mathsf{K}_n)$ does not yield any TMC spanning double star, then its RC subgraphs have diameter at most 2. If the coloring uses $\binom{n-2}{2} + 2$ colors, then the size of its RC subgraphs is $\binom{n}{2} - \binom{n-2}{2} - 2 = 2n - 5$. Therefore if H is an SRC subgraph with $\delta(\mathsf{H}) = 3$, then, by Theorem 6, H is the Petersen graph. On the other hand, by Lemma 3, if an SRC subgraph of any coloring of $\mathsf{E}(\mathsf{K}_n)$ is not empty (that is, if the coloring uses less than $\binom{n}{2}$ colors) then it cannot be regular, thus we have the desired contradiction.

For proving Theorem 6 we need the following lemma.

Lemma 4 Let n be an integer with $n \ge 2$, and let G be a graph of order n, diameter at most 2 and degree sequence d_1, d_2, \ldots, d_n . Then

$$\sum_{i=1}^n d_i^2 \ge n^2 - n,$$

and equality holds iff either the girth of G is 5 or $G = K_{1,n-1}$.

Proof. Let $V(G) = \{v_1, v_2, ..., v_n\}$ and d_i be degree of v_i for every i. Count the walks of length 2 in G, that is, the ordered triples (v_i, v_j, v_k) of vertices with $v_i v_j, v_j v_k \in E(G)$ (i = k being allowed). For a given j the number is obviously d_i^2 , therefore the total number is $\sum_{i=1}^n d_i^2$.

Now we show that there is an injection f from the set of ordered pairs of distinct vertices to the set of these walks. For $v_i v_j \notin E(G)$, let $f(v_i, v_j) = (v_i, v_k, v_j)$ with arbitrary k such that $v_i v_k, v_k v_j \in E(G)$ (such j exists since the diameter of G is at most 2). For $v_i v_j \in E(G)$, let $f(v_i, v_j) = (v_i, v_j, v_i)$. f is an injection since for $i \neq k$, (v_i, v_j, v_k) can only be the image of (v_i, v_k) , and for i = k, it can only be the image of (v_i, v_j) .

Since the number of ordered pairs of distinct vertices is $n^2 - n$, $\sum_{i=1}^{n} d_i^2 \ge n^2 - n$. Equality holds iff f is surjective, that is, iff there is exactly one k with $v_i v_k, v_k v_j \in E(G)$ for every i, j with $v_i v_j \notin E(G)$ and there is no such k for any i, j with $v_i v_j \in E(G)$. In other words, iff G contains neither C₃ nor C₄, that is, G is either a forest or its girth is at least 5. The only forest with order n and diameter at most 2 is $K_{1,n-1}$. Since the distance of two vertices of a shortest cycle of a graph is obviously the length of the shorter arc of this cycle connecting them, a girth of a graph with diameter 2 cannot be greater than 5.

Proof. [Proof of Theorem 6.] Let e = |E(G)|. By Lemma 4, it is sufficient to show that if the degree sequence of G is d_1,d_2,\ldots,d_n and

$$\sum_{i=1}^{n} d_i^2 \ge n^2 - n,$$
(10)

then

$$e \ge \frac{(\delta+1)n - \delta^2 - 1}{2},\tag{11}$$

and equality holds in (11) only if it holds in (10).

First assume that $n \leq \delta^2 + 1$. In this case, since obviously $e \geq \delta n/2$, (11)

First assume that $n \leq \delta^2 + 1$. In this case, since obviously $e \geq \delta n/2$, (11) is trivially true. Equality holds iff $n = \delta^2 + 1$ and $d_i = \delta$ for every i. Then $\sum_{i=1}^{n} d_i^2 = n\delta^2 = n(n-1) = n^2 - n$. Now assume $n > \delta^2 + 1$. Because of convexity of the function x^2 , if x_1, \ldots, x_n and s are real numbers such that, for every i, $x_i \geq \delta$, $s \geq \delta n$ and $\sum_{i=1}^{n} x_i = s$, then $\sum_{i=1}^{n} x_i^2 \leq (n-1)\delta^2 + (s - (n-1)\delta)^2$, and equality holds iff there is at most one i with $x_i > \delta$. Hence if $\sum_{i=1}^{n} d_i^2 \geq n^2 - n$ then

$$(n-1)\delta^2 + (2e - (n-1)\delta)^2 \ge n^2 - n,$$

that is,

$$\begin{array}{rll} 4e^2 - 4\delta(n-1)e + \delta^2((n-1)^2 + n - 1) & \geq & n(n-1), \\ e^2 - \delta(n-1)e + \frac{\delta^2 - 1}{4}n(n-1) & \geq & 0, \end{array}$$

therefore

$$\begin{aligned} 2e &\geq \delta(n-1) + \sqrt{\delta^2(n-1)^2 - (\delta^2 - 1)n(n-1)} \\ &= \delta(n-1) + \sqrt{(n-1)(n-\delta^2)}. \end{aligned}$$

Thus, it suffices to show

$$\delta(n-1)+\sqrt{(n-1)(n-\delta^2)}>(\delta+1)n-\delta^2-1,$$

that is,

$$\delta^2 - \delta + 1 > n - \sqrt{(n-1)(n-\delta^2)}.$$
(12)

It is easy to verify that the right-hand side is strictly decreasing in n. Since equality holds in (12) for $n = \delta^2 + 1$, the inequality is valid for $n > \delta^2 + 1$.

The author would like to thank Béla Bollobás and Miklós Simonovits for valuable comments and remarks, and Miklós Simonovits for calling my attention to this problem.

References

- Alon, N., On a conjecture of Erdős, Simonovits and Sós concerning Anti-Ramsey theorems, J. Graph Theory, 7 (1983), 91–94.
- Bannai, E., Ito, T., On finite Moore graphs, J. Fac. Sci. Tokyo, Sect. 1A 20 (1973), 191–208.
- [3] Bialostocki, A., personal communication
- [4] Bialostocki, A., Dierker, P., Voxman, W., Either a graph or its complement is connected: A continuing saga, *Mathematics Magazine*, to appear.
- [5] Bialostocki, A., Voxman, W., On the Anti-Ramsey numbers for spanning trees, Bull. Inst. Combin. Appl., 32 (2001), 23–26.
- [6] Damerell, R.M., On Moore graphs, Proc. Cambridge Phil. Soc., 74 (1973), 227–236.
- [7] Dirac, G.A., Extensions of Turán's theorem on graphs, Acta Math. Acad. Sci. Hungar., 14 (1963), 418–422.
- [8] Erdős, P., Rényi, A., On the problem in the theory of graphs (in Hungarian), Publ. Math. Inst. Hung. Acad. Sci., 7/A (1962), 623–641.
- [9] Erdős, P., Rényi, A., Sós, V.T., On a problem of graph theory, Studia. Sci. Math. Hungar., 1 (1966), 215–235.
- [10] Erdős, P., Simonovits, M., Sós, V.T., Anti-Ramsey theorems. In: Infinite and finite sets, (Colloq. Keszthely, 1973), pp. 633–642. Coll. Math. Soc. J. Bolyai, Vol. 10, Amsterdam: North-Holland 1975.
- [11] Hardy, G.H., Ramanujan, S., Asymptotic formulae in combinatory analysis, Proc. London Math. Soc., 17 (1918), 75–115.
- [12] Hoffman, A.J., Singleton, R.R., On Moore graphs with diameter 2 and 3, *IBM J. Res. Develop.*, 4 (1960), 497–504.

- [13] Jiang, T., Edge-colorings with no large polychromatic stars, Graphs Combin., 18 (2002), 303–308.
- [14] Jiang, T., Schiermeyer, I., West, D., The Erdős–Simonovits–Sós conjecture for $k\leq 7,$ in preparation.
- [15] Jiang, T., West, D., Edge-colorings of complete graphs that avoid polychromatic trees, *Discrete Math.*, 274 (2004), 137–145.
- [16] Manoussakis, Y., Spyratos, M., Tuza, Zs., Voigt, M., Minimal colorings for properly colored subgraphs, *Graphs Combin.*, **12** (1996), 345–360.
- [17] Montágh, B., Anti-Ramsey numbers of long paths, in preparation.
- [18] Montellano-Ballesteros, J.J., Neumann-Lara, V., An Anti-Ramsey theorem, *Combinatorica*, (2002) 22, 445–449.
- [19] Schiermeyer, I., Rainbow numbers for matchings and complete graphs, Discrete Math., 286 (2004), 157–162.
- [20] Simonovits, M., Sós, V.T., On restricted colourings of K_n , Combinatorica, 4 (1984), 101–110.
- [21] Singleton, R.R.: There is no irregular Moore graph, American Math. Monthly, 75 (1968), 42–43.

Received: October 1, 2008