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On a quadratic type functional equation on locally compact abelian groups

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Abstract. Let $(G,+)$ be a locally compact abelian Hausdorff group, $\mathcal{K}$ is a finite automorphism group of G, $\kappa = card\mathcal{K}$ and let μ be a regular compactly supported complex-valued Borel measure on G such that $\mu(G) = \frac{1}{\kappa}$. We find the continuous solutions $f, g : G \to \mathbb{C}$ of the functional equation

$$\sum_{k\in\mathcal{K}}\sum_{\lambda\in\mathcal{K}}\int_G f(x + k\cdot y + \lambda\cdot s)d\mu(s) = g(y) + \kappa f(x), \ x, y \in G,$$

in terms of k-additive mappings. This equations provides a common generalization of many functional equations (quadratic, Jensen's, Cauchy equations).

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1 Introduction

Throughout this paper, Let $(G, +)$ be a locally compact abelian Hausdroff group, $\mathcal{K}$ be a finite automorphism group of G, $\kappa := card\mathcal{K}$, μ be a regular compactly supported complex-valued Borel measure on G such that $\mu(G) = \int_G d\mu(t) = \frac{1}{\kappa}$, $M_C(G)$ be the space of all regular compactly supported complexe-valued Borel measures on G. By $C(G)$ we mean the algebra of all continuous functions from G into $\mathbb{C}$. All terminology in this paper concerning harmonic analysis according to the monograph in [4].
The following generalization of Cauchy and quadratic functional equation is

$$\sum_{n=0}^{N-1} f(x + w^n y) = Nf(x) + g(y) \ , x, y \in \mathbb{C}, \tag{1}$$

where $N \in \{2, 3, ...\}$ and w is a primitive N^{th} root of unity, $f, g : \mathbb{C} \to \mathbb{C}$ are continuous, was solved by Stetkær [7]. Łukasik [6] derived an explicit formula solutions of the following functional equation

$$\sum_{k \in \mathcal{K}} f(x + k \cdot y) = \kappa f(x) + h(y) \ , x, y \in G, \tag{2}$$

where $\kappa := card\mathcal{K}$. such that $g(y) - g(0) = h(y)$.
The purpose of this paper is to derive an explicit solutions of the following integral-functional equation

$$\sum_{k \in \mathcal{K}} \sum_{\lambda \in \mathcal{K}} \int_G f(x + k \cdot y + \lambda \cdot s) d\mu(s) = g(y) + \kappa f(x) \ , x, y \in G. \tag{3}$$

This equation is a generalization of (1) and (2). In fact, Eq. (2) results from (3) by taking $\mu = \frac{1}{\kappa}\delta_a$, where δ_a denotes the Dirac measure concentrated at a. Furthermore, using our main result Theorem 2, we find the solutions of the following functional equations:

A/

$$f(x + y + a) + f(x + \sigma(y) + a) = 2f(x) + g(y) \ , x, y \in G,$$

when we take $\mu = \frac{1}{2}\delta_a, \mathcal{K} = \{I, \sigma\}$ where σ is an automorphism of the abelian group G such that $\sigma^2 = id_G, f, g : G \to \mathbb{C}$ where investigated by Stetkær [7].

B/

$$\sum_{k=1}^{n} f(x + y + a_k) = nf(x) + ng(y) \ , x, y \in G,$$

when we take $\mu = \frac{\sum_{k=1}^{n} \delta_{a_k}}{n}, \mathcal{K} = \{I\}$ and $a_1, \ldots, a_n \in G$.

C/

$$f(x + y + a) = f(x) + g(y) \ , x, y \in G$$

when we take $\mu = \delta_a, \mathcal{K} = \{I\}$.

Furthermore, we find the continuous solutions of some functional equations by using our main result.

2 Notations and preliminary results

In this section, we need to introduce some notions and notations which we will need in the sequel.

A function $A : G \to \mathbb{C}$ is said to be additive provided if $A(x + y) = A(x) + A(y)$ for all $x, y \in G$. In this case, it is easily seen that $A(rx) = rA(x)$ for all $x \in G$ and all $r \in \mathbb{Z}$.

Let $k \in \mathbb{N}$ and $A_k : G^k \to \mathbb{C}$ be a function, then we say that A_k is k-additive if it is additive in each variable. In addition, we say that A is symmetric if

$$A(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(k)}) = A(x_1, x_2, \ldots, x_k)$$

whenever $x_1, x_2, \ldots, x_k \in G$ and σ is a permutation of $\{1, 2, \ldots, k\}$.

Let $A_k : G^k \to \mathbb{C}$ be symmetric and k-additive and let $A_k(x) = A(x, x, \ldots, x)$ for $x \in G$ and note that $A_k(rx) = r^k A_k(x)$ whenever $x \in G$ and $r \in \mathbb{Z}$.

In this way a function $A_k : G^k \longrightarrow \mathbb{C}$ which satisfies for all $r \in \mathbb{Z}$ and $x \in G$, $A_k(rx) = r^k A_k$ will be called a $\mathbb{Z}$-homogeneous form of degree k, (assuming $A_k \neq 0$).

A function $p : G \to \mathbb{C}$ is called a generalized polynomial of degree at most $N \in \mathbb{N}$ if there exist $a_0 \in G$ and $\mathbb{Z}$-homogeneous forms $A_k : G \to \mathbb{C}$ (for $1 \leq k \leq N$) of degree k, such that

$$p(x) = a_0 + \sum_{k=1}^{N} A_k(x), \quad x \in G.$$

The following theorem were proved by Łukasik in [6].

Theorem 1 *Let $(G, +)$ be an abelian topological group. The functions $f, h \in G \to \mathbb{C}$ satisfy the functional equation (2) if and only if there exist symmetric $k-$additive mappings $A_k : G^k \to \mathbb{C}$, $k \in \{1, ..., \kappa\}$ and $A_0, B_0 \in \mathbb{C}$ such that*

$$f(x) = A_0 + A_1(x) + \ldots + A_\kappa(x, \ldots, x), \ x \in G,$$

$$h(x) = B_0 + \sum_{\lambda \in \mathcal{K}} A_1(\lambda \cdot x) + \ldots + \sum_{\lambda \in \mathcal{K}} A_\kappa(\lambda \cdot x, \ldots, \lambda \cdot x), \;\; x \in G,$$

$$\binom{j}{i} \sum_{\lambda \in \mathcal{K}} A_j(x, \ldots x, \underbrace{\lambda \cdot y, \ldots, \lambda \cdot y}_{i}) = 0, \;\; x, y \in G, 1 \leq i \leq j-1, 2 \leq j \leq \kappa.$$

3 Main results

In this section, we describe the solutions of Eq.(3).

Theorem 2 *Let* $(G, +)$ *be a locally compact abelian Hausdorff group and let* $\mu \in M_C(G)$ *such that* $\mu(G) = \frac{1}{\kappa}$. *The functions* $f, g \in C(G)$ *satisfy the functional equation (3) if and only if there exist symmetric* $k-$*additive mappings* $A_k \in C(G)$, $k \in \{1, ..., \kappa\}$ *such that*

$$f(x) = f(0) + P(x) = f(0) + A_1(x) + \ldots + A_\kappa(x, \ldots, x), x \in G,$$

$$g(x) = g(0) + \sum_{\lambda \in \mathcal{K}} A_1(\lambda \cdot x) + \ldots + \sum_{\lambda \in \mathcal{K}} A_k(\lambda \cdot x, \ldots, \lambda \cdot x), x \in G,$$

where

$$g(0) = \kappa \sum_{\lambda \in \mathcal{K}} \int_G P(\lambda \cdot s) d\mu(s),$$

$$\binom{j}{i} \sum_{\lambda \in \mathcal{K}} A_j(x, \ldots x, \underbrace{\lambda \cdot y, \ldots, \lambda \cdot y}_{i}) = 0, \;\; x, y \in G, 1 \leq i \leq j-1, 2 \leq j \leq \kappa.$$

Proof. Assume that the functions $f, g \in C(G)$ satisfy the functional equation (3). It is easy to check that if we put $y = 0$ in (3), we get

$$\sum_{k \in \mathcal{K}} \sum_{\lambda \in \mathcal{K}} \int_G f(x + \lambda \cdot s) d\mu(s) = g(0) + \kappa f(x).$$

Therefore,

$$\kappa \sum_{\lambda \in \mathcal{K}} \int_G f(x + \lambda \cdot s) d\mu(s) = g(0) + \kappa f(x), \tag{4}$$

for all $x \in G$. Replacing x by $x + k \cdot y$ in (4), we obtain

$$\sum_{\lambda \in \mathcal{K}} \int_G f(x + k \cdot y + \lambda \cdot s) d\mu(s) = \frac{1}{\kappa} g(0) + f(x + k \cdot y),$$

from which we infer that

$$\sum_{k\in\mathcal{K}}\sum_{\lambda\in\mathcal{K}}\int_G f(x+k\cdot y+\lambda\cdot s)d\mu(s)=\sum_{k\in\mathcal{K}} f(x+k\cdot y)+g(0). \tag{5}$$

By using (3) and (5), we observe that

$$\sum_{k\in\mathcal{K}} f(x+k\cdot y)=g(y)+\kappa f(x)-g(0). \tag{6}$$

Putting $g(y)-g(0)=h(y)$ in (6), we obtain

$$\sum_{k\in\mathcal{K}} f(x+k\cdot y)=\kappa f(x)+h(y),$$

which means that the functions $f, h \in C(G)$ satisfy the equation (2). According to Theorem (1), there exist symmetric k–additive mappings $A_k \in C(G)$, $k \in \{1, ..., \kappa\}$ such that

$$\begin{aligned} f(x) &= f(0)+P(x)=f(0)+A_1(x)+\ldots+A_\kappa(x,\ldots,x), x\in G,\\ g(x) &= g(0)+\sum_{\lambda\in\mathcal{K}} A_1(\lambda\cdot x)+\ldots+\sum_{\lambda\in\mathcal{K}} A_k(\lambda\cdot x,\ldots,\lambda\cdot x), x\in G, \end{aligned}$$

We compute the left hand side of (3) to be

$$\begin{aligned} &\sum_{k\in\mathcal{K}}\sum_{\lambda\in\mathcal{K}}\int_G f(x+k\cdot y+\lambda\cdot s)d\mu(s)\\ &\qquad=\kappa f(0)+\sum_{k\in\mathcal{K}}\sum_{\lambda\in\mathcal{K}}\int_G P(x+k\cdot y+\lambda\cdot s)d\mu(s)\\ &\qquad=\kappa f(0)+\kappa P(x)+g(y)\\ &\qquad=\kappa f(0)+\kappa P(x)+g(0)+\sum_{\lambda\in\mathcal{K}} P(\lambda\cdot y). \end{aligned}$$

In put $x=y=0$, we get

$$\kappa\sum_{\lambda\in\mathcal{K}}\int_G P(\lambda\cdot s)d\mu(s)=2\kappa P(0)+g(0)$$

Hence

$$g(0)=\kappa\sum_{\lambda\in\mathcal{K}}\int_G P(\lambda\cdot s)d\mu(s)$$

Conversely, assume that there exist symmetric k-additive mappings $A_k \in C(G)$, $k \in \{1, ..., \kappa\}$ such that

$$f(x) = f(0) + P(x) = f(0) + A_1(x) + \ldots + A_\kappa(x, \ldots, x), x \in G,$$

where

$$P(x) = \sum_{k=1}^{\kappa} A_k(x),$$

$$g(x) = g(0) + \sum_{\lambda \in \mathcal{K}} A_1(\lambda \cdot x) + \ldots + \sum_{\lambda \in \mathcal{K}} A_k(\lambda \cdot x, \ldots, \lambda \cdot x), x \in G,$$

where

$$g(0) = \kappa \sum_{\lambda \in \mathcal{K}} \int_G P(\lambda \cdot s) d\mu(s),$$

$$\binom{j}{i} \sum_{\lambda \in \mathcal{K}} A_j(x, \ldots x, \underbrace{\lambda \cdot y, \ldots, \lambda \cdot y}_{i}) = 0, \; x, y \in G, 1 \leq i \leq j-1, 2 \leq j \leq \kappa.$$

Then $f, g \in C(G)$ and a small computation shows that

$$\sum_{k \in \mathcal{K}} P(x + k \cdot y) = \kappa P(x) + \sum_{k \in \mathcal{K}} P(k \cdot y), \; P(0) = 0. \tag{7}$$

using (7) at the second equality (3) is given as follows

$$\begin{aligned}
&\sum_{k \in \mathcal{K}} \sum_{\lambda \in \mathcal{K}} \int_G f(x + k \cdot y + \lambda \cdot s) d\mu(s) \\
&= \sum_{k \in \mathcal{K}} \int_G \sum_{\lambda \in \mathcal{K}} \left[f(0) + P(x + k \cdot y + \lambda \cdot s) d\mu(s) \right] \\
&= \kappa f(0) + \sum_{k \in \mathcal{K}} P(x + k \cdot y) + \sum_{k \in \mathcal{K}} \sum_{\lambda \in \mathcal{K}} \int_G P(\lambda \cdot s) d\mu(s) \\
&= \kappa f(0) + \kappa P(x) + \sum_{k \in \mathcal{K}} P(k \cdot y) + \kappa \sum_{\lambda \in \mathcal{K}} \int_G P(\lambda \cdot s) d\mu(s) \\
&= \kappa f(x) + g(y).
\end{aligned}$$

Hence, the functions $f, g \in C(G)$ satisfy the functional equation (3). □

4 Applications

Corollary 1 *Let* $(G,+)$ *be a locally compact abelian Hausdorff group. The function* $f \in C(G)$ *satisfies the functional equation*

$$\sum_{k \in \mathcal{K}} f(x + k \cdot y) = \kappa f(x)$$

if and only if there exist symmetric $k-$*additive mappings* $A_k \in C(G^k), k \in \{0, ..., \kappa - 1\}$ *such that*

$$f(x) = f(0) + A_1(x) + \ldots + A_{\kappa-1}(x, \ldots, x), \; x \in G.$$

$$\binom{j}{i} \sum_{\lambda \in \mathcal{K}} A_j(x, \ldots x, \underbrace{\lambda \cdot y, \ldots, \lambda \cdot y}_{i}) = 0, \; x, y \in G, 1 \leq i \leq j, 2 \leq j \leq \kappa - 1.$$

Proof. By putting $\mu = \frac{\delta_0}{\kappa}$ and $g = 0$ in Theorem 2 and from [[6], Theorem 5] we get the desired result. □

Corollary 2 *Let* $(G,+)$ *be a locally compact abelian Hausdorff group, and choose an arbitrarily element* $a \in G$*. The functions* $f, g \in C(G)$ *satisfy the functional equation*

$$f(x + y + a) = f(x) + g(y)$$

if and only if there exists a mapping $A \in C(G)$, *such that*

$$f(x) = f(0) + A(x), \; x \in G$$

$$g(x) = g(0) + A(x), \; x \in G.$$

Proof. By similar the method, we put $\mu = \delta_a$, $\mathcal{K} = \{I\}$ in Theorem 2 and by a simple calculation we get $g(0) = P(a)$. Hence we get the desired result.

□

Corollary 3 *Let* $(G,+)$ *be a locally compact abelian Hausdorff group, and choose an arbitrarily element* $a \in G$*. The functions* $f, g \in C(G)$ *satisfy the functional equation*

$$f(x + y + a) + f(x + \sigma(y) + a) = 2f(x) + g(y)$$

if and only if there exists a symmetric bi-additive mapping $A_k \in C(G^k), k \in \{1,2\}$ *such that*

$$f(x) = f(0) + A_1(x) + A_2(x,x), \; x \in G$$

$$g(x) = g(0) + A_1(x) + A_1(\sigma(x)) + A_2(x,x) + A_2(\sigma(x), \sigma(x)), \; x \in G$$

$$A_2(x,y) + A_2(x, \sigma(y)) = 0, \; g(0) = 2P(a).$$

Proof. Using Theorem 2, by a simple calculation we get the desired result by putting $\mu = \frac{\delta_a}{2}$, $\mathcal{K} = \{I, \sigma\}$.

□

Corollary 4 *Let* $(G,+)$ *be a locally compact abelian Hausdorff group,* $\mu \in M_C(G)$ *and choose arbitrarily elements* $a_1, \ldots, a_n \in G$.

The functions $f, g \in C(G)$ *satisfy the functional equation*

$$\sum_{k=1}^{n} f(x + y + a_k) = nf(x) + ng(y)$$

if and only if there exists a symmetric additive mapping $A \in C(G)$ *such that*

$$f(x) = f(0) + A(x), \; x \in G,$$

$$g(x) = g(0) + A(x), \; x \in G,$$

$$g(0) = \frac{1}{n} \sum_{k=1}^{n} P(a_k)$$

Proof. It easy to prove the result by tacking $\mu = \frac{\sum_{k=1}^{n} \delta_{a_k}}{n}$ and $\mathcal{K} = \{I\}$ in Theorem 2. □

Corollary 5 *Let* $(G,+)$ *be a locally compact abelian Hausdorff group, let* $\mu \in M_C(G)$ *and choose arbitrarily element* $a \in G$. *The functions* $f, g \in C(G)$ *satisfy the functional equation*

$$\sum_{k \in \mathcal{K}} f(x + k \cdot y + a) = \kappa f(x) + g(y).$$

if and only if there exists a symmetric k-additive mapping $A \in C(G)$ *such that*

$$f(x) = f(0) + A_1(x) + \ldots + A_\kappa(x, \ldots, x), \; x \in G,$$

$$g(x) = g(0) + \sum_{\lambda \in \mathcal{K}} A_1(\lambda \cdot x) + \ldots + \sum_{\lambda \in \mathcal{K}} A_\kappa(\lambda \cdot x, \ldots, \lambda \cdot x), x \in G$$

$$g(0) = \kappa P(a)$$

$$\binom{j}{i} \sum_{\lambda \in \mathcal{K}} A_j(x, \ldots x, \underbrace{\lambda \cdot y, \ldots, \lambda \cdot y}_{i}) = 0, \ \ x, y \in G, 1 \leq i \leq j, 2 \leq j \leq \kappa - 1.$$

Proof. By similar method, we get the result by putting $\mu = \frac{\delta_a}{\kappa}$ in Theorem 2. □

Corollary 6 *Let* $(G, +)$ *be a locally compact abelian Hausdorff group, and let* $\mu \in M_C(G)$. *The functions* $f, g \in C(G)$ *satisfy the functional equation*

$$\int_G f(x + y + t) d\mu(t) = f(x) + g(y)$$

if and only if there exists a symmetric additive mapping $A : G \to \mathbb{C}$ $A \in C(G)$ *such that*

$$f(x) = f(0) + A(x) \ , x \in G,$$

$$g(x) = g(0) + A(x) \ , x \in G,$$

References

[1] S. Czerwik, *Functional Equations and Inequalities in Several Variables,* World Scientific, New Jersey, London, Singapore, Hong Kong, (2002).

[2] B. Fadli, D. Zeglami, S. Kabbaj, On Gajda's type quadratic equation on locally compact abelian group, indagationes. *Math.* **26** (4) (2015), 660–668.

[3] Ż. Fechner, L. Székelyhidi, *A generalization of Gajda's equation on commutative topological groups* (9 Mar 2014), arXiv:1403.2052[math.FA].

[4] E. Hewitt, K. A. Ross, *Abstract harmonic analysis. Vol. I: Structure of topological groups. Integration theory. Group representations*, Die Grundlehren der Mathematischen Wissenschaften, Band 115. Springer-Verlag, Berlin-Gottingen-Heidelberg, 1963.

[5] P. Kannappan, *Functional Equations and Inequalities with Applications*, in: Springer Monographs in Mathematics, Springer, New York, 2009.

[6] R. Łukasik, Some generalization of Cauchy's and the quadratic functional equations, *Aequationes Math.*, **83** (2012), 75–86.

[7] H. Stetkær,Functional equations involving means of functions on the complex plane, *Aequationes. Math.* **56** (1998), 47–62.

[8] H. Stetkær,Functional equations on abelian groups with involution, *II, Aequationes. Math.* **55** (1998), 227–240.

[9] H. Stetkær, *Functional Equations on Groups*, World Scientific Publishing Co., Singapore, 2013.

[10] L. Székelyhidi, *Convolution Type Functional Equations on Topological Abelian Groups*, World Scientific, Singapore, 1991.

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