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Some properties of analytic functions related with Booth lemniscate

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Abstract. The object of the present paper is to study of two certain subclass of analytic functions related with Booth lemniscate which we denote by $\mathcal{BS}(\alpha)$ and $\mathcal{BK}(\alpha)$. Some properties of these subclasses are considered.

1 Introduction

Let Δ be the open unit disk in the complex plane $\mathbb{C}$ and $\mathcal{A}$ be the class of normalized and analytic functions. Easily seen that any $f \in \mathcal{A}$ has the following form:

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \qquad (z \in \Delta). \tag{1}$$

Further, by $\mathcal{S}$ we will denote the class of all functions in $\mathcal{A}$ which are univalent in Δ. The set of all functions $f \in \mathcal{A}$ that are starlike univalent in Δ will be denote by $\mathcal{S}^*$ and the set of all functions $f \in \mathcal{A}$ that are convex univalent in

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Δ will be denote by $\mathcal{K}$. Analytically, the function $f \in \mathcal{A}$ is a starlike univalent function, if and only if

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \qquad (z \in \Delta).$$

Also, $f \in \mathcal{A}$ belongs to the class $\mathcal{K}$, if and only if

$$\mathfrak{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0 \qquad (z \in \Delta).$$

For more details about this functions, the reader may refer to the book of Duren [2]. Define by $\mathfrak{B}$ the class of analytic functions $w(z)$ in Δ with $w(0) = 0$ and $|w(z)| < 1$, $(z \in \Delta)$. Let f and g be two functions in $\mathcal{A}$. Then we say that f is subordinate to g, written $f(z) \prec g(z)$, if there exists a function $w \in \mathfrak{B}$ such that $f(z) = g(w(z))$ for all $z \in \Delta$. Furthermore, if the function g is univalent in Δ, then we have the following equivalence:

$$f(z) \prec g(z) \Leftrightarrow (f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta)).$$

Recently, the authors [10, 11], (see also [5]) have studied the function

$$F_\alpha(z) := \frac{z}{1 - \alpha z^2} = \sum_{n=1}^{\infty} \alpha^{n-1} z^{2n-1} \qquad (z \in \Delta,\ 0 \le \alpha \le 1). \tag{2}$$

We remark that the function $F_\alpha(z)$ is a starlike univalent function when $0 \le \alpha < 1$. In addition $F_\alpha(\Delta) = D(\alpha)$ $(0 \le \alpha < 1)$, where

$$D(\alpha) = \left\{x + iy \in \mathbb{C} : \ \left(x^2 + y^2\right)^2 - \frac{x^2}{(1-\alpha)^2} - \frac{y^2}{(1+\alpha)^2} < 0\right\}$$

and

$$F_1(\Delta) = \mathbb{C} \backslash \{(-\infty, -i/2] \cup [i/2, \infty)\}.$$

For $f \in \mathcal{A}$ we denote by $\mathrm{Area}\, f(\Delta)$, the area of the multi-sheeted image of the disk $\Delta_r = \{z \in \mathbb{C} : |z| < r\}$ $(0 < r \le 1)$ under f. Thus, in terms of the coefficients of f, $f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$ one gets with the help of the classical Parseval-Gutzmer formula (see [2]) the relation

$$\mathrm{Area}\, f(\Delta) = \iint_{\Delta_r} |f'(z)|^2 dx dy = \pi \sum_{n=1}^{\infty} n|a_n|^2 r^{2n}, \tag{3}$$

which is called the Dirichlet integral of f. Computing this area is known as the area problem for the functions of type f. Thus, a function has a finite Dirichlet integral exactly when its image has finite area (counting multiplicities). All polynomials and, more generally, all functions $f \in \mathcal{A}$ for which f' is bounded on Δ are Dirichlet finite. Now by (2), (3) and since $\sum_{n=1}^{\infty} n r^{2(n-1)} = 1/(1-r^2)^2$ we get:

Corollary 1 *Let* $0 \leq \alpha < 1$. *Then*

$$\mathrm{Area}\{F_\alpha(\Delta)\} = \frac{\pi}{(1-\alpha^2)^2}.$$

Let $\mathcal{BS}(\alpha)$ be the subclass of $\mathcal{A}$ which satisfy the condition

$$\left(\frac{zf'(z)}{f(z)} - 1\right) \prec F_\alpha(z) \qquad (z \in \Delta). \tag{4}$$

The function class $\mathcal{BS}(\alpha)$ was studied extensively by Kargar $et\ al.$ [5]. The function

$$\tilde{f}(z) = z\left(\frac{1+z\sqrt{\alpha}}{1-z\sqrt{\alpha}}\right)^{\frac{1}{2\sqrt{\alpha}}}, \tag{5}$$

is extremal function for several problems in the class $\mathcal{BS}(\alpha)$. We note that the image of the function $F_\alpha(z)$ $(0 \leq \alpha < 1)$ is the Booth lemniscate. We remark that a curve described by

$$\left(x^2+y^2\right)^2 - \left(n^4+2m^2\right)x^2 - \left(n^4-2m^2\right)y^2 = 0 \qquad (x,y) \neq (0,0),$$

(is a special case of Persian curve) was studied by Booth and is called the Booth lemniscate [1]. The Booth lemniscate is called elliptic if $n^4 > 2m^2$ while, for $n^4 < 2m^2$, it is termed hyperbolic. Thus it is clear that the curve

$$\left(x^2+y^2\right)^2 - \frac{x^2}{(1-\alpha)^2} - \frac{y^2}{(1+\alpha)^2} = 0 \qquad (x,y) \neq (0,0),$$

is the Booth lemniscate of elliptic type. Thus the class $\mathcal{BS}(\alpha)$ is related to the Booth lemniscate.

In the present paper some properties of the class $\mathcal{BS}(\alpha)$ including, the order of strongly satarlikeness, upper and lower bound for $\mathfrak{Ref}(z)$, distortion and grow theorems and some sharp inequalities and logarithmic coefficients inequalities are considered. Also at the end, we introduce a certain subclass of convex functions.

2 Main results

Our first result is contained in the following. Further we recall that (see [12]) the function f is strongly starlike of order γ and type β in the disc Δ, if it satisfies the following inequality:

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}-\beta\right\}\right|<\frac{\pi\gamma}{2}\qquad (0\le\beta\le 1, 0<\gamma\le 1). \tag{6}$$

Theorem 1 *Let* $0\le\alpha\le 1$ *and* $0<\varphi<2\pi$*. If* $f\in\mathcal{BS}(\alpha)$*, then* f *is strongly starlike function of order* $\gamma(\alpha,\varphi)$ *and type 1 where*

$$\gamma(\alpha,\varphi):=\frac{2}{\pi}\arctan\left(\frac{1+\alpha}{1-\alpha}|\tan\varphi|\right).$$

Proof. Let $z=re^{i\varphi}(r<1)$ and $\varphi\in(0,2\pi)$. Then we have

$$\begin{aligned}F_\alpha(re^{i\varphi})&=\frac{re^{i\varphi}}{1-\alpha r^2e^{2i\varphi}}\cdot\frac{1-\alpha r^2e^{-2i\varphi}}{1-\alpha r^2e^{-2i\varphi}}\\&=\frac{r(1-\alpha r^2)\cos\varphi+ir(1+\alpha r^2)\sin\varphi}{1-2\alpha r^2\cos 2\varphi+\alpha^2r^4}.\end{aligned}$$

Hence

$$\left|\frac{\mathfrak{Im}\{F_\alpha(re^{i\varphi})\}}{\mathfrak{Re}\{F_\alpha(re^{i\varphi})\}}\right|=\left|\frac{(1+\alpha r^2)\sin\varphi}{(1-\alpha r^2)\cos\varphi}\right|=\frac{1+\alpha r^2}{1-\alpha r^2}|\tan\varphi|\qquad(\varphi\in(0,2\pi)). \tag{7}$$

For such r the curve $F_\alpha(re^{i\varphi})$ is univalent in $\Delta_r=\{z:|z|<r\}$. Therefore

$$\left[\left(\frac{zf'(z)}{f(z)}-1\right)\prec F_\alpha(z),\quad z\in\Delta_r\right]\Leftrightarrow\left[\left(\frac{zf'(z)}{f(z)}-1\right)\in F_\alpha(\Delta_r),\quad z\in\Delta_r\right]. \tag{8}$$

Then by (7) and (8), we have

$$\begin{aligned}\left|\arg\left\{\frac{zf'(z)}{f(z)}-1\right\}\right|&=\left|\arctan\frac{\mathfrak{Im}[(zf'(z)/f(z))-1]}{\mathfrak{Re}[(zf'(z)/f(z))-1]}\right|\\&\le\left|\arctan\frac{\mathfrak{Im}(F_\alpha(re^{i\varphi}))}{\mathfrak{Re}(F_\alpha(re^{i\varphi}))}\right|\\&<\arctan\left(\frac{1+\alpha r^2}{1-\alpha r^2}|\tan\varphi|\right),\end{aligned}$$

and letting $r \to 1^-$, the proof of the theorem is completed. □

In the sequel we define an analytic function $\mathcal{L}(z)$ by

$$\mathcal{L}(z) = \exp \int_0^z \frac{1+F_\alpha(t)}{t} dt \qquad (0 \le \alpha \le 3-2\sqrt{2}, t \neq 0), \tag{9}$$

where F_α is given by (2). Since the function F_α is convex univalent for $0 \le \alpha \le 3-2\sqrt{2}$, thus as result of (cf. [9]), the function $\mathcal{L}(z)$ is convex univalent function in Δ.

Theorem 2 *Let* $0 \le \alpha \le 3-2\sqrt{2}$. *If* $f \in \mathcal{BS}(\alpha)$, *then*

$$\mathcal{L}(-r) \le \Re\mathfrak{e}\{f(z)\} \le \mathcal{L}(r) \qquad (|z| = r < 1),$$

where $\mathcal{L}(.)$ *defined by* (9).

Proof. Suppose that $f \in \mathcal{BS}(\alpha)$. Then by Lindelöf's principle of subordination [4], we get

$$\begin{aligned} \inf_{|z|\le r} \Re\mathfrak{e}\{\mathcal{L}(z)\} \le \inf_{|z|\le r} \Re\mathfrak{e}\{f(z)\} &\le \sup_{|z|\le r} \Re\mathfrak{e}\{f(z)\} \\ &\le \sup_{|z|\le r} \Re\mathfrak{e}\{|f(z)|\} \le \sup_{|z|\le r} \Re\mathfrak{e}\{\mathcal{L}(z)\}. \end{aligned} \tag{10}$$

Because F_α is a convex univalent function for $0 \le \alpha \le 3-2\sqrt{2}$ and has real coefficients, hence $F_\alpha(\Delta)$ is a convex domain with respect to real axis. Moreover we have

$$\sup_{|z|\le r} \Re\mathfrak{e}\{\mathcal{L}(z)\} = \sup_{-r\le z\le r} \mathcal{L}(z) = \mathcal{L}(r)$$

and

$$\inf_{|z|\le r} \Re\mathfrak{e}\{\mathcal{L}(z)\} = \inf_{-r\le z\le r} \mathcal{L}(z) = \mathcal{L}(-r).$$

The proof of Theorem 2 is thus completed. □

Theorem 3 *Let* $f \in \mathcal{BS}(\alpha)$, $0 < \alpha \le 3-2\sqrt{2}$, $r_s(\alpha) = \frac{\sqrt{1+4\alpha}-1}{2\alpha} \le 0.8703$,

$$F_\alpha(r_s(\alpha)) = \max_{|z|=r_s(\alpha)<1} |F_\alpha(z)| \quad \text{and} \quad F_\alpha(-r_s(\alpha)) = \min_{|z|=r_s(\alpha)<1} |F_\alpha(z)|.$$

Then we have

$$\frac{1}{1+r_s^2(\alpha)}(F_\alpha(r_s(\alpha))-1) \le |f'(z)| \le \frac{1}{1-r_s^2(\alpha)}(F_\alpha(r_s(\alpha))+1) \tag{11}$$

and

$$\int_0^{r_s(\alpha)} \frac{F_\alpha(t)}{1+t^2} dt - \arctan r_s(\alpha) \le |f(z)| \le \frac{1}{2}\log\left(\frac{1+r_s(\alpha)}{1-r_s(\alpha)}\right) + \int_0^{r_s(\alpha)} \frac{F_\alpha(t)}{1-t^2} dt \tag{12}$$

Proof. Let $f \in \mathcal{BS}(\alpha)$. Then by definition of subordination we have

$$\frac{zf'(z)}{f(z)} = 1 + F_\alpha(w(z)), \tag{13}$$

where $w(z)$ is an analytic function $w(0) = 0$ and $|w(z)| < 1$. From [6, Corollary 2.1], if $f \in \mathcal{BS}(\alpha)$, then f is starlike univalent function in $|z| < r_s(\alpha)$, where $r_s(\alpha) = \frac{\sqrt{1+4\alpha}-1}{2\alpha}$. Thus if we define $q(z) : \Delta_{r_s(\alpha)} \to \mathbb{C}$ by the equation $q(z) := f(z)$, where $\Delta_{r_s(\alpha)} := \{z : |z| < r_s(\alpha)\}$, then $q(z)$ is starlike univalent function in $\Delta_{r_s(\alpha)}$ and therefore

$$\frac{r_s(\alpha)}{1 + r_s^2(\alpha)} \le |q(z)| \le \frac{r_s(\alpha)}{1 - r_s^2(\alpha)} \qquad (|z| = r_s(\alpha) < 1).$$

Now by (13), we have

$$zf'(z) = q(z)(F_\alpha(z) + 1) \qquad |z| = r_s(\alpha) < 1.$$

Since $w(\Delta_{r_s(\alpha)}) \subset \Delta_{r_s(\alpha)}$ and by the maximum principle for harmonic functions, we get

$$\begin{aligned} |f'(z)| &= \frac{|q(z)|}{|z|}|F_\alpha(w(z)) + 1| \\ &\le \frac{1}{1 - r_s^2(\alpha)}(|F_\alpha(w(z))| + 1) \\ &\le \frac{1}{1 - r_s^2(\alpha)}\left(\max_{|z|=r_s(\alpha)} |F_\alpha(w(z))| + 1\right) \\ &\le \frac{1}{1 - r_s^2(\alpha)}(F_\alpha(r_s(\alpha)) + 1). \end{aligned}$$

With the same proof we obtain

$$|f'(z)| \ge \frac{1}{1 + r_s^2(\alpha)}(F_\alpha(r_s(\alpha)) - 1).$$

Since the function f is a univalent function, the inequality for $|f(z)|$ follows from the corresponding inequalities for $|f'(z)|$ by Privalov's Theorem [4, Theorem 7, p. 67]. □

Theorem 4 *Let* $F_\alpha(z)$ *be given by* (2)*. Then we have*

$$\frac{1}{1 + \alpha} \le |F_\alpha(z)| \le \frac{1}{1 - \alpha} \qquad (z \in \Delta - \{0\}, 0 < \alpha < 1). \tag{14}$$

Proof. It is sufficient that to consider $|F_\alpha(z)|$ on the boundary

$$\partial F_\alpha(\Delta) = \left\{F_\alpha(e^{i\theta}) : \theta \in [0, 2\pi]\right\}.$$

A simple check gives us

$$x = \mathfrak{Re}\left\{F_\alpha(e^{i\theta})\right\} = \frac{(1-\alpha)\cos\theta}{1+\alpha^2-2\alpha\cos 2\theta} \tag{15}$$

and

$$y = \mathfrak{Im}\left\{F_\alpha(e^{i\theta})\right\} = \frac{(1+\alpha)\sin\theta}{1+\alpha^2-2\alpha\cos 2\theta}. \tag{16}$$

Therefore, we have

$$\left|F_\alpha(e^{i\theta})\right|^2 = \frac{1}{1+\alpha^2-2\alpha\cos 2\theta} \tag{17}$$

$$= \frac{1}{1+\alpha^2-2\alpha(2t^2-1)} =: H(t) \qquad (t = \cos\theta). \tag{18}$$

Since $0 \le t \le 1$, it is easy to see that $H'(t) \le 0$ when $-1 \le t \le 0$ and $H'(t) \ge 0$ if $0 \le t \le 1$. Thus

$$\frac{1}{(1+\alpha)^2} \le H(t) \le \frac{1}{(1-\alpha)^2} \qquad (-1 \le t < 0)$$

and

$$\frac{1}{(1+\alpha)^2} \le H(t) \le \frac{1}{(1-\alpha)^2} \qquad (0 < t \le 1).$$

This completes the proof. □

A simple consequence of Theorem 4 as follows.

Theorem 5 *If* $f \in \mathcal{BS}(\alpha)$ $(0 < \alpha < 1)$, *then*

$$\frac{1}{1+\alpha} \le \left|\frac{zf'(z)}{f(z)} - 1\right| \le \frac{1}{1-\alpha} \qquad (z \in \Delta).$$

The inequalities are sharp for the function $\tilde{f}$ *defined by* (5).

Proof. By definition of subordination, and by using of Theorem 4, the proof is obvious. For the sharpness of inequalities consider the function $\widetilde{f}$ which defined by (5). It is easy to see that

$$\left|\frac{z\widetilde{f}'(z)}{\widetilde{f}(z)} - 1\right| = \left|\frac{z}{1-\alpha z^2}\right| = |F_\alpha(z)|$$

and concluding the proof. □

The logarithmic coefficients γ_n of $f(z)$ are defined by

$$\log\left\{\frac{f(z)}{z}\right\} = \sum_{n=1}^{\infty} 2\gamma_n z^n \qquad (z \in \Delta). \tag{19}$$

This coefficients play an important role for various estimates in the theory of univalent functions. For example, consider the Koebe function

$$k(z) = \frac{z}{(1-\mu z)^2} \qquad (\mu \in \mathbb{R}).$$

Easily seen that the above function $k(z)$ has logarithmic coefficients $\gamma_n(k) = \mu^n/n$ where $|\mu| = 1$ and $n \geq 1$. Also for $f \in \mathcal{S}$ we have

$$\gamma_1 = \frac{a_2}{2} \quad \text{and} \quad \gamma_2 = \frac{1}{2}\left(a_3 - \frac{a_2^2}{2}\right)$$

and the sharp estimates

$$|\gamma_1| \leq 1 \quad \text{and} \quad |\gamma_2| \leq \frac{1}{2}(1 + 2e^{-2}) \approx 0.635\ldots,$$

hold. Also, sharp inequalities are known for sums involving logarithmic coefficients. For instance, the logarithmic coefficients γ_n of every function $f \in \mathcal{S}$ satisfy the sharp inequality

$$\sum_{n=1}^{\infty} |\gamma_n|^2 \leq \frac{\pi^2}{6} \tag{20}$$

and the equality is attained for the Koebe function (see [3, Theorem 4]).

The following lemma will be useful for the next result.

Lemma 1 (see [5, Theorem 2.1]) *Let* $f \in \mathcal{A}$ *and* $0 \leq \alpha < 1$. *If* $f \in \mathcal{BS}(\alpha)$, *then*

$$\log\frac{f(z)}{z} \prec \int_0^z \frac{P_\alpha(t) - 1}{t} dt \qquad (z \in \Delta), \tag{21}$$

where

$$P_\alpha(z) - 1 = \frac{2}{\pi(1-\alpha)} i \log\left(\frac{1 - e^{\pi i (1-\alpha)^2} z}{1 - z}\right) \qquad (z \in \Delta) \tag{22}$$

and

$$\widetilde{P}_\alpha(z) = \int_0^z \frac{P_\alpha(t) - 1}{t} dt \qquad (z \in \Delta), \tag{23}$$

are convex univalent in Δ.

We remark that an analytic function $P_{\mu,\beta} : \Delta \to \mathbb{C}$ by

$$P_{\mu,\beta}(z) = 1 + \frac{\beta-\mu}{\pi} i \log\left(\frac{1-e^{2\pi i\frac{1-\mu}{\beta-\mu}}z}{1-z}\right), \quad (\mu < 1 < \beta). \tag{24}$$

is a convex univalent function in Δ, and has the form:

$$P_{\mu,\beta}(z) = 1 + \sum_{n=1}^{\infty} B_n z^n,$$

where

$$B_n = \frac{\beta-\mu}{n\pi} i \left(1 - e^{2n\pi i\frac{1-\mu}{\beta-\mu}}\right), \qquad (n = 1, 2, \ldots). \tag{25}$$

The above function $P_{\mu,\beta}(z)$ was introduced by Kuroki and Owa [7] and they proved that $P_{\mu,\beta}$ maps Δ onto a convex domain

$$P_{\mu,\beta}(\Delta) = \{w \in \mathbb{C} : \mu < \mathfrak{Re}\{w\} < \beta\}, \tag{26}$$

conformally. Note that if we take $\mu = 1/(\alpha-1)$ and $\beta = 1/(1-\alpha)$ in (24), then we have the function P_α which defined by (22). Now we have the following result about logarithmic coefficients.

Theorem 6 *Let $f \in \mathcal{A}$ belongs to the class $\mathcal{BS}(\alpha)$ and $0 < \alpha < 1$. Then the logarithmic coefficients of f satisfy the inequality*

$$\sum_{n=1}^{\infty} |\gamma_n|^2 \leq \frac{1}{(1-\alpha)^2}\left[\frac{\pi^2}{45} - \frac{1}{\pi^2}\left(\mathrm{Li}_4\left(e^{\pi(\alpha-2)i}\right) + \mathrm{Li}_4\left(e^{\pi(2-\alpha)i}\right)\right)\right], \tag{27}$$

where Li_4 is as following

$$\mathrm{Li}_4(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^4} = -\frac{1}{2}\int_0^1 \frac{\log^2(1/t)\log(1-tz)}{t} dt. \tag{28}$$

The inequality is sharp.

Proof. If $f \in \mathcal{BS}(\alpha)$, then by using Lemma 1 and with a simple calculation we get

$$\log\frac{f(z)}{z} \prec \sum_{n=1}^{\infty} \frac{2}{\pi n^2(1-\alpha)} i\left(1 - e^{\pi n(2-\alpha)i}\right) z^n \qquad (z \in \Delta). \tag{29}$$

Now, by putting (19) into the last relation we have

$$\sum_{n=1}^{\infty} 2\gamma_n z^n \prec \sum_{n=1}^{\infty} \frac{1}{\pi n^2(1-\alpha)} i\left(1-e^{\pi n(2-\alpha)i}\right) z^n \qquad (z\in\Delta). \tag{30}$$

Again, by Rogosinski's theorem [2, 6.2], we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}|\gamma_n|^2 &\le \sum_{n=1}^{\infty}\left|\frac{1}{\pi n^2(1-\alpha)} i\left(1-e^{\pi n(2-\alpha)i}\right)\right|^2\\ &= \frac{2}{\pi^2(1-\alpha)^2}\left(\sum_{n=1}^{\infty}\frac{1}{n^4}-\sum_{n=1}^{\infty}\frac{\cos\pi(2-\alpha)n}{n^4}\right)\end{aligned}$$

It is a simple exercise to verify that $\sum_{n=1}^{\infty}\frac{1}{n^4}=\pi^4/90$ and

$$\sum_{n=1}^{\infty}\frac{\cos\pi(2-\alpha)n}{n^4}=\frac{1}{2}\left\{\mathrm{Li}_4\left(e^{-i(2-\alpha)\pi}\right)+\mathrm{Li}_4\left(e^{i(2-\alpha)\pi}\right)\right\}$$

and thus the desired inequality (27) follows. For the sharpness of the inequality, consider

$$F(z)=z\exp\widetilde{P}(z). \tag{31}$$

It is easy to see that the function $F(z)$ belongs to the class $\mathcal{BS}(\alpha)$. Also, a simple check gives us

$$\gamma_n(F(z))=\frac{1}{\pi n^2(1-\alpha)} i\left(1-e^{\pi n(2-\alpha)i}\right).$$

Therefore the proof of this theorem is completed. □

Theorem 7 *Let* $f\in\mathcal{BS}(\alpha)$. *Then the logarithmic coefficients of* f *satisfy*

$$|\gamma_n|\le\frac{1}{2n}\quad (n\ge 1).$$

Proof. If $f\in\mathcal{BS}(\alpha)$, then by definition $\mathcal{BS}(\alpha)$, we have

$$\frac{zf'(z)}{f(z)}-1=z\left(\log\left\{\frac{f(z)}{z}\right\}\right)'\prec F_\alpha(z).$$

Thus

$$\sum_{n=1}^{\infty}2n\gamma_n z^n\prec\sum_{n=1}^{\infty}\alpha^{n-1}z^{2n-1}.$$

Applying the Rogosinski theorem [8], we get the inequality $2n|\gamma_n|\le 1$. This completes the proof. □

3 The class $\mathcal{BK}(\alpha)$

In this section we introduce a new class. Our principal definition is the following.

Definition 1 *Let $0 \leq \alpha < 1$ and F_α be defined by (2). Then $f \in \mathcal{A}$ belongs to the class $\mathcal{BK}(\alpha)$ if f satisfies the following:*

$$\frac{zf''(z)}{f'(z)} \prec F_\alpha(z) \qquad (z \in \Delta). \tag{32}$$

Remark 1 *By Alexander's lemma $f \in \mathcal{BK}(\alpha)$, if and only if $zf'(z) \in \mathcal{BS}(\alpha)$. Thus, if $f \in \mathcal{A}$ belongs to the class $\mathcal{BK}(\alpha)$, then*

$$\frac{\alpha}{\alpha - 1} < \mathfrak{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < \frac{2-\alpha}{1-\alpha} \qquad (z \in \Delta).$$

The following theorem provides us a method of finding the members of the class $\mathcal{BK}(\alpha)$.

Theorem 8 *A function $f \in \mathcal{A}$ belongs to the class $\mathcal{BK}(\alpha)$ if and only if there exists a analytic function q, $q(z) \prec F_\alpha(z)$ such that*

$$f(z) = \int_0^z \left(\exp \int_0^\zeta \frac{q(t)}{t}\right) d\zeta. \tag{33}$$

Proof. First, we let $f \in \mathcal{BK}(\alpha)$. Then from (32) and by definition of subordination there exists a function $\omega \in \mathfrak{B}$ such that

$$\frac{zf''(z)}{f'(z)} = F_\alpha(\omega(z)) \qquad (z \in \Delta). \tag{34}$$

Now we define $q(z) = F_\alpha(\omega(z))$ and so $q(z) \prec F_\alpha(z)$. The equation (34) readily gives

$$\{\log f'(z)\}' = \frac{q(z)}{z}$$

and moreover

$$f'(z) = \exp\left(\int_0^\zeta \frac{q(t)}{t} dt\right),$$

which upon integration yields (33). Conversely, by simple calculations we see that if f satisfies (33), then $f \in \mathcal{BK}(\alpha)$ and therefore we omit the details. $\square$

If we apply Theorem 8 with $q(z) = F_\alpha(z)$, then (33) with some easy calculations becomes

$$\hat{f}_\alpha(z) := z + \frac{z^2}{2} + \frac{1}{6}z^3 + \frac{1}{12}\left(\alpha + \frac{1}{2}\right)z^4 + \frac{1}{60}\left(4\alpha + \frac{1}{2}\right)z^5 + \cdots. \tag{35}$$

Theorem 9 *If a function $f(z)$ defined by (1) belongs to the class $\mathcal{BK}(\alpha)$, then*

$$|a_2| \le \frac{1}{2} \quad \text{and} \quad |a_3| \le \frac{1}{6}.$$

The equality occurs for $\hat{f}$ given in (35).

Proof. Assume that $f \in \mathcal{BK}(\alpha)$. Then from (32) we have

$$\frac{zf''(z)}{f'(z)} = \frac{\omega(z)}{1 - \alpha\omega^2(z)}, \tag{36}$$

where $\omega \in \mathfrak{B}$ and has the form $\omega(z) = b_1z + b_2z^2 + b_3z^3 + \cdots$. It is fairly well-known that if $|\omega(z)| = |b_1z + b_2z^2 + b_3z^3 + \cdots| < 1$ $(z \in \Delta)$, then for all $k \in \mathbb{N} = \{1, 2, 3, \ldots\}$ we have $|b_k| \le 1$. Comparing the initial coefficients in (36) gives

$$2a_2 = b_1 \quad \text{and} \quad 6a_3 - 4a_2^2 = b_2. \tag{37}$$

Thus $|a_2| \le 1/2$ and $6a_3 = b_1^2 + b_2$. Since $|b_1|^2 + |b_2| \le 1$, therefore the assertion is obtained. □

Corollary 2 *It is well known that for $\omega(z) = b_1z + b_2z^2 + b_3z^3 + \cdots \in \mathfrak{B}$ for all $\mu \in \mathbb{C}$, we have $|b_2 - \mu b_1^2| \le \max\{1, |\mu|\}$. Therefore the Fekete-Szegö inequality i.e. estimates of $|a_3 - \mu a_2^2|$ for the class $\mathcal{BK}(\alpha)$ is equal to*

$$|a_3 - \mu a_2^2| \le \frac{1}{6}\max\left\{1, \left|\frac{3\mu}{2} - 1\right|\right\} \qquad (\mu \in \mathbb{C}).$$

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors of the manuscript have read and agreed to its content and are accountable for all aspects of the accuracy and integrity of the manuscript.

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