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Real vector space with scalar product of quasi-triangular fuzzy numbers

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Abstract. The construction of membership function of fuzzy numbers is an important problem in vagueness modeling. Theoretically, the shape of fuzzy numbers must depend on the applied triangular space. The membership function must defined in a such a way that the change of the triangular norm modifies the shape of fuzzy number, but the calculus with them remain valid. The quasi-triangular fuzzy numbers introduced by M. Kovacs in 1992 are satisfied this requirement. The shortage that not any quasi-triangular fuzzy number has opposite (inverse) can be solved if the set of quasi-triangular fuzzy numbers is included isomorphically in an extended set and this extended set with addition forms a group. In the present paper we formulate the extended set of the quasi-triangular fuzzy numbers, being also shown that the extended set is a real vector space with scalar product.

1 Introduction

The concept of quasi-triangular fuzzy numbers generated by a continuous decreasing function was introduced first by M. Kovács in 1992. The shortage that not any quasi-triangular fuzzy number has opposite (inverse) but only the ones with spread zero, can be solved if the set of quasi-triangular fuzzy numbers is included isomorphically in an extended set and this extended set with addition forms a group. In section 3 this group is constructed and in

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section 4 it is shown that the extended set with addition and multiplication with a scalar is a real vector space. In the section 5 we construct the real vector space with scalar product of quasi-triangular fuzzy numbers.

In the study of algebraic structures for fuzzy numbers many results since the 1970s have been obtained. For example D. Dubois and H. Prade (1978) investigates the operations with fuzzy numbers and theirs properties, R. Goetschel and W. Voxman (1986) and S. Gähler (1999) continue this work and M. Kovacs and L. H. Tran (1991) constructs and studies the set of centered M-fuzzy numbers. M. Kovacs (1992) introduces a notion of quasi-triangular fuzzy number which was used in the fuzzy linear programming by Z. Mako (2006). The properties of another class of quasi-triangular fuzzy numbers were investigated by M. Mares (1992,1992/1993, 1993, 1997), J. Dombi and N. Győrbíró (2006) and D. H. Hong (2007) obtains some properties of the operations with fuzzy numbers. A. M. Bica (2007) investigates the operations over the class of fuzzy numbers.

2 Preliminaries

The fuzzy set concept was introduced in mathematics by K. Menger in 1942 and reintroduced in the system theory by L. A. Zadeh in 1965. L. A. Zadeh has introduced this notion to measure quantitatively the vague of the linguistic variable. The basic idea was: if X is a set, then all A subsets of X can be identified with its characteristic function $\chi_A : X \to \{0,1\}$, $\chi_A(x) = 1 \Leftrightarrow x \in A$ and $\chi_A(x) = 0 \Leftrightarrow x \notin A$.

The notion of fuzzy set is another approach of the subset notion. There exist continue and transitory situations in which we have to sugest that an element belongs to a set by different level. This fact we indicate with the membership degree.

Definition 1 *Let* X *be a set. A mapping* $\mu : X \to [0,1]$ *is called* membership function*, and the set* $\bar{A} = \{(x, \mu(x)) \ / x \in X\}$ *is called* fuzzy set *on* X*. The membership function of* $\bar{A}$ *is denoted by* $\mu_{\bar{A}}$*.*

The collection of all fuzzy subsets of X we will denote by $\mathcal{F}(X)$. We place a bar over a symbol if it represents a fuzzy set. If $\bar{A}$ is a fuzzy set of X, then $\mu_{\bar{A}}(x)$ represents the membership degree of x to X. The empty fuzzy set is denoted by $\bar{\emptyset}$, where $\mu_{\bar{\emptyset}}(x) = 0$ for all $x \in X$. The total fuzzy set is denoted by $\bar{X}$, where $\mu_{\bar{X}}(x) = 1$ for all $x \in X$.

Definition 2 *The height of* $\bar{A}$ *is defined as* $\mathrm{hgt}\left(\bar{A}\right) = \sup_{x \in X} \mu_{\bar{A}}(x)$. *The* support *of* $\bar{A}$ *is the subset of* X *given by* $supp\bar{A} = \{x \in X \ / \ \mu_{\bar{A}}(x) > 0\}$.

Definition 3 *Let* X *be a topological space. The* α–level *of* $\bar{A}$ *is defined as*

$$\left[\bar{A}\right]^{\alpha} = \begin{cases} \{x \in X / \ \mu_{\bar{A}}(x) \geq \alpha\} & \text{if} \quad \alpha > 0, \\ \mathrm{cl}\left(supp\bar{A}\right) & \text{if} \quad \alpha = 0. \end{cases}$$

where $\mathrm{cl}\left(supp\bar{A}\right)$ *is closure of the support of* $\bar{A}$.

Definition 4 *A fuzzy set* $\bar{A}$ *on vector space* X *is* convex, *if all* α *-levels are convex subsets of* X, *and it is* normal *if* $\left[\bar{A}\right]^{1} \neq \emptyset$.

In many situations people are only able to characterize imprecisely numerical data. For example people use terms like: "about 100" or "near 10". These are examples of what are called fuzzy numbers.

Definition 5 *A convex, normal fuzzy set on the real line* $\mathbb{R}$ *with upper semi-continuous membership function will be called* fuzzy number.

Triangular norms and co-norms were introduced by K. Menger (1942) and studied first by B. Schweizer and A. Sklar (1961, 1963, 1983) to model distances in probabilistic metric spaces. In fuzzy sets theory triangular norms and co-norms are extensively used to model logical connection and and or. In the fuzzy literatures, these concepts was studied e. g. in E. Creţu (2001), J. Dombi (1982), D. Dubois and H. Prade (1985), J. Fodor (1991, 1999), S. Jenei (1998, 1999, 2000, 2001, 2004), V. Radu (1974, 1984, 1992).

Definition 6 *The function* $N : [0, 1] \to [0, 1]$ *is a* negation operation *if:*

(i) $N(1) = 0$ *and* $N(0) = 1$;

(ii) N *is continuous and strictly decreasing;*

(iii) $N(N(x)) = x$, *for all* $x \in [0, 1]$.

Definition 7 *Let* N *be a negation operation. The mapping* $T : [0, 1] \times [0, 1] \to [0, 1]$ *is a* triangular norm *(briefly* t-norm*) if satisfies the properties:*

$$\begin{aligned} \textit{Symmetry} &: T(x, y) = T(y, x), \quad \forall x, y \in [0, 1]; \\ \textit{Associativity} &: T(T(x, y), z) = T(x, T(y, z)), \quad \forall x, y, z \in [0, 1]; \\ \textit{Monotonicity} &: T(x_1, y_1) \leq T(x_2, y_2) \quad \mathrm{if} x_1 \leq x_2 \mathrm{and} y_1 \leq y_2; \\ \textit{One identity} &: \ T(x, 1) = x, \ \forall x \in [0, 1] \end{aligned}$$

and the mapping $S : [0,1] \times [0,1] \to [0,1]$*,*

$$S(x,y) = N(T(N(x), N(y)))$$

is a triangular co-norm (the dual of T *given by* N*).*

Definition 8 *The t-norm* T *is* Archimedean *if* T *is continuous and* $T(x,x) < x$*, for all* $x \in (0,1)$.

Definition 9 *The t-norm* T *is called* strict *if* T *is strictly increasing in both arguments.*

Theorem 1 ([22]) *Every Archimedean t-norm* T *is representable by a continuous and decreasing function* $g : [0,1] \to [0,+\infty]$ *with* $g(1) = 0$ *and*

$$T(x,y) = g^{[-1]}(g(x) + g(y)),$$

where

$$g^{[-1]}(x) = \begin{cases} g^{-1}(x) & \text{if} \quad 0 \le x < g(0), \\ 0 & \text{if} \quad x \ge g(0). \end{cases}$$

If g_1*and* g_2 *are the generator function of* T*, then there exist* $c > 0$ *such that* $g_1 = cg_2$.

Remark 1 *If the Archimedean t-norm* T *is strict, then* $g(0) = +\infty$ *otherwise* $g(0) = p < \infty$.

Theorem 2 ([38]) *An application* $N : [0,1] \to [0,1]$ *is a negation if and only if there exist an increasing and continuous function* $e : [0,1] \to [0,1]$*, with* $e(0) = 0$*,* $e(1) = 1$ *such that*$N(x) = e^{-1}(1 - e(x))$*, for all* $x \in [0,1]$.

Remark 2 *The generator function of negation* $N(x) = 1 - x$ *is* $e(x) = x$*. Another negation generator function is*

$$e_\lambda(x) = \frac{\ln(1 + \lambda x)}{\ln(1 + \lambda)},$$

where $\lambda > -1$*,* $\lambda \neq 0$.

Remark 3 *Examples to t-norm are following:*

- *minim:* $\min(x,y) = \min\{x,y\}$;

- *product:* $P(x,y) = xy$, *the generator function is* $g(x) = -\ln x$;
- *weak:* $W(x,y) = \begin{cases} \min\{x,y\} & \text{if } \max\{x,y\} = 1, \\ 0 & \text{otherwise.} \end{cases}$

If the negation operation is $N(x) = 1 - x$, *then the dual of these t-norms are:*

- *maxim:* $\max(x,y) = \max\{x,y\}$;
- *probability:* $S_P(x,y) = x + y - xy$;
- *strong:* $S_W(x,y) = \begin{cases} \max\{x,y\} & \text{if } \min\{x,y\} = 0, \\ 1 & \text{otherwise.} \end{cases}$

Proposition 1 *If* T *is a t-norm and* S *is the dual of* T, *then*

$$W(x,y) \le T(x,y) \le \min\{x,y\},$$
$$\max\{x,y\} \le S(x,y) \le S_W(x,y),$$

for all $x, y \in [0,1]$.

Let X be a nonempty set, T be a t-norm, N be a negation operation and S the dual of T given by N. The intersection, union, complement and Cartesian product of fuzzy sets may be defined in the following way.

Definition 10 *The* $T-$intersection*'s membership function of fuzzy sets* $\bar{A}$ *and* $\bar{B}$ *is defined as*

$$\mu_{\bar{A}\sqcap\bar{B}}(x) = T(\mu_{\bar{A}}(x), \mu_{\bar{B}}(x)), \quad \forall x \in X.$$

The $S-$union*'s membership function of fuzzy sets* $\bar{A}$ *and* $\bar{B}$ *is defined as*

$$\mu_{\bar{A}\sqcup\bar{B}}(x) = S(\mu_{\bar{A}}(x), \mu_{\bar{B}}(x)), \quad \forall x \in X.$$

The $N-$complement*'s membership function of fuzzy sets* $\bar{A}$ *and* $\bar{B}$ *is defined as*

$$\mu_{\neg\bar{A}}(x) = N(\mu_{\bar{A}}(x)), \quad \forall x \in X.$$

Definition 11 *The* T-Cartesian product*'s membership function of fuzzy sets* $\bar{A}_i \in \mathcal{F}(X_i)$, $i = 1, ..., n$ *is defined as*

$$\mu_{\bar{A}}(x_1, x_2, ..., x_n) = T\left(\mu_{\bar{A}_1}(x_1), T\left(\mu_{\bar{A}_2}(x_2), T\left(...T\left(\mu_{\bar{A}_{n-1}}(x_{n-1}), \mu_{\bar{A}_n}(x_n)\right)...\right)\right)\right),$$

for all $(x_1, x_2, ..., x_n) \in X_1 \times X_2 \times ... \times X_n$.

In order to use fuzzy sets and relations in any intelligent system we must be able to perform arithmetic operations. In fuzzy theory the extension of arithmetic operations to fuzzy sets was formulated by L.A. Zadeh in 1965. Using any t-norm the extension is possible to generalize.

Definition 12 (Generalized Zadeh's extension principle) *Let* T *be a t-norm and let* $X_1, X_2, ..., X_n$ *(*$n \geq 2$*) and* Y *be a family of sets. Assume that* $f : X_1 \times X_2 \times ... \times X_n \to Y$ *is a mapping. On the basis of the* generalized extension principle *(*sup-T extension principle*) to* f *a mapping* $F : \mathcal{F}(X_1) \times \mathcal{F}(X_2) \times ... \times \mathcal{F}(X_n) \to \mathcal{F}(Y)$ *is ordered such that for all* $\left(\bar{A}_1, \bar{A}_2, ..., \bar{A}_n\right) \in \mathcal{F}(X_1) \times \mathcal{F}(X_2) \times ... \times \mathcal{F}(X_n)$ *the membership function of* $F\left(\bar{A}_1, \bar{A}_2, ..., \bar{A}_n\right)$ *is*

$$\mu_{F(\bar{A}_1,\bar{A}_2,...,\bar{A}_n)}(y) = \begin{cases} \sup\limits_{(x_1,...,x_n)\in f^{-1}(y)} \left\{ T\left(\mu_{\bar{A}_1}(x_1), T\left(...T\left(\mu_{\bar{A}_{n-1}}(x_{n-1}), \mu_{\bar{A}_n}(x_n)\right)...\right)\right)\right\} & \\ & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \\ & \text{if } f^{-1}(y) = \emptyset. \end{cases}$$

If $n = 1$, *then*

$$\mu_{F(\bar{A}_1)}(y) = \begin{cases} \sup_{x_1 \in f^{-1}(y)} \left\{ \mu_{\bar{A}_1}(x_1) \right\} & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{if } f^{-1}(y) = \emptyset. \end{cases}$$

If we use the generalized Zadeh's extension principle, the operations on $\mathcal{F}(X)$ are uniquely determined by T, N and the corresponding operations of X.

Definition 13 *The triplet* $(\mathcal{F}(X), T, N)$ *will be called* fuzzy triangular space.

If T is a t-norm and "$*$" is a binary operation on $\mathbb{R}$, then "$*$" can be extended to fuzzy quantities in the sense of the generalized extension principle of Zadeh.

Definition 14 *Let* $\bar{A}$ *and* $\bar{B}$ *be two fuzzy numbers. Then the membership function of fuzzy set* $\bar{A} * \bar{B} \in \mathcal{F}(\mathbb{R})$ *is*

$$\mu_{\bar{A}*\bar{B}}(y) = \sup\{T(\mu_{\bar{A}}(x_1), \mu_{\bar{B}}(x_2)) \ / \ x_1 * x_2 = y\}, \tag{1}$$

for all $y \in \mathbb{R}$.

If we replace "$*$" with operations "$+$", "$-$","$\cdot$", or "$/$", then we get the membership functions of sum, difference, product or fraction.

3 Additive group of quasi-triangular fuzzy numbers

The construction of membership function of fuzzy numbers is an important problem in vagueness modeling. Theoretically, the shape of fuzzy numbers must depend on the applied triangular space.

We noticed that, if the model constructed on the computer does not comply the requests of the given problem, then we choose another norm. The membership function must defined in a such a way that the change of the t-norm modifies the shape of fuzzy number, but the calculus with them remain valid. This desideratum is satisfied, for instance if quasi-triangular fuzzy numbers introduced by M. Kovacs [21] are used.

Let $p \in [1, +\infty]$ and $g : [0,1] \to [0,\infty]$ be a continuous, strictly decreasing function with the boundary properties $g(1) = 0$ and $\lim_{t\to 0} g(t) = g_0 \leq \infty$. The quasi-triangular fuzzy number we define in the fuzzy triangular space $(\mathcal{F}(\mathbb{R}), T_{gp}, N)$, where

$$T_{gp}(x,y) = g^{[-1]}\left((g^p(x) + g^p(y))^{\frac{1}{p}} \right) \tag{2}$$

is an Archimedean t-norm generated by g and

$$N(x) = \begin{cases} 1 - x & \text{if } \; g_0 = +\infty, \\ g^{-1}(g_0 - g(x)) & \text{if } \; g_0 \in \mathbb{R}. \end{cases} \tag{3}$$

is a negation operation.

Definition 15 *The* set of quasi-triangular fuzzy numbers *is*

$$\begin{aligned} \mathcal{N}_g = \Big\{ \bar{A} \in \mathcal{F}(\mathbb{R}) \; &/ \textit{ there is } a \in \mathbb{R}, d > 0 \textit{ such that} \\ \mu_{\bar{A}}(x) &= g^{[-1]}\left(\frac{|x-a|}{d} \right) \textit{ for all } x \in \mathbb{R} \Big\} \bigcup \\ \Big\{ \bar{A} \in \mathcal{F}(\mathbb{R}) \; &/ \textit{ there is } a \in \mathbb{R} \textit{ such that} \\ \mu_{\bar{A}}(x) &= \chi_{\{a\}}(x) \textit{ for all } x \in \mathbb{R} \Big\}, \end{aligned} \tag{4}$$

where χ_A *is characteristic function of the set* A. *The elements of* $\mathcal{N}_g$ *will be called* quasi-triangular fuzzy numbers *generated by* g *with center* λ *and spread* d *and we will denote them with* $< \lambda, d >$.

Remark 4 *The quasi-triangular fuzzy numbers* $< a_1,\ d_1 >$ *and* $< a_2,\ d_2 >$ *are equal if and only if* $a_1 = a_2$ *and* $d_1 = d_2$.

Remark 5 *If* $<\lambda, d>\in \mathcal{N}_g$ *and* $d>0$, *then*

$$[<\lambda, d>]^{\alpha} = [\lambda - dg(\alpha), \lambda + dg(\alpha)]$$

and if $d=0$, *then* $[<\lambda, d>]^{\alpha} = \{\lambda\}$, *for all* $\alpha \in [0,1]$.

Example 1 *Let* $g:(0,1] \to [0,\infty)$ *be a function given by* $g(t) = \sqrt{-2\ln t}$. *Then the membership function of quasi-triangular fuzzy numbers* $<a, d>$ *is*

$$\mu(t) = e^{-\frac{(t-a)^2}{2d^2}} \quad \text{if } d>0, \quad \text{and}$$

$$\mu(t) = \begin{cases} 1 & \text{if } t = a, \\ 0 & \text{if } t \neq a \end{cases} \quad \text{if } d = 0.$$

Suppose $\bar{A}$ and $\bar{B}$ are fuzzy sets on $\mathbb{R}$. Then using the generalized Zadeh's extension principle we get:

Definition 16 *If* $p \in [1, +\infty)$, *then the* T_{gp}-sum *of* $\bar{A}$ *and* $\bar{B}$ *is defined by*

$$\mu_{\bar{A}+\bar{B}}(z) = \sup_{x+y=z} \left[g^{[-1]} \left(\left[g^p\left(\mu_{\bar{A}}(x)\right) + g^p\left(\mu_{\bar{B}}(y)\right) \right]^{\frac{1}{p}} \right) \right],$$

for all $z \in \mathbb{R}$.

If $p = +\infty$, *then the* T_{gp}-sum *of* $\bar{A}$ *and* $\bar{B}$ *is defined by*

$$\mu_{\bar{A}+\bar{B}}(z) = \sup_{x+y=z} \min\{\mu_{\bar{A}}(x), \mu_{\bar{B}}(y)\},$$

for all $z \in \mathbb{R}$.

M. Kovács and T. Keresztfalvi in [19] proved the formula (5) for the T_{gp}-sum of quasi-triangular fuzzy numbers.

Theorem 3 *Let* $p \in [1, +\infty]$. *If* $\bar{A} = \langle a, d \rangle$ *and* $\bar{B} = \langle b, e \rangle$ *are quasi-triangular fuzzy numbers, then* $\bar{A} + \bar{B}$ *is quasi-triangular fuzzy number too, and*

$$\bar{A} + \bar{B} = \left\langle a + b, \left(d^q + e^q\right)^{\frac{1}{q}} \right\rangle, \tag{5}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 4 ([23]) $(\mathcal{N}_g, +)$ *is a commutative monoid with element zero* $\bar{0} = <0, 0>$ *and if* $p \in (1, +\infty]$, *then it possesses the simplification property.*

As follows from the theorem 4, the quasi-triangular fuzzy numbers do not form an additive group. This fact can complicate some theoretical considerations or applied procedures. This deficiency can be removed if the set of quasi-triangular fuzzy numbers is included isomorphically in an extended set and this extended set forms an additive group with T_{gp}-sum. In this section we construct this group if $p > 1$.

As follows from the definition of T_{gp}-Cartesian product, the membership function of the pair $(< a_1, d_1 >, < a_2, d_2 >)$ is

$$\mu_{(\langle a_1,d_1\rangle,\langle a_2,d_2\rangle)}(x,y) = T_{gp}\left(\mu_{\langle a_1,d_1\rangle}(x), \mu_{\langle a_2,d_2\rangle}(y)\right), \tag{6}$$

for all $(x,y) \in \mathbb{R}\times\mathbb{R}$. We denote the set of pairs by $\mathfrak{I}_{gp}$.

Definition 17 *Let* $(< a_1, d_1 >, < a_2, d_2 >)$, $(< a_3, d_3 >, < a_4, d_4 >) \in \mathfrak{I}_{gp}$. *Then we say that*
$(< a_1, d_1 >, < a_2, d_2 >)$ equivalent *to* $(< a_3, d_3 >, < a_4, d_4 >)$, *and write* $(< a_1, d_1 >, < a_2, d_2 >) \sim (< a_3, d_3 >, < a_4, d_4 >)$ *if*

$$a_1 + a_4 = a_2 + a_3,$$
$$\left(d_1^q + d_4^q\right)^{1/q} = \left(d_2^q + d_3^q\right)^{1/q}.$$

It can be easily seen that "~" is an equivalence relation. This relation generates in $\mathfrak{I}_{gp}$ a division on equivalence class.

Definition 18 *The* factor *set is*

$$\mathfrak{I}_{gp}/_\sim = \left\{\overline{(< a_1, d_1 >, < a_2, d_2 >)} \ / \ < a_1, d_1 >, < a_2, d_2 > \in \mathcal{N}_g\right\},$$

where

$$\overline{(< a_1, d_1 >, < a_2, d_2 >)} = \{(< a_3, d_3 >, < a_4, d_4 >) \ / \ < a_3, d_3 >, < a_4, d_4 > \in \mathcal{N}_g \ \text{and} \ a_1 + a_4 = a_2 + a_3 \ , \ \left(d_1^q + d_4^q\right)^{1/q} = \left(d_2^q + d_3^q\right)^{1/q}\right\}.$$

Definition 19 *The* addition *operation in* $\mathfrak{I}_{gp}/_\sim$ *is defined by*

$$\overline{(< a_1, d_1 >, < a_2, d_2 >)} \oplus \overline{(< a_3, d_3 >, < a_4, d_4 >)} = \overline{\left(\left\langle a_1 + a_3, \left(d_1^q + d_3^q\right)^{\frac{1}{q}}\right\rangle, \left\langle a_2 + a_4, \left(d_2^q + d_4^q\right)^{\frac{1}{q}}\right\rangle\right)},$$

for all $\overline{(< a_1, d_1 >, < a_2, d_2 >)}, \overline{(< a_3, d_3 >, < a_4, d_4 >)} \in \mathfrak{I}_{gp}/_\sim$.

Because the commutative monoid $(\mathcal{N}_g, +)$ possesses simplification property if $p > 1$, it follows that:

Theorem 5 *If* $p > 1$, *then* $(\mathfrak{I}_{gp}/_\sim, \oplus)$ *is an additive commutative group.*

The *opposite* of $\overline{(<a_1, d_1>, <a_2, d_2>)}$ we denote by $\ominus\overline{(<a_1, d_1>, <a_2, d_2>)}$.

Proposition 2 *If* $<x, y>, <a, d> \in \mathcal{N}_g$, *then*

$$\overline{(<a, d> + <x, y>, <a, d>)} = \overline{(<x, y>, <0, 0>)}.$$

Proposition 3 ([24]) *If* $p > 1$, *then the function* $F : \mathcal{N}_g \to \mathfrak{I}_{gp}/_\sim$ *with* $F(<x, y>) = \overline{(<x, y>, <0, 0>)}$ *is a homomorphism.*

Theorem 6 ([24]) $(\mathcal{N}_g, +)$ *is isomorphic to* $(F(\mathcal{N}_g), \oplus)$.

The consequence of Theorem 6 $\overline{(<x, y>, <0, 0>)}$ is identical with $<x, y>$, if we consider this isomorphism. Using this property we introduce the following notations:

By the Theorem 6 it follows that $\overline{(<x, y>, <0, 0>)}$ is identical with $<x, y>$, if we consider the isomorphism in Theorem 6. Using this property we introduce the following notations:

We denote by $[x, y] = \overline{(<x, y>, <0, 0>)}$ the *quasi-triangular fuzzy number with center* x and spread y, and its *opposite* by $\ominus [x, y] = \overline{(<0, 0>, <x, y>)}$.

Definition 20 *If* $p > 1$, *then the extended set of quasi-triangular fuzzy number is* $\mho_{gp} = \mho_{gp}^{\oplus} \cup \mho_{gp}^{\ominus}$, *where*

$$\mho_{gp}^{\oplus} = \{[x, y] \ / \ <x, y> \in \mathcal{N}_g\} \text{ and } \mho_{gp}^{\ominus} = \{\ominus [x, y] \ / \ <x, y> \in \mathcal{N}_g\}.$$

Theorem 7 ([24]) *If* $p > 1$, *then* $\mathfrak{I}_{gp}/_\sim = \mho_{gp}$.

If we introduce the notation $[x_1, y_1] \ominus [x_2, y_2] = [x_1, y_1] \oplus (\ominus [x_2, y_2])$, then from Theorem 7 it follows:

Theorem 8 *If* $p > 1$, *then* $(\mho_{gp}, \oplus)$ *is an additive commutative group.*

Corollary 1 **(i)** *If* $[x, y] \in \mho_{gp}$, *then* $[0, 0] \ominus [x, y] = \ominus [x, y]$.

(ii) *If* $[x, y] \in \mho_{gp}$, *then* $\ominus (\ominus [x, y]) = [x, y]$.

(iii) *If* $[x_1, y_1], [x_2, y_2] \in \mho_{gp}$, *then*

$$(\ominus [x_1, y_1]) \oplus (\ominus [x_2, y_2]) = \ominus ([x_1, y_1] \oplus [x_2, y_2]) .$$

(iv) *If* $[x_1, y_1], [x_2, y_2] \in \mho_{gp}$, *then*

$$[x_1, y_1] \ominus [x_2, y_2] = \begin{cases} \left[x_1 - x_2, \left(y_1^q - y_2^q\right)^{\frac{1}{q}}\right] & \textit{if} \quad y_1 \geq y_2, \\ \ominus \left[x_2 - x_1, \left(y_2^q - y_1^q\right)^{\frac{1}{q}}\right] & \textit{if} \quad y_2 > y_1. \end{cases}$$

(v) *If* $[x_1, y_1], [x_2, y_2], [x_3, y_3], [x_4, y_4] \in \mho_{gp}$ *and*

$$[x_1, y_1] \ominus [x_2, y_2] = [x_3, y_3] \ominus [x_4, y_4] ,$$

then

$$[x_1, y_1] \oplus [x_4, y4] = [x_3, y_3] \oplus [x_2, y_2] .$$

4 Real vector space of quasi-triangular fuzzy numbers

In this section we construct the vector space of quasi-triangular fuzzy numbers if $p > 1$. We know that $2 \cdot [x, y] = [x, y] \oplus [x, y] = \left[2x, 2^{\frac{1}{q}} y\right]$ for all $[x, y] \in \mho_{gp}$. We generalize this property as follows.

Definition 21 *For all* $[x, y] \in \mho_{gp}^{\oplus}$ *and for all* $a \in \mathbb{R}$ *the* scalar multiplication $a [x, y]$ *is defined by*

$$a [x, y] = \begin{cases} \left[ax, a^{\frac{1}{q}} y\right] & \textit{if} \quad a \geq 0, \\ \ominus \left[-ax, (-a)^{\frac{1}{q}} y\right] & \textit{if} \quad a < 0, \end{cases}$$

and for all $\ominus [x, y] \in \mho_{gp}^{\ominus}$ *the* scalar multiplication $a (\ominus [x, y])$ *is defined by*

$$a (\ominus [x, y]) = \ominus (a [x, y]) .$$

Remark 6 *For all* $\ominus [x, y] \in \mho_{gp}^{\ominus}$ *and for all* $a \in \mathbb{R}$ *we have*

$$a (\ominus [x, y]) = \begin{cases} \ominus \left[ax, a^{\frac{1}{q}} y\right] & \textit{if} \quad a \geq 0, \\ \left[-ax, (-a)^{\frac{1}{q}} y\right] & \textit{if} \quad a < 0. \end{cases}$$

Similarly, for all $[x,y] \in \mho_{gp}^{\oplus}$ *and* $a \geq 0$ *we have*

$$(-a)[x,y] = \ominus\left[ax, a^{\frac{1}{q}}y\right] = a(\ominus[x,y]) \text{ and}$$
$$(-a)(\ominus[x,y]) = a[x,y].$$

Theorem 9 *If* $p > 1$, *then the triple* $(\mho_{gp}, \oplus, \cdot)$ *is a real vector space.*

Proof. Since $(\mho_{gp}, \oplus)$ is an additive commutative group the following properties must be proved.

(i) If $a, b \in \mathbb{R}$ and $Z \in \mho_{gp}$, then $(a+b)Z = aZ \oplus bZ$.
If $Z = [x,y] \in \mho_{gp}^{\oplus}$, $a \geq 0$ and $b \geq 0$, then

$$\begin{aligned}(a+b)[x,y] &= \left[(a+b)x, (a+b)^{\frac{1}{q}}y\right] \\ &= \left[ax, a^{\frac{1}{q}}y\right] \oplus \left[bx, b^{\frac{1}{q}}y\right] \\ &= a[x,y] \oplus b[x,y].\end{aligned}$$

If $Z = \ominus[x,y] \in \mho_{gp}^{\ominus}$, $a \geq 0$ and $b \geq 0$, then

$$\begin{aligned}(a+b)(\ominus[x,y]) &= \ominus((a+b)[x,y]) \\ &= \ominus(a[x,y] \oplus b[x,y]) \\ &= (\ominus a[x,y]) \oplus (\ominus b[x,y]) \\ &= a(\ominus[x,y]) \oplus b(\ominus[x,y]).\end{aligned}$$

If $Z = [x,y] \in \mho_{gp}^{\oplus}$, $a \geq 0$ $b < 0$ and $a + b \geq 0$, then

$$\begin{aligned}a[x,y] \oplus b[x,y] &= \left[ax, a^{\frac{1}{q}}y\right] \ominus \left[-bx, (-b)^{\frac{1}{q}}y\right] \\ &= \left[ax - (-b)x, (a-(-b))^{\frac{1}{q}}y\right] \\ &= (a+b)[x,y].\end{aligned}$$

If $Z = \ominus[x,y] \in \mho_{gp}^{\ominus}$, $a \geq 0$ $b < 0$ and $a + b \geq 0$, then

$$\begin{aligned}a(\ominus[x,y]) \oplus b(\ominus[x,y]) &= (\ominus a[x,y]) \oplus ((-b)[x,y]) \\ &= \ominus a[x,y] \ominus b[x,y] \\ &= \ominus(a[x,y] \oplus b[x,y]) \\ &= (a+b)(\ominus[x,y]).\end{aligned}$$

If $Z = [x, y] \in \mho_{gp}^{\oplus}$, $a \geq 0$ $b < 0$ and $a + b < 0$, then

$$\begin{aligned} a\,[x,y] \oplus b\,[x,y] &= \left[ax, a^{\frac{1}{q}} y\right] \ominus \left[-bx, (-b)^{\frac{1}{q}}\, y\right] \\ &= \ominus \left[-ax + (-b)\,x, (-b-a)^{\frac{1}{q}}\, y\right] \\ &= (a+b)\,[x,y]\,. \end{aligned}$$

If $Z = \ominus [x, y] \in \mho_{gp}^{\ominus}$, $a \geq 0$ $b < 0$ and $a + b < 0$, then

$$\begin{aligned} a\,(\ominus [x,y]) \oplus b\,(\ominus [x,y]) &= (\ominus a\,[x,y]) \oplus ((-b)\,[x,y]) \\ &= \ominus a\,[x,y] \ominus b\,[x,y] \\ &= \ominus\,(a\,[x,y] \oplus b\,[x,y]\ ) \\ &= (a+b)\,(\ominus [x,y])\,. \end{aligned}$$

If $Z = [x, y] \in \mho_{gp}^{\oplus}$, $a < 0$ and $b < 0$, then

$$\begin{aligned} a\,[x,y] \oplus b\,[x,y] &= \ominus \left[-ax, (-a)^{\frac{1}{q}}\, y\right] \ominus \left[-bx, (-b)^{\frac{1}{q}}\, y\right] \\ &= \ominus \left[-ax - bx, (-a-b)^{\frac{1}{q}}\, y\right] \\ &= (a+b)\,[x,y]\,. \end{aligned}$$

If $Z = \ominus [x, y] \in \mho_{gp}^{\ominus}$, $a < 0$ and $b < 0$, then

$$\begin{aligned} a\,(\ominus [x,y]) \oplus b\,(\ominus [x,y]) &= ((-a)\,[x,y]) \oplus ((-b)\,[x,y]) \\ &= (-a-b)\,[x,y] \\ &= (a+b)\,(\ominus [x,y])\,. \end{aligned}$$

(ii) If $Z_1, Z_2 \in \mho_{gp}$ and $a \in \mathbb{R}$, then $a\,(Z_1 \oplus Z_2) = aZ_1 \oplus aZ_2$.
If $Z_1 = [x_1, y_1] \in \mho_{gp}^{\oplus}$, $Z_2 = [x_2, y_2] \in \mho_{gp}^{\oplus}$ and $a \geq 0$, then

$$\begin{aligned} a\,(Z_1 \oplus Z_2) &= a\left[x_1 + x_2, \left(y_1^q + y_2^q\right)^{\frac{1}{q}}\right] \\ &= \left[ax_1 + ax_2, \left(ay_1^q + ay_2^q\right)^{\frac{1}{q}}\right] \\ &= a\,[x_1, y_1] \oplus a\,[x_2, y_2] \\ &= aZ_1 \oplus aZ_2. \end{aligned}$$

If $Z_1 = [x_1, y_1] \in \mho_{gp}^{\oplus}$, $Z_2 = \ominus [x_2, y_2] \in \mho_{gp}^{\ominus}$, $y_1 \geq y_2$ and $a \geq 0$, then

$$\begin{aligned} a\,(Z_1 \oplus Z_2) &= a\left[x_1 - x_2, \left(y_1^q - y_2^q\right)^{\frac{1}{q}}\right] \\ &= \left[ax_1 - ax_2, \left(ay_1^q - ay_2^q\right)^{\frac{1}{q}}\right] \\ &= a\,[x_1, y_1] \ominus a\,[x_2, y_2] \\ &= aZ_1 \oplus aZ_2. \end{aligned}$$

If $Z_1 = [x_1, y_1] \in \mho_{gp}^{\oplus}$, $Z_2 = \ominus [x_2, y_2] \in \mho_{gp}^{\ominus}$, $y_1 < y_2$ and $a \geq 0$, then

$$\begin{aligned} a\,(Z_1 \oplus Z_2) &= (-a)\left[x_2 - x_1, \left(y_2^q - y_1^q\right)^{\frac{1}{q}}\right] \\ &= \ominus\left[ax_2 - ax_1, \left(ay_2^q - ay_1^q\right)^{\frac{1}{q}}\right] \\ &= a\,[x_1, y_1] \ominus a\,[x_2, y_2] \\ &= aZ_1 \oplus aZ_2. \end{aligned}$$

If $Z_1 = \ominus [x_1, y_1]$, $Z_2 = \ominus [x_2, y_2] \in \mho_{gp}^{\ominus}$ and $a \geq 0$, then

$$\begin{aligned} a\,(Z_1 \oplus Z_2) &= (-a)\left[x_1 + x_2, \left(y_1^q + y_2^q\right)^{\frac{1}{q}}\right] \\ &= \ominus\left[ax_1 + ax_2, \left(ay_1^q + ay_2^q\right)^{\frac{1}{q}}\right] \\ &= a\,(\ominus [x_1, y_1]) \oplus a\,(\ominus [x_2, y_2]) \\ &= aZ_1 \oplus aZ_2. \end{aligned}$$

If $a < 0$, then

$$\begin{aligned} aZ_1 \oplus aZ_2 &= \ominus\left(\left(-a\right) Z_1\right) \ominus \left(\left(-a\right) Z_2\right) \\ &= \ominus\left(\left(-a\right) Z_1 \oplus \left(-a\right) Z_2\right) \\ &= \ominus\ \left(-a\right)\left(Z_1 \oplus Z_2\right) \\ &= a\left(Z_1 \oplus Z_2\right)\ . \end{aligned}$$

(iii) If $a, b \in \mathbb{R}$ and $Z \in \mho_{gp}$, then $(ab)\,Z = a\,(bZ)$.
If $Z = [x, y] \in \mho_{gp}^{\oplus}$, $a \geq 0$ and $b \geq 0$, then

$$\begin{aligned} (ab)\,[x, y] &= \left[(ab)\,x, (ab)^{\frac{1}{q}}\,y\right] \\ &= a\left[bx, b^{\frac{1}{q}}y\right] \\ &= a\,(b\,[x, y])\,. \end{aligned}$$

If $Z = [x, y] \in \mho_{gp}^{\oplus}$, $a \geq 0$ and $b < 0$, then

$$\begin{aligned}(ab)[x, y] &= \ominus \left[(-ab)x, (-ab)^{\frac{1}{q}} y\right] \\ &= a\left(\ominus \left[-bx, (-b)^{\frac{1}{q}} y\right]\right) \\ &= a(b[x, y]).\end{aligned}$$

If $Z = [x, y] \in \mho_{gp}^{\oplus}$, $a < 0$ and $b < 0$, then

$$\begin{aligned}(ab)[x, y] &= \left[(-a)(-b)x, ((-a)(-b))^{\frac{1}{q}} y\right] \\ &= (-a)((-b)[x, y]) \\ &= a(\ominus((-b)[x, y])) \\ &= a(b[x, y]).\end{aligned}$$

If $Z = \ominus [x, y] \in \mho_{gp}^{\ominus}$, then

$$\begin{aligned}(ab)(\ominus [x, y]) &= \ominus ((ab)[x, y]) \\ &= \ominus (a(b[x, y])) \\ &= a(\ominus (b[x, y])) \\ &= a(b(\ominus [x, y])).\end{aligned}$$

(iv) If $Z \in \mho_{gp}$, then $1 \cdot Z = Z$.
If $Z = [x, y] \in \mho_{gp}^{\oplus}$, then $1[x, y] = [x, y]$.
If $Z = \ominus [x, y] \in \mho_{gp}^{\ominus}$, then $1(\ominus [x, y]) = \ominus [x, y]$.

■

5 Scalar product of quasi-triangular fuzzy numbers

In this section we construct the real vector space with scalar product of quasi-triangular fuzzy numbers.

Definition 22 *The* product *of the classes*

$$\overline{(< a_1, d_1 >, < a_2, d_2 >)}, \overline{(< a_3, d_3 >, < a_4, d_4 >)} \in \mathfrak{I}_{gp}/_\sim$$

is defined by

$$\begin{aligned}&\overline{(< a_1, d_1 >, < a_2, d_2 >)} \cdot \overline{(< a_3, d_3 >, < a_4, d_4 >)} \\ &= (a_1 - a_2)(a_3 - a_4) + \left(d_1^q - d_2^q\right)\left(d_3^q - d_4^q\right).\end{aligned} \tag{7}$$

Theorem 10 $(\mho_{gp}, \oplus, \cdot)$ *is a real vector space with scalar product given by (7).*

Proof. Let

$$(<a_5, d_5>, <a_6, d_6>) \in \overline{(<a_1, d_1>, <a_2, d_2>)} \quad \text{and}$$
$$(<a_7, d_7>, <a_8, d_8>) \in \overline{(<a_3, d_3>, <a_4, d_4>)}.$$

Since

$$a_5 - a_6 = a_1 - a_2, a_7 - a_8 = a_3 - a_4, d_5^q - d_6^q = d_1^q - d_2^q, d_7^q - d_8^q = d_3^q - d_4^q$$

follows that the (7) does not depend on choice of the elements.
Let

$$\overline{(<a_1, d_1>, <a_2, d_2>)}, \ \overline{(<a_3, d_3>, <a_4, d_4>)},$$
$$\overline{(<a_5, d_5>, <a_6, d_6>)} \in \mathfrak{I}_{gp}/_\sim.$$

(i) The scalar product is commutative since:

$$\overline{(<a_1, d_1>, <a_2, d_2>)} \cdot \overline{(<a_3, d_3>, <a_4, d_4>)} =$$
$$\overline{(<a_3, d_3>, <a_4, d_4>)} \cdot \overline{(<a_1, d_1>, <a_2, d_2>)}.$$

(ii) For all $\lambda \geq 0$ we have

$$\begin{aligned}
&\left(\lambda\overline{(<a_1, d_1>, <a_2, d_2>)}\right) \cdot \overline{(<a_3, d_3>, <a_4, d_4>)} \\
&= (\lambda([a_1, d_1] \ominus [a_2, d_2])) \cdot \overline{(<a_3, d_3>, <a_4, d_4>)} \\
&= \overline{(<\lambda a_1, \lambda^{1/q} d_1>, <\lambda a_2, \lambda^{1/q} d_2>)} \cdot \overline{(<a_3, d_3>, <a_4, d_4>)} \\
&= (\lambda a_1 - \lambda a_2)(a_3 - a_4) + \left(\lambda d_1^q - \lambda d_2^q\right)\left(d_3^q - d_4^q\right) \\
&= \lambda\left[(a_1 - a_2)(a_3 - a_4) + \left(d_1^q - d_2^q\right)\left(d_3^q - d_4^q\right)\right] \\
&= \lambda\overline{(<a_1, d_1>, <a_2, d_2>)} \cdot \overline{(<a_3, d_3>, <a_4, d_4>)}.
\end{aligned}$$

For all $\lambda < 0$ we have

$$\begin{aligned}
&\left(\lambda\overline{(<a_1,d_1>,<a_2,d_2>)}\right)\cdot\overline{(<a_3,d_3>,<a_4,d_4>)}\\
&=(\lambda([a_1,d_1]\ominus[a_2,d_2]))\cdot\overline{(<a_3,d_3>,<a_4,d_4>)}\\
&=\overline{\left(<-\lambda a_2,(-\lambda)^{1/q}d_2>,<-\lambda a_1,(-\lambda)^{1/q}d_1>\right)}\cdot\\
&\quad\overline{(<a_3,d_3>,<a_4,d_4>)}\\
&=(-\lambda a_2+\lambda a_1)(a_3-a_4)+\left(-\lambda d_2^q+\lambda d_1^q\right)\left(d_3^q-d_4^q\right)\\
&=\lambda\left[(a_1-a_2)(a_3-a_4)+\left(d_1^q-d_2^q\right)\left(d_3^q-d_4^q\right)\right]\\
&=\lambda\overline{(<a_1,d_1>,<a_2,d_2>)}\cdot\overline{(<a_3,d_3>,<a_4,d_4>)}.
\end{aligned}$$

(iii) The distributivity follows by

$$\begin{aligned}
&\left(\overline{(<a_1,d_1>,<a_2,d_2>)}\oplus\overline{(<a_3,d_3>,<a_4,d_4>)}\right)\cdot\\
&\quad\overline{(<a_5,d_5>,<a_6,d_6>)}\\
&=\overline{\left(<a_1+a_3,\left(d_1^q+d_3^q\right)^{1/q}>,<a_2+a_4,\left(d_2^q+d_4^q\right)^{1/q}>\right)}\cdot\\
&\quad\overline{(<a_5,d_5>,<a_6,d_6>)}\\
&=(a_1+a_3-a_2-a_4)(a_5-a_6)+\left(d_1^q+d_3^q-d_2^q-d_4^q\right)\left(d_5^q-d_6^q\right)\\
&=(a_1-a_2)(a_5-a_6)+\left(d_1^q-d_2^q\right)\left(d_5^q-d_6^q\right)+\\
&\quad(a_3-a_4)(a_5-a_6)+\left(d_3^q-d_4^q\right)\left(d_5^q-d_6^q\right)\\
&=\left(\overline{(<a_1,d_1>,<a_2,d_2>)}\cdot\overline{(<a_5,d_5>,<a_6,d_6>)}\right)\\
&\quad\oplus\left(\overline{(<a_3,d_3>,<a_4,d_4>)}\cdot\overline{(<a_5,d_5>,<a_6,d_6>)}\right).
\end{aligned}$$

(iv) The positivity also satisfied:

$$\begin{aligned}
&\overline{(<a_1,d_1>,<a_2,d_2>)}\cdot\overline{(<a_1,d_1>,<a_2,d_2>)}\\
&=(a_1-a_2)^2+\left(d_1^q-d_2^q\right)^2\geq 0.
\end{aligned}$$

If

$$\overline{(<a_1,d_1>,<a_2,d_2>)}\cdot\overline{(<a_1,d_1>,<a_2,d_2>)}=0,$$

then $a_1=a_2$ and $d_1=d_2$. In conclusion $\overline{(\langle a_1,d_1\rangle,\langle a_2,d_2\rangle)}$ is the zero element. ■

Proposition 4 *For all* $[a_1, d_1], [a_2, d_2] \in \mho_{gp}$ *we have*

$$\begin{aligned} \ominus [a_1, d_1] \cdot (\ominus [a_2, d_2]) &= [a_1, d_1] \cdot [a_2, d_2], \\ \ominus [a_1, d_1] \cdot [a_2, d_2] &= -[a_1, d_1] \cdot [a_2, d_2]. \end{aligned}$$

Proof. Since

$$\begin{aligned} [a_1, d_1] &= \overline{(< a_1, d_1 >, < 0, 0 >)}, \\ \ominus [a_1, d_1] &= \overline{(< 0, 0 >, < a_1, d_1 >)}, \\ [a_2, d_2] &= \overline{(< a_2, d_2 >, < 0, 0 >)}, \\ \ominus [a_2, d_2] &= \overline{(< 0, 0 >, < a_2, d_2 >)} \end{aligned}$$

it follows that

$$\begin{aligned} [a_1, d_1] \cdot [a_2, d_2] &= a_1 a_2 + d_1^q d_2^q, \\ \ominus [a_1, d_1] \cdot [a_2, d_2] &= -a_1 a_2 - d_1^q d_2^q \\ &= -[a_1, d_1] \cdot [a_2, d_2], \\ \ominus [a_1, d_1] \cdot (\ominus [a_2, d_2]) &= a_1 a_2 + d_1^q d_2^q \\ &= [a_1, d_1] \cdot [a_2, d_2]. \end{aligned}$$

■

Definition 23 *In the real vector space* $\mho_{gp}$ *the* norm *of* $[a, d] \in \mho_{gp}^{\oplus}$ *and* $\ominus [a, d] \in \mho_{gp}^{\ominus}$ *is defined by*

$$\begin{aligned} \|[a, d]\| &= \sqrt{a^2 + d^{2q}}, \\ \|\ominus [a, d]\| &= \sqrt{a^2 + d^{2q}}. \end{aligned}$$

Definition 24 *In the real vector space* $\mho_{gp}$ *the* distance *of* $C_1, C_2 \in \mho_{gp}$ *is defined by*

$$d(C_1, C_2) = \|C_1 \ominus C_2\|.$$

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