Hadamard product of GCD matrices

Antal Bege
Sapientia–Hungarian University of Transylvania
Department of Mathematics and Informatics,
Târgu Mureş, Romania
email: abege@ms.sapientia.ro

Abstract. Let f be an arithmetical function. The matrix $[f(i,j)]_{n\times n}$ given by the value of f in greatest common divisor of (i,j), $f\big((i,j)\big)$ as its i, j entry is called the greatest common divisor (GCD) matrix. We consider the Hadamard product of this matrices and we calculate the Hadamard product and the determinant of Hadamard product of two GCD matrices.

1 Introduction

The classical Smith determinant introduced by H. J. Smith [6] is

$$\det[(i,j)]_{n\times n} = \begin{vmatrix} (1,1) & (1,2) & \cdots & (1,n) \\ (2,1) & (2,2) & \cdots & (2,n) \\ \cdots & \cdots & \cdots & \cdots \\ (n,1) & (n,2) & \cdots & (n,n) \end{vmatrix} = \varphi(1)\cdot\varphi(2)\cdots\varphi(n), \qquad (1)$$

where (i,j) is the greatest common divisor of i and j, and $\varphi(n)$ is Euler's totient function.

The GCD matrix with respect to f is

$$[f(i,j)]_{n\times n} = \begin{bmatrix} f((1,1)) & f((1,2)) & \cdots & f((1,n)) \\ f((2,1)) & f((2,2)) & \cdots & f((2,n)) \\ \cdots & \cdots & \cdots & \cdots \\ f((n,1)) & f((n,2)) & \cdots & f((n,n)) \end{bmatrix}$$

AMS 2000 subject classifications: 11C20, 11A25, 15A36
Key words and phrases: GCD determinant, arithmetical function, Hadamard product

If we consider the GCD matrix $[f(i,j)]_{n\times n}$ where

$$f(n)=\sum_{d|n} g(d),$$

H. J. Smith proved that

$$\det[f(i,j)]_{n\times n}=g(1)\cdot g(2)\cdots g(n).$$

For $g=\varphi$

$$f(i,j)=\sum_{d|(i,j)} \varphi(d)=(i,j).$$

this formula reduces to (1).
If we consider the GCD matrix $[f(i,j)]_{n\times n}$ where $f(n)=\sum_{d|n} g(d)$ Pólya and Szegő [5] proved that

$$[f(i,j)]_{n\times n}=G\cdot C^T, \tag{2}$$

where G and A are lower triangular matrices given by

$$g_{ij}=\begin{cases} g(j), & j\mid i \\ 0, & \text{otherwise}\end{cases}$$

and

$$c_{ij}=\begin{cases} 1, & j\mid i \\ 0, & \text{otherwise}\end{cases}$$

L. Carlitz [2] in 1960 gave a new form of (2)

$$[f(i,j)]_{n\times n}=C\ \mathrm{diag}\big(g(1),\ g(2),\ldots,g(n)\big)\ C^T, \tag{3}$$

where $C=[c_{ij}]_{n\times n}$,

$$c_{ij}=\begin{cases} 1, & \text{if} \quad j\mid i \\ 0, & \text{if} \quad j\nmid i\end{cases},$$

$D=[d_{ij}]_{n\times n}$ diagonal matrix

$$d_{ij}=\begin{cases} g(i), & \text{ha} \quad i=j \\ 0, & \text{ha} \quad i\neq j\end{cases}.$$

From (3) follows that the value of the determinant is

$$\det[f(i,j)]_{n\times n}=g(1)g(2)\cdots g(n). \tag{4}$$

Here we present some examples which are relevant in our study.

Example 1 *If*

$$g(n)=\beta(n)=\sum_{i=1}^{n}(i,n)$$

the Pillai function then

$$f(n)=\sum_{d|n}\beta(n)=n\tau(n),$$

where $\tau(n)$ *the number of divisors. The GCD matrix and determinant in this case have the following form:*

$$[(i,j)\tau(i,j)]_{n\times n}=C\ diag\big(\beta(1),\ \beta(2),\dots,\beta(n)\big)\ C^T, \tag{5}$$

$$\det[(i,j)\tau(i,j)]_{n\times n}=\beta(1)\beta(2)\cdots\beta(n). \tag{6}$$

Example 2 *If* $g(n)=\dfrac{\varphi(n)}{n}$*, then*

$$f(n)=\sum_{d|n}\frac{\varphi(d)}{d}=\frac{\beta(n)}{n},$$

$$\left[\frac{\beta(i,j)}{(i,j)}\right]_{n\times n}=C\ diag\left(\frac{\varphi(1)}{1},\ \frac{\varphi(2)}{2},\dots,\frac{\varphi(n)}{n}\right)\ C^T, \tag{7}$$

$$\det\left[\frac{\beta(i,j)}{(i,j)}\right]_{n\times n}=\frac{\varphi(1)\varphi(2)\cdots\varphi(n)}{n!}.$$

For other contributions, we mention the papers of S. Beslin and S. Ligh [1], P. Haukkanen, J. Wang and J. Sillanpää [3].
We introduce the concept of Hadamard product (see F. Zhang [7]).

Definition 1 *The Hadamard product* $C=A\circ B=[c_{ij}]_{n\times n}$ *of two matrices* $A=[a_{ij}]_{n\times n}$ *and* $B=[b_{ij}]_{n\times n}$ *is simply their elementwise product,*

$$c_{ij}=a_{ij}b_{ij},\quad i,j\in\{1,2,\dots,b\}.$$

A. Ocal [4] establishes various results concerning GCD matrices and least common multiple (LCM) matrices. In examples 1 and 2 appears Hadamard products of special GCD matrices:

$$\det\Big[[\tau(i,j)]_{n\times n}\circ[(i,j)]_{n\times n}\Big]_{n\times n}=\beta(1)\beta(2)\cdots\beta(n),$$

$$\det\left[[\beta(i,j)]_{n\times n}\circ\left[\frac{1}{(i,j)}\right]_{n\times n}\right]_{n\times n}=\frac{\varphi(1)\varphi(2)\cdots\varphi(n)}{n!}.$$

Let f and g be two arithmetical functions. In this paper we calculate the Hadamard product and the determinant of Hadamard product of $[f(i,j)]_{n\times n}$ and $[g(i,j)]_{n\times n}$.

2 Main results

Theorem 1 *Let* h *and* g *be two arithmetical functions and* g *totally multiplicative. If*

$$f(n)=\sum_{d|n}h(d)g\left(\frac{n}{d}\right), \tag{8}$$

then

1.

$$\left[[f(i,j)]_{n\times n}\circ\left[\frac{1}{g(i,j)}\right]_{n\times n}\right]_{n\times n}=C\ \mathit{diag}\left(\frac{h(1)}{g(1)},\frac{h(2)}{g(2)},\dots,\frac{h(n)}{g(n)}\right)C^T,$$

where $C=[c_{ij}]_{n\times n}$*,*

$$c_{ij}=\begin{cases}1, & \text{if}\quad j\mid i\\ 0, & \text{if}\quad j\nmid i\end{cases},$$

2.

$$\det\left[[f(i,j)]_{n\times n}\circ\left[\frac{1}{g(i,j)}\right]_{n\times n}\right]_{n\times n}=\frac{h(1)}{g(1)}\frac{h(2)}{g(2)}\cdots\frac{h(n)}{g(n)}, \tag{9}$$

3. *Exists* $H(n)$ *and* $G(n)$ *arithmetical functions such that*

$$\det\left[[f(i,j)]_{n\times n}\circ\left[\frac{1}{g(i,j)}\right]_{n\times n}\right]_{n\times n}=\frac{\det[H(i,j)]}{\det[G(i,j)]}.$$

Proof. Let

$$A=[a_{ij}]_{n\times n}=\left[[f(i,j)]_{n\times n}\circ\left[\frac{1}{g(i,j)}\right]_{n\times n}\right]_{n\times n}.$$

By the definition of Hadamard product we have

$$a_{ij}=\frac{f(i,j)}{g(i,j)}.$$

If we calculate

$$B = [b_{ij}]_{n\times n} = C \text{ diag}\left(\frac{h(1)}{g(1)}, \frac{h(2)}{g(2)}, \ldots, \frac{h(n)}{g(n)}\right)$$

we have

$$b_{ij} = \sum_{k|j}\sum_{k|i} \frac{h(k)}{g(k)}.$$

But taking in consideration that g is totally multiplicative and by (8) we can deduce that

$$\begin{aligned} b_{ij} &= \sum_{k|(j,i)} \frac{h(k)}{g(k)} = \sum_{k|(j,i)} \frac{h(k)g\left(\frac{(i,j)}{k}\right)}{g(k)g\left(\frac{(i,j)}{k}\right)} = \frac{1}{g\big((i,j)\big)} \sum_{k|(j,i)} h(k)g\left(\frac{(i,j)}{k}\right) = \\ &= \frac{f(i,j)}{g(i,j)} = a_{ij}, \end{aligned}$$

which means that $A = B$.
If we calculate the determinant of both parts we have (9).
Let

$$H(n) = \sum_{d|n} h(d)$$

and

$$G(n) = \sum_{d|n} g(d).$$

By (4) we have

$$\det[H(i,j)]_{n\times n} = h(1)h(2)\cdots h(n)$$

and

$$\det[G(i,j)]_{n\times n} = g(1)g(2)\cdots g(n).$$

which means that

$$\det\left[[f(i,j)]_{n\times n} \circ \left[\frac{1}{g(i,j)}\right]_{n\times n}\right]_{n\times n} = \frac{\det[H(i,j)]}{\det[G(i,j)]}.$$

■

Example 3 *If* $g(n) = n$ *then*

$$f(n) = \sum_{d|n} h(d)\frac{n}{d}.$$

and

$$\left[\frac{f(i,j)}{(i,j)}\right]_{n\times n} = \left[[f(i,j)]_{n\times n} \circ \left[\frac{1}{(i,j)}\right]_{n\times n}\right]_{n\times n} =$$

$$= C \ \mathit{diag}\left(\frac{h(1)}{1}, \frac{h(2)}{2}, \dots, \frac{h(n)}{n}\right) \ C^T$$

$$\det \left[\frac{f(i,j)}{(i,j)}\right]_{n\times n} = \det \left[[f(i,j)]_{n\times n} \circ \left[\frac{1}{(i,j)}\right]_{n\times n}\right]_{n\times n} = \frac{h(1)h(2)\cdots h(n)}{n!}.$$

Example 4 *If* $g(n) = \dfrac{1}{n}$ *then*

$$f(n) = \sum_{d|n} h(d)\frac{d}{n}$$

and

$$[f(i,j)(i,j)]_{n\times n} = C \ \mathit{diag}(h(1)1, h(2)2, \dots, h(n)n) \ C^T,$$

$$\det [f(i,j)(i,j)]_{n\times n} = \det \left[[f(i,j)]_{n\times n} \circ [(i,j)]_{n\times n}\right]_{n\times n} = h(1)\cdots h(n)n!.$$

If we want to apply this theorem to given f and g, by Mobius inversion formula we have

$$h(n) = \sum_{d|n} \mu(d)g(d)f\left(\frac{n}{d}\right)$$

where $\mu(n)$ is the usual Mobius function and we can formulate the following result.

Theorem 2 *Let* f *and* g *be two arithmetical functions and* g *totally multiplicative. We have*

$$\begin{aligned}\left[\frac{f(i,j)}{g(i,j)}\right]_{n\times n} &= \left[[f(i,j)]_{n\times n} \circ \left[\frac{1}{g(i,j)}\right]_{n\times n}\right]_{n\times n} = \\ &= C \ \mathit{diag}\left(\frac{f(1)}{g(1)}, \dots, \frac{\sum_{d|n} \mu(d)g(d)f\left(\frac{n}{d}\right)}{g(n)}\right) \ C^T,\end{aligned}$$

and

$$\det \left[[f(i,j)]_{n\times n} \circ \left[\frac{1}{g(i,j)}\right]_{n\times n}\right]_{n\times n} = \frac{f(1)}{g(1)} \cdots \frac{\sum_{d|n} \mu(d)g(d)f\left(\frac{n}{d}\right)}{g(n)}.$$

Example 5 *If* f *power free multiplicative arithmetical function (*$f(p^\alpha) = f(p)$*)*

$$\det\left[f(i,j)(i,j)\right]_{n\times n} = \prod_{k=1}^{n} \varphi(k) f(k),$$

in particular if $f(n) = \gamma(n)$ *the greatest square free divisor of* n

$$\det\left[\gamma((i,j))(i,j)\right]_{n\times n} = \prod_{k=1}^{n} \varphi(k)\gamma(k).$$

Example 6 *For a power GCD matrix and determinant we have*

$$[(i,j)^s]_{n\times n} = C \;\mathit{diag}\big(J_s(1),\; J_s(2),\dots,J_s(n)\big)\; C^T,$$

$$\det[(i,j)^s]_{n\times n} = J_s(1)J_s(2)\cdots J_s(n).$$

where $J_s(n)$ *the jordan totient function.*

References

[1] S. Beslin, S. Ligh, GCD-closed sets and the determinants of GCD matrices, *Fibonacci Quart.,* **30** (1992), 157–160.

[2] Carlitz, L. Some matrices related to the gratest integer function, *J. Elisha Mitchell Sci. Soc.,* **76** (1960), 5–7.

[3] P. Haukkanen, J. Wang, J. Silanpää, On Smith's determinant, *Linear Algebra Appl.,* **258** (1997) 251–269.

[4] A. A. Öçal, On the Hadamard products of GCD and LCM matrices, *Commun. Fac. Sci. Univ. Ank., Sér. A1, Math. Stat.,* **52** (2003), 1–12.

[5] Gy. Pólya, G. Szegő, *Problems and theorems in analysis*, Vol. II, Springer, Berlin, Heidelberg, New York, 1998.

[6] H. J. S. Smith, On the value of a certain arithmetical determinant, *Proc. London Math. Soc.,* **7** (1875/76), 208–212.

[7] F. Zhang, *Matrix theory*, Springer, New York, Berlin, Heidelberg, 1999.

Received: December 3, 2008