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On generalized nonlinear Euler-Bernoulli Beam type equations

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Abstract. This paper is devoted to the study of a nonlinear Euler-Bernoulli Beam type equation involving both left and right Caputo fractional derivatives. Differently from the approaches of the other papers where they established the existence of solution for the linear Euler-Bernoulli Beam type equation numerically, we use the lower and upper solutions method with some new results on the monotonicity of the right Caputo derivative. Furthermore, we give the explicit expression of the upper and lower solutions. A numerical example is given to illustrate the obtained results.

1 Introduction

Fractional differential equations containing a composition of left and right fractional derivatives occur in the fractional theoretical mechanics and may arise naturally in the study of variational problems such as the fractional Euler-Lagrange equations, see [1–5, 8, 12]. The presence of both left and right fractional derivatives in the nonlinear differential equation poses many complications when trying to apply the existing methods, for this reason, most

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studies focus on the linear cases and use numerical analysis, we refer the reader to [1, 5, 12].

In [12] the authors discussed a linear fractional differential equation involving the right Caputo derivative and the left Riemann-Liouville fractional derivative and describing the height of granular material decreasing over time in a silo:

$$^{C}D_{L^-}^{\alpha} D_{0^+}^{\alpha} u(t) + bu(t) = 0, 0 \leq t \leq L, 0 < \alpha \leq 1.$$

Recently in [5], the author solved analytically and numerically a linear fractional Euler–Bernoulli beam equation containing left and right fractional Caputo derivatives:

$$\begin{aligned} ^{C}D_{L^-}^{\alpha\,C} D_{0^+}^{\alpha} u(t) &= f(t), 0 \leq t \leq L, 1 < \alpha \leq 2 \\ u(0) &= u'(0) = u(L) = u'(L) = 0, \end{aligned}$$

that is derived by using a variational approach.

In [8] the authors proved existence of solutions for a nonlinear fractional oscillator equation containing both left Riemann–Liouville and right Caputo fractional derivatives

$$\begin{aligned} -^{C}D_{1^-}^{\alpha} D_{0^+}^{\beta} u(t) + \omega^2 u(t) &= f(t, u(t)), \\ 0 &\leq t \leq 1, \omega \in \mathbb{R}, \omega \neq 0, 0 < \alpha, \beta < 1 \\ u(0) &= 0, D_{0^+}^{\beta} u(1) = 0. \end{aligned}$$

The main tools for this study was the upper and lower solutions method.

Nonlinear fractional differential equations has been studied by different methods such fixed point theorems, upper and lower solutions method, successive approximations,... see [1–10, 13, 14].

In this paper we focus on a nonlinear Euler-Bernoulli Beam equation involving both left and right Caputo fractional derivatives

$$^{C}D_{1^-}^{\alpha\,C} D_{0^+}^{\beta} u(t) = f(t, u(t)), 0 \leq t \leq 1, \tag{1}$$

with the boundary conditions

$$u(0) = u'(0) = u(1) = D_{0^+}^{\beta+1} u(1) = 0, \tag{2}$$

Where $1 < \alpha, \beta < 2$, $^{C}D_{1^-}^{\alpha}$ and $^{C}D_{0^+}^{\beta}$ denote respectively the right and the left sides Caputo derivatives and $D_{0^+}^{\beta+1}$ denotes the left Riemann–Liouville fractional derivative. Denote by (P1) the problem (1)-(2).

Note that the presence of both left and right fractional derivatives leads to great difficulties. To solve problem (P1) we apply the lower and upper solutions method and a new result on the monotonicity for the right Caputo derivative. To succeed with such approach, we transform the problem (P1) to a right Caputo fractional boundary value problem of lower order. After constructing the explicit expressions of the lower and upper solutions, we define a sequence of modified problems that we solve by Schauder fixed point theorem. An example is presented to illustrate the main theorem.

2 Preliminaries

In this section, we recall necessary definitions of fractional operators and their properties [11, 15, 16].

The left and right Caputo derivatives of order $n-1 < p < n$ are respectively defined as

$$\begin{aligned} {}^{C}D_{0+}^{p}g(t) &= \left(I_{0+}^{1-p}D^{n}g\right)(t), \\ {}^{C}D_{1-}^{p}g(t) &= -\left(I_{1-}^{1-p}D^{n}g\right)(t), \end{aligned}$$

and the Riemann -Liouville fractional derivative is defined as

$$D_{0+}^{p}g(t) = D^{n}\left(I_{0+}^{1-p}g\right)(t),$$

where D^{n} is the the classical derivative operator of order n and the operators I_{0+}^{p} and I_{1-}^{p} are respectively the left and right fractional Riemann–Liouville integrals of order $p > 0$ defined by

$$\begin{aligned} I_{0+}^{p}g(t) &= \frac{1}{\Gamma(p)}\int_{0}^{t}\frac{g(s)}{(t-s)^{1-p}}ds, \\ I_{1-}^{p}g(t) &= \frac{1}{\Gamma(p)}\int_{t}^{1}\frac{g(s)}{(s-t)^{1-p}}ds. \end{aligned}$$

The composition rules of the fractional operators (for $n-1 < p < n$) are:

1- $I_{0+}^{p}{}^{C}D_{0+}^{p}g(t) = g(t) - \sum_{k=0}^{n-1}\frac{g^{(k)}(0)}{k!}t^{k}$.

2- $I_{1-}^{p}{}^{C}D_{1-}^{p}g(t) = g(t) - \sum_{k=0}^{n-1}\frac{(-1)^{k}g^{(k)}(1)}{k!}(1-t)^{k}$.

3- ${}^{C}D_{1-}^{p}Dg(t) = -{}^{C}D_{1-}^{p+1}g(t)$.

4- $DD_{0+}^{p}g(t) = D_{0+}^{p+1}g(t)$.

Next we give some results on the Caputo derivative of monotone functions.

Theorem 1 [8] *Assume that* $g \in C^1[0,1]$ *is such that* ${}^C D^{\gamma}_{1-} g(t) \geq 0$ *for all* $t \in [0,1]$ *and all* $\gamma \in (p,1)$ *with some* $p \in (0,1)$. *Then* g *is monotone decreasing. Similarly, if* ${}^C D^{\gamma}_{1-} g(t) \leq 0$ *for all* t *and* γ *mentioned above, then* g *is monotone increasing.*

3 Main results

To reduce the problem (P1) to an equivalent right Caputo fractional boundary value problem of lower order, we use the following Lemma

Lemma 1 *If a function* g *satisfies* $g(0) = g'(0)$, *then we have*

$$ {}^C D^{\alpha}_{1-} {}^C D^{\beta}_{0+} g(t) = -{}^C D^{\alpha-1}_{1-} D^{\beta+1}_{0+} g(t). $$

where $D^{\beta+1}_{0+}$ *denotes the Riemann -Liouville fractional derivative.*

Proof. The proof is based on the composition rules of the fractional operators. □

From Lemma 2, equation (1) can be written as

$$ -{}^C D^{\alpha-1}_{1-} D^{\beta+1}_{0+} u(t) = f(t, u(t)), 0 \leq t \leq 1. $$

Denote by (P2) the auxiliary problem:

$$ (P2) \begin{cases} D^{\beta+1}_{0+} u(t) = v(t), 0 \leq t \leq 1 \\ u(0) = u'(0) = u(1) = 0. \end{cases} $$

In the next lemma, we give the solution of (P2).

Lemma 2 *If* $1 < \beta < 2$, *then problem (P2) has a unique solution given by*

$$ u(t) = I^{\beta+1}_{0+} v(t) - t^{\beta} I^{\beta+1}_{0+} v(1). $$

Let E denotes the Banach space $C([0,1], \mathbb{R})$ equipped with the uniform norm $\|u\| = \max_{t \in [0,1]} |u(t)|$. Define the operator T on E by

$$ Tv(t) = I^{\beta+1}_{0+} v(t) - t^{\beta} I^{\beta+1}_{0+} v(1), t \in [0,1], $$

thus

$$ u(t) = Tv(t), t \in [0,1]. $$

From the boundary condition $D_{0^+}^{\beta+1}u(1) = 0$, we show that the problem (P1) is equivalent to the following right Caputo boundary value problem of order $0 < \alpha - 1 < 1$:

$$(P3)\begin{cases} -{}^{C}D_{1^-}^{\alpha-1}v(t) = f(t, Tv(t)), 0 \le t \le 1 \\ v(1) = 0. \end{cases}$$

Let us make the following hypotheses:

$(H1)$ There exists a constant $A \geq 0$ such that

$$f(t, x) \geq \frac{-A}{\Gamma(2-\alpha)}(1-t)^{1-\gamma},$$

for $0 \leq t \leq 1$, for all $\gamma \in [\alpha - 1, 1)$ and $\frac{-A(\beta+1)}{\Gamma(3+\beta)} \leq x \leq 0$.

$(H2)$ There exists a constant $B \leq 0$ such that $A \geq |B|$ and

$$f(t, x) \leq \frac{-B}{\Gamma(2-\alpha)}(1-t)^{1-\gamma},$$

for $0 \leq t \leq 1$, for all $\gamma \in [\alpha - 1, 1)$ and $0 \leq x \leq \frac{-B(\beta+1)}{\Gamma(3+\beta)}$.

To use Theorem 1, we have to adapt the definition of the lower and upper solutions for problem (P1) as follows:

Definition 1 *The functions $\underline{\sigma}$, $\overline{\sigma} \in AC^4[0,1]$ are called lower and upper solutions of problem (P1) respectively, if*

a) $-{}^{C}D_{1^-}^{\gamma} D_{0^+}^{\beta+1}\underline{\sigma}(t) \geq f(t, \underline{\sigma}(t))$,*for all* $t \in [0,1]$ *and for all* $\gamma \in [\alpha - 1, 1)$ *and*

$\underline{\sigma}(0) \geq 0$, $\underline{\sigma}'(0) \geq 0$, $\underline{\sigma}(1) \geq 0$ *and* $D_{0^+}^{\beta+1}\underline{\sigma}(1) \geq 0$.

b) $-{}^{C}D_{1^-}^{\gamma} D_{0^+}^{\beta+1}\overline{\sigma}(t) \leq f(t, \overline{\sigma}(t))$,*for all* $t \in [0,1]$ *and for all* $\gamma \in [\alpha - 1, 1)$ *and*

$\overline{\sigma}(0) \leq 0$, $\overline{\sigma}'(0) \leq 0$, $\overline{\sigma}(1) \leq 0$ *and* $D_{0^+}^{\beta+1}\overline{\sigma}(1) \leq 0$.

Where

$$AC^4[0,1] = \left\{u \in C^3[0,1], u^{(3)} \textit{ absolutely continuous function on } [0,1]\right\}.$$

Functions $\underline{\sigma}$ and $\overline{\sigma}$ are lower and upper solutions in reverse order if $\underline{\sigma}(t) \geq \overline{\sigma}(t), 0 \leq t \leq 1$.

Lemma 3 *Under the hypotheses (H1) and (H2), the problem (P1) has a lower and an upper solutions.*

Proof. Define $\varphi(t) = A(1-t)$, then we get

$$\begin{aligned} 0 &\geq T\varphi(t) = I_{0^+}^{\beta+1}\varphi(t) - t^\beta I_{0^+}^{\beta+1}\varphi(1) \\ &= \frac{At^\beta}{\Gamma(3+\beta)}\left(-t^2 + t(\beta+2) - (\beta+1)\right) \geq \frac{-A(\beta+1)}{\Gamma(3+\beta)}. \end{aligned}$$

By computation we obtain for $\gamma \in [\alpha-1, 1)$

$$^C D_{1^-}^{\gamma}\varphi(t) = \frac{A}{\Gamma(2-\gamma)}(1-t)^{1-\gamma},$$

Now, we show that $\overline{\sigma}(t) = T\varphi(t)$ is an upper solution of problem (P1). By the help of hypothesis (H1), we have for all $t \in [0,1]$ and for all $\gamma \in [\alpha-1, 1)$

$$\begin{aligned} -{}^C D_{1^-}^{\gamma} D_{0^+}^{\beta+1}\overline{\sigma}(t) &= -{}^C D_{1^-}^{\gamma}\varphi(t) = \frac{-A}{\Gamma(2-\gamma)}(1-t)^{1-\gamma} \\ &\leq f(t, T\varphi(t)) = f(t, \overline{\sigma}(t)) \end{aligned}$$

in addition $\overline{\sigma}(0) \leq 0$, $\overline{\sigma}'(0) \leq 0, \overline{\sigma}(1) \leq 0$ and $D_{0^+}^{\beta+1}\overline{\sigma}(1) \leq 0$, consequently $\overline{\sigma}(t) = T\varphi(t)$ is an upper solution of problem (P1).

Similarly, setting $\psi(t) = B(1-t)$ and taking hypothesis (H2) into account, we show that $\underline{\sigma}(t) = T\psi(t)$ is a lower solution of problem (P1). Finally we write the explicit expressions of the upper and lower solutions as

$$\begin{aligned} \overline{\sigma}(t) &= T\varphi(t) = \frac{At^\beta}{\Gamma(3+\beta)}\left(-t^2 + t(\beta+2) - (\beta+1)\right) \leq 0, \\ \underline{\sigma}(t) &= T\psi(t) = \frac{Bt^\beta}{\Gamma(3+\beta)}\left(-t^2 + t(\beta+2) - (\beta+1)\right) \geq 0, \end{aligned}$$

and then $\underline{\sigma}$ and $\overline{\sigma}$ are lower and upper solutions in reverse order, i.e

$$\overline{\sigma}(t) \leq \underline{\sigma}(t), 0 \leq t \leq 1.$$

The proof is completed. □

Let us introduce the following sequence of modified problems $\left((P4)_\gamma\right)$, for $\gamma \in [\alpha-1, 1)$:

$$\left((P4)_\gamma\right) \begin{cases} -{}^C D_{1^-}^{\gamma} v(t) = Fv(t), 0 \leq t \leq 1 \\ v(1) = 0, \end{cases}$$

where the operator $F : E \to E$, is defined by

$$Fv(t) = f(t, T(\min(\varphi, \max(v, \psi)))), 0 \leq t \leq 1.$$

The relation between the solution of the sequence of modified problem $\left((\mathsf{P4})_\gamma\right)$ and the solution of problem (P1) is given in the following lemma:

Lemma 4 *If* v *is a solution of problem* $\left((\mathsf{P4})_{\alpha-1}\right)$, *then* $u = Tv$ *is a solution of problem (P1) satisfying*

$$\overline{\sigma}(t) \leq u(t) \leq \underline{\sigma}(t), 0 \leq t \leq 1. \tag{3}$$

Proof. Let v_γ be a solution of problem $\left((\mathsf{P4})_\gamma\right)$ for $\gamma \in [\alpha-1, 1)$, we shall prove that

$$\psi(t) \leq v_\gamma(t) \leq \varphi(t), 0 \leq t \leq 1. \tag{4}$$

For this purpose, set $\epsilon(t) = v_\gamma(t) - \varphi(t)$. It's clear that $\epsilon(1) = 0$. Suppose the contrary, i.e. there exists $t_1 \in (0,1]$ such that $\epsilon(t_1) > 0$, since ϵ is continuous, then there exist $a \in [0, t_1]$ and $b \in (t_1, 1]$ such that $\epsilon(b) = 0$ and $\epsilon(t) \geq 0$, for all $t \in [a, b]$. By taking the right Caputo derivative of ϵ, it yields

$$\begin{aligned} {}^C D^\gamma_{1^-} \epsilon(t) &= {}^C D^\gamma_{1^-} v_\gamma(t) - {}^C D^\gamma_{1^-} \varphi(t) \\ &= -f(t, T(\min[\varphi, (\max(v_\gamma, \psi))])) - {}^C D^\gamma_{1^-} \varphi(t) \\ &= -f(t, T\varphi(t)) - {}^C D^\gamma_{1^-} \varphi(t). \end{aligned}$$

Taking in to account that $\overline{\sigma} = T\varphi(t)$ is an upper solution, we conclude that ${}^C D^\gamma_{1^-} \epsilon(t) \leq 0$, $t \in [a, b]$, therefore, ϵ is increasing on $[a, b]$ by Theorem 1. Since $\epsilon(b) = 0$, then $\epsilon(t) \leq 0$, for all $t \in [a, b]$ and thus $v_\gamma(t) \leq \varphi(t)$, $t \in [a, b]$ that leads to a contradiction. Proceeding by the same way, we prove that $\psi(t) \leq v_\gamma(t)$, $t \in [0, 1]$.

Now, let v be a solution of problem $((\mathsf{P4})_{\alpha-1})$, then thanks to inequalities (4) we have

$$-{}^C D^{\alpha-1}_{1^-} v(t) = Fv(t) = f(t, Tv(t)),$$

hence v is a solution of (P3) and consequently $u = Tv$ is a solution of (P1). Let us rewrite the operator T as

$$Tv(t) = \frac{1}{\Gamma(1+\beta)} \int_0^1 G(t,s) v(s)\, ds$$

where the Green function G given by

$$G(t,s) = \begin{cases} (t-s)^\beta - t^\beta (1-s)^\beta, s \leq t \\ -t^\beta (1-s)^\beta, s \geq t \end{cases}$$

is negative for $0 \leq s, t \leq 1$, consequently, by applying the operator T to (4) it yields

$$T\varphi(t) \leq Tv(t) \leq T\psi(t), 0 \leq t \leq 1,$$

thus (3) holds. This achieves the proof. □

Now we are ready to prove our main results for problem (P1):

Theorem 2 *Under the hypotheses (H1) and (H2), the problem (P1) has at least one solution* u *satisfying*

$$\overline{\sigma}(t) \leq u(t) \leq \underline{\sigma}(t), 0 \leq t \leq 1.$$

Proof. Define the operator R on E, by

$$\begin{aligned} Rv(t) &= -I_{1^-}^{\alpha-1} Fv(t) \\ &= -I_{1^-}^{\alpha-1} f(t, T(\min(\varphi, \max(v, \psi)))), 0 \leq t \leq 1. \end{aligned}$$

Let us remark that if R has a fixed point v then $u = Tv$ is a solution of (P1). Set

$$\Omega = \{v \in C[0,1], \|v\| \leq \frac{M}{\Gamma(\alpha)}\}.$$

where

$$M = \max\{|f(t,x)|, \overline{\sigma}(t) \leq x \leq \underline{\sigma}(t), 0 \leq t \leq 1\}.$$

Let us prove that $R(\Omega)$ is uniformly bounded, equicontinuous and $R(\Omega) \subset \Omega$. Let $v \in \Omega$, then $\overline{\sigma}(t) \leq T(\min(\varphi, \max(v, \psi)))(t) \leq \underline{\sigma}(t)$ we get

$$\begin{aligned} |Rv(t)| &\leq I_{1^-}^{\alpha-1} |f(t, T(\min(\varphi, \max(v, \psi)))(t))| \\ &= \frac{1}{\Gamma(\alpha-1)} \int_t^1 \frac{|f(s, T(\min(\varphi, \max(v, \psi)))(s))|}{(s-t)^{2-\alpha}} ds \\ &\leq \frac{M}{\Gamma(\alpha)}, \end{aligned}$$

therefore $R(\Omega)$ is uniformly bounded and $R(\Omega) \subset \Omega$. Let $0 \leq t_1 < t_2 \leq 1$, for simplicity we denote $g(t) = f(t, T(\min(\varphi, \max(v, \psi)))(t))$, we have

$$\begin{aligned} |Rv(t_1) - Rv(t_2)| &\leq \left| I_{1^-}^{\alpha-1} g(t_1) - I_{1^-}^{\alpha-1} g(t_2) \right| \\ &\leq \frac{1}{\Gamma(\alpha-1)} \int_{t_1}^{t_2} (s-t_1)^{\alpha-2} |g(s)| ds + \\ &\quad \frac{1}{\Gamma(\alpha-1)} \int_{t_2}^{1} \left((s-t_1)^{\alpha-2} - (s-t_2)^{\alpha-2} \right) |g(s)| ds \\ &\leq \frac{M}{\Gamma(\alpha)} \left((1-t_1)^{\alpha-1} - (1-t_2)^{\alpha-1} \right) \to 0, t_1 \to t_2, \end{aligned}$$

hence, $R(\Omega)$ is equicontinuous. By Arzela-Ascoli Theorem we conclude that R is completely continuous. Finally an application of Schauder fixed point theorem implies that R has a fixed point $v \in \Omega$, and so $u = Tv$ is a solution of (P1) satisfying from Lemma 7, $\underline{\sigma}(t) \leq u(t) \leq \overline{\sigma}(t)$, $0 \leq t \leq 1$. The proof is completed. □

Next, we present an example to illustrate the obtained results.

Example 1 *Consider the problem (P1) with* $\alpha = \frac{5}{3}$, $\beta = \frac{3}{2}$ *and*

$$f(t,x) = \frac{2\Gamma\left(\frac{9}{2}\right)}{5\Gamma\left(\frac{1}{3}\right)} x(1-t)^{\frac{1}{3}}, 0 \leq t \leq 1, x \in \mathbb{R}.$$

If we choose $A = 1$ *and* $B = -1$*, then Hypotheses (H1) and (H2) are satisfied, in fact for for*$\gamma \in [\frac{2}{3}, 1), 0 \leq t \leq 1, \frac{-5}{2\Gamma(\frac{9}{2})} \leq x \leq 0$, *we have*

$$\begin{aligned} f(t,x) &= \frac{2\Gamma\left(\frac{9}{2}\right)}{5\Gamma\left(\frac{1}{3}\right)} x(1-t)^{\frac{1}{3}} = \frac{2\Gamma\left(\frac{9}{2}\right)}{5\Gamma\left(\frac{1}{3}\right)} x(1-t)^{1-\frac{2}{3}} \\ &\geq \frac{-1}{\Gamma\left(\frac{1}{3}\right)}(1-t)^{1-\frac{2}{3}} \geq \frac{-1}{\Gamma\left(\frac{1}{3}\right)}(1-t)^{1-\gamma}. \end{aligned}$$

and if $\gamma \in [\frac{2}{3}, 1), 0 \leq t \leq 1, 0 \leq x \leq \frac{5}{2\Gamma(\frac{9}{2})}$, *it yields*

$$\begin{aligned} f(t,x) &= \frac{2\Gamma\left(\frac{9}{2}\right)}{5\Gamma\left(\frac{1}{3}\right)} x(1-t)^{\frac{1}{3}} = \frac{2\Gamma\left(\frac{9}{2}\right)}{5\Gamma\left(\frac{1}{3}\right)} x(1-t)^{1-\frac{2}{3}} \\ &\leq \frac{1}{\Gamma\left(\frac{1}{3}\right)}(1-t)^{1-\frac{2}{3}} \leq \frac{1}{\Gamma\left(\frac{1}{3}\right)}(1-t)^{1-\gamma}. \end{aligned}$$

The expressions of lower and upper solutions are

$$\begin{aligned} \overline{\sigma}(t) &= T\varphi(t) = \frac{t^{\frac{3}{2}}}{\Gamma\left(\frac{9}{2}\right)}\left(-t^2 + \frac{7}{2}t - \frac{5}{2}\right) \leq 0, \\ \underline{\sigma}(t) &= T\psi(t) = \frac{-t^{\frac{3}{2}}}{\Gamma\left(\frac{9}{2}\right)}\left(-t^2 + \frac{7}{2}t - \frac{5}{2}\right) \geq 0. \end{aligned}$$

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