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Haar measure is not approximable by balls

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Abstract. We construct a compact topological Abelian group with an invariant metric and a compact subset of Haar measure zero that cannot be covered by balls of total measure less than 1. This answers a question of Christensen.

1 Introduction

J.P.R. Christensen asked in 1979 in a conference in Oberwolfach (see [2]) the following question: *let X be a compact Abelian group with invariant metric and let* μ *be its Haar measure. Is it true that for any subset* $A \subset X$*,* $\mu(A)$ *is the infimum of* $\sum_i \mu(B_i)$*, where* B_i*'s are balls covering* A*?*

It follows from Christensen's paper, and it was also proved independently by P. Mattila (in the same proceedings [2], page 270, Remark 3.7) that this is true if we only require the balls to cover *almost all* of A. Mattila showed that this latter statement holds for any uniformly distributed measure on a metric space (i.e. for any Borel regular measure for which balls of the same size have the same, positive and finite measure). An interesting example of R.O. Davies shows (see [1]) that even if we require the balls to cover only almost all of A, it is not always possible to choose the balls to be disjoint.

The question of Christensen is equivalent to the following problem: *is it true that if* $\mu(A) = 0$*, then* A *can be covered by balls of arbitrary small total measure?* It turns out that the answer to this question is negative. In this

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note we construct a compact Abelian group X (we simply take a product of finite sets, i.e. X is homeomorphic to a Cantor set) and we choose an invariant metric on X, so that there is a compact subset $A \subset X$ of Haar measure zero for which $\sum_i \mu(B_i) \geq 1$ for any covering by balls $A \subset \bigcup_i B_i$. The main idea of our proof is to find a metric on a finite set so that all non-degenarate balls (i.e. balls of more than one element) overlap so badly that one needs many balls, of very large total measure, to cover half of the points. One can of course always use in a finite metric space the degenarate balls only, and so cover any set by balls whose total measure does not exceed the measure of the set. However, by taking the product of these finite sets, we obtain a counterexample to Christensen's problem.

2 Construction

Let $n_0 = N_0 = 1$, and choose inductively for each $k \in \mathbb{N}$ positive integers n_k, N_k for which

$$\frac{n_k}{2}\left(1 - \left(\frac{n_k - 1}{n_k}\right)^{N_k}\right) \geq \prod_{j<k} n_j^{N_j} \tag{1}$$

and n_k is even. Let Z_{n_k} denote the additive group mod n_k, and let $Y_k = Z_{n_k}^{N_k}$ be the product group on the product space $\{0, 1, \ldots, n_{k-1}\}^{N_k}$ equipped with the discrete topology. The Haar measure μ_k on Y_k is uniformly distributed, each element of Y_k has measure $1/n_k^{N_k}$.

We define an invariant metric on Y_k as follows. If $x = (x_1, x_2, \ldots, x_{N_k})$ and $y = (y_1, y_2, \ldots, y_{N_k})$ are two elements of Y_k, let

$$d_k(x, y) = \begin{cases} 0 & \text{if } x_j = y_j \text{ for all } j \\ 2 & \text{if } x_j \neq y_j \text{ for all } j \\ 1 & \text{otherwise.} \end{cases}$$

This is indeed a metric, since for any three distinct points x, y, z the distance between any two of them is 1 or 2, so the triangle-inequality is automatically satisfied. It is also immediate to see that d_k is invariant under the group actions.

Now let X be the direct product of the groups Y_k, $k \in \mathbb{N}$. This is a compact Abelian group on the product space $\prod_{k=1}^{\infty}\{0, 1, \ldots, n_{k-1}\}^{N_k}$ and its Haar measure is the product measure $\mu = \prod_{k=1}^{\infty} \mu_k$. For two distinct points

$x = (x_1, x_2, \dots) \in X$, $y = (y_1, y_2, \dots) \in X$ define

$$d(x,y) = \frac{d_k(x_k, y_k)}{2^k},$$

where k is the least index for which the coordinates $x_k, y_k \in Y_k$ are different. This defines a metric on X. Indeed, if $x = (x_1, x_2, \dots)$, $y = (y_1, y_2, \dots)$, $z = (z_1, z_2, \dots)$ are three distinct points, let k be the least index such that x_k, y_k, z_k are not all the same. If they are all different, then they satisfy the triangle-inequality since d_k satisfies it. If two of them are the same, say, $x_k = y_k \neq z_k$, then $d(x,z) = d(y,z) \in \{1/2^k, 1/2^{k-1}\}$ and $d(x,y) \in \{1/2^\ell, 1/2^{\ell-1}\}$ for some $\ell \geq k+1$, so again, x, y, z satisfy the triangle-inequality. The metric d is invariant under the group actions, since each d_k is invariant.

Since the distance between any two points of X is a power of 1/2, each (open or closed) ball in X coincides with a closed ball whose radius is $1/2^k$ for some $k \in \mathbb{N}$. For $x = (x_1, x_2, \dots) \in X$, the closed ball $B(x, 1)$ of centre x and radius 1 is the whole space X, and for $k \geq 1$, the closed ball $B(x, 1/2^k)$ of centre x and radius $1/2^k$ is

$$\{(y_1, y_2, \dots) \in X : \ x_j = y_j \text{ for } j < k, \text{ and } d_k(x_k, y_k) = 0 \text{ or } 1\}. \tag{2}$$

Therefore each ball is a (finite) union of cylinder sets, and each cylinder set of X is a finite union of balls. This shows that indeed the topology induced by the metric d is the product topology of X.

Let

$$A_k = \{(x_1, x_2, \dots, x_{N_k}) \in Y_k : \ 0 \leq x_1 < n_k/2\} \subset Y_k,$$

and let $A = \prod_{k=1}^{\infty} A_k \subset X$. Then $\mu_k(A_k) = 1/2$ for each k, hence $\mu(A) = \prod_{k=1}^{\infty} 1/2 = 0$. We show that $\sum_i \mu(B_i) \geq 1$ for any balls B_i whose union covers A.

Suppose that $A \subset \bigcup B_i$ for some balls B_i. The set A is compact since it is a product of the compact sets A_k, therefore we can assume that $A \subset \bigcup B_i$ is a finite cover. Without loss of generality we can also assume that each B_i meets A, and that none of the balls B_i is contained in any of the others. Finally, we can assume that all the balls are closed and their radius is a power of 1/2.

Take the smallest ball $B_i = B(x, 1/2^k)$. If it has radius 1, then $\mu(B_i) = \mu(X) = 1$. If its radius is less than 1, then it is of the form given by (2). Since B_i meets A, therefore $x_j \in A_j$ for all $j < k$. Consider the cylinder set

$$C = \{(y_1, y_2, \dots) \in X : \ x_j = y_j \text{ for } j < k\}.$$

If a ball $B(y, 1/2^\ell)$ meets C and it has radius larger than $1/2^k$, then it covers our ball B_i, so all the balls of our cover that meet C must have the same radius $1/2^k$. Consider the projections of these balls to the kth coordinate. These are closed balls $B'_i \subset Y_k$ of d_k-radius 1, and their union covers A_k. It follows from the definition of the set A_k and of the distance d_k that A_k cannot be covered by less than $n_k/2$ balls of d_k-radius 1. Indeed, for any given set $E \subset Y_k$ of less than $n_k/2$ points, one can always find a point in Y_k whose first coordinate is less than $n_k/2$, and whose d_k-distance from E is 2.

One checks easily that for each B'_i, $(n_k - 1)^{N_k}$ points of Y_k are in the complement of B'_i, hence

$$\mu_k(B'_i) = 1 - \left(\frac{n_k - 1}{n_k}\right)^{N_k}$$

and so

$$\begin{aligned} \mu(B_i) &= \left(1 - \left(\frac{n_k - 1}{n_k}\right)^{N_k}\right) \Big/ \prod_{j<k} n_j^{N_j} \\ \sum_i \mu(B_i) &\geq \frac{n_k}{2} \cdot \left(1 - \left(\frac{n_k - 1}{n_k}\right)^{N_k}\right) \Big/ \prod_{j<k} n_j^{N_j} \geq 1 \end{aligned}$$

by (1). ∎

References

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[2] *Measure theory, Oberwolfach 1979.* Proceedings of a Conference held at Oberwolfach, July 1–7, 1979. Edited by Dietrich Kölzow. Lecture Notes in Mathematics, 794. Springer, Berlin, 1980.

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