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On the $\mathcal{L}$-duality of a Finsler space with exponential metric $\alpha e^{\beta/\alpha}$

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Abstract. The (α, β)-metrics are the most studied Finsler metrics in Finsler geometry with Randers, Kropina and Matsumoto metrics being the most explored metrics in modern Finsler geometry. The $\mathcal{L}$-dual of Randers, Kropina and Matsumoto space have been introduced in [3, 4, 5], also in recent the $\mathcal{L}$-dual of a Finsler space with special (α, β)-metric and generalized Matsumoto spaces have been introduced in [16, 17] . In this paper, we find the $\mathcal{L}$-dual of a Finsler space with an exponential metric $\alpha e^{\beta/\alpha}$, where α is Riemannian metric and β is a non-zero one form.

1 Introduction

The concept of $\mathcal{L}$-duality between Lagrange and Finsler spaces was introduced by R. Miron [8] in 1987. Since then it has been studied intensively by many Finsler geometers [3, 4, 5]. The $\mathcal{L}$-duals of a Finsler spaces with some special (α, β)-metrics have been obtained in [14, 15]. The concept of Finslerian and Lagrangian structures were introduced in the papers [9, 13] and the theory of higher order Lagrange and Hamilton spaces were discussed in [10, 11, 12]. Further, the geometry of higher order Finsler spaces have been studied in

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[1, 7, 11].
The importance of $\mathcal{L}$-duality is not limited to computing only the dual of some Finsler fundamental functions but many other geometrical problems have been solved by taking the $\mathcal{L}$-duals of Finsler spaces. In fact, duality has been used to solve the complex Zermelo nevigation problem of classifying Randers metrics of constant flag curvature [2] and it has been also used to study the geometry of a Cartan space [4]. In general, duality can be used to solve the geometrical problems of (α, β) metrics. Here, we study the $\mathcal{L}$-dual of the Finsler space associated with the exponential metric $\alpha e^{\beta/\alpha}$, where α is Riemannian metric and β is a non-zero one form.

2 The Legendre transformation

A Finsler space $F^n = (M, F(x, y))$ is said to have an (α, β)-metric if F is a positively homogeneous function of degree one in two variables α and β, where $\alpha^2 = a(y, y) = a_{ij}y^i y^j$, $y = y^i \frac{\partial}{\partial x^i}|_x \in T_x M$, α is Riemannian metric, and $\beta = b_i(x)y^i$ is a 1-form on $\widetilde{TM} = TM \setminus \{0\}$. A Finsler space with the fundamental function:

$$F(x, y) = \alpha(x, y) + \beta(x, y)$$

is called a *Randers space* [6].

A Finsler space having the fundamental function:

$$F(x, y) = \frac{\alpha^2(x, y)}{\beta(x, y)}$$

is called a *Kropina space* and one with

$$F(x, y) = \frac{\alpha^2(x, y)}{\alpha(x, y) - \beta(x, y)}$$

is called a *Matsumoto space.*

A Finsler space with the fundamental function:

$$F(x, y) = \alpha e^{\beta/\alpha} \tag{1}$$

is called a *Finsler space with exponential metric.*

Definition 1 *A Cartan space* C^n *is a pair* (M, H) *which consists of a real* n*-dimensional* C^∞*-manifold* M *and a Hamiltonian function* $H : T^*M \setminus \{0\} \to \mathfrak{R}$*,* *where* (T^*M, π^*, M) *is the cotangent bundle of* M *such that* $H(x,p)$ *has the following properties:*

1. *It is two homogeneous with respect to* $p_i \quad (i = 1, 2, \dots, n)$.
2. *The tensor field* $g^{ij}(x,p) = \frac{1}{2}\frac{\partial^2 H}{\partial p_i \partial p_j}$ *is nondegenerate.*

Let $C^n = (M, K)$ be an n-dimensional Cartan space having the fundamental function $K(x,p)$. We can also consider Cartan spaces having the metric functions of the following forms

$$K(x,p) = \sqrt{a^{ij}(x)p_i p_j} + b^i(x)p_i$$

or

$$K(x,p) = \frac{a^{ij}p_i p_j}{b^i(x)p_i}$$

and we will again call these spaces Randers and Kropina spaces respectively on the cotangent bundle T^*M.

Definition 2 *A regular Lagrangian* $L(x,y)$ *on a domain* $D \subset TM$ *is a real smooth function* $L : D \to R$ *and a regular Hamiltonian* $H(x,p)$ *on a domain* $D^* \subset T^*M$ *is a real smooth function* $H : D^* \to R$ *such that the matrices with entries*

$$g_{ab}(x,y) = \dot{\partial}_a \dot{\partial}_b L(x,y) \quad \text{and}$$
$$g^{*ab}(x,p) = \dot{\partial}^a \dot{\partial}^b H(x,p)$$

are everywhere nondegenerate on D *and* D^* *respectively.*

Examples. (a) Every Finsler space $F^n = (M, F(x,y))$ is a Lagrange manifold with $L = \frac{1}{2}F^2$.

(b) Every Cartan space $C^n = (M, \bar{F}(x,p))$ is a Hamilton manifold with $H = \frac{1}{2}\bar{F}^2$. (Here $\bar{F}$ is positively 1-homogeneous in p_i and the tensor $\bar{g}^{ab} = \frac{1}{2}\dot{\partial}_a \dot{\partial}_b \bar{F}^2$ is nondegenerate).

(c) (M, L) and (M, H) with

$$L(x,y) = \frac{1}{2}a_{ij}(x)y^i y^j + b_i(x)y^i + c(x)$$

and

$$H(x,p) = \frac{1}{2}\bar{a}^{ij}(x)p_i p_j + \bar{b}^i(x)p_i + \bar{c}(x), \quad \text{where} \quad \bar{c} = b_i b^i - c,$$

are Lagrange and Hamilton manifolds respectively (Here $a_{ij}(x), \bar{a}^{ij}$ are the fundamental tensors of Riemannian manifold, b_i are components of covector field, $\bar{b}^i$ are the components of a vector field, C and $\bar{C}$ are the smooth functions on M).

Let $L(x, y)$ be a regular Lagrangian on a domain $D \subset TM$ and let $H(x, p)$ be a regular Hamiltonian on a domain $D^* \subset T^*M$. If $L \in F(D)$ is a differential map, we can consider the fiber derivative of L, locally given by the diffeomorphism between the open set $U \subset D$ and $U^* \subset D^*$

$$\psi(x, y) = (x^i, \dot{\partial}_a L(x, y)),$$

which will be called the *Legendre transformation*.

It is easily seen that L is a regular Lagrangian if and only if ψ is a local diffeomorphism.

In the same manner if $H \in F(D^*)$ the fiber derivative is given locally by

$$\varphi(x, y) = (x^i, \dot{\partial}^a H(x, y)),$$

which is a local diffeomorphism if and only if H is regular.

Let us consider a regular Lagrangian L. Then ψ is a diffeomorphism between the open sets $U \subset D$ and $U^* \subset D^*$. We can define in this case the function:

$$H : U^* \to R, \ H(x, p) = p_a y^a - L(x, y), \tag{2}$$

where $y = (y^a)$ is the solution of the equations $p_a = \dot{\partial}_a L(x, y)$.

Also, if H is a regular Hamiltonian on M, ϕ is a diffeomorphism between same open sets $U^* \subset D^*$ and $U \subset D$, we can consider the function

$$L : U \to R, \ L(x, y) = p_a y^a - H(x, p), \tag{3}$$

where $y = (p_a)$ is the solution of the equations

$$y^a = \dot{\partial}^a H(x, p).$$

The Hamiltonian $H(x, p)$ given by (2) is the *Legendre transformation of the Lagrangian* L and the Lagrangian given by (3) is called the *Legendre transformation of the Hamiltonian* H.

If (M, K) is a Cartan space, then (M, H) is a Hamilton manifold [10, 13], where $H(x, p) = \frac{1}{2}K^2(x, p)$ is 2-homogenous on a domain of T^*M. So we get the following transformation of H on U:

$$L(x, y) = p_a y^a - H(x, p) = H(x, p). \tag{4}$$

Theorem 1 *The scalar field* $L(x, y)$ *given by* (4) *is a positively* 2*-homogeneous regular Lagrangian on* U*.*

Therefore, we get Finsler metric F *of* U*, so that*

$$L = \frac{1}{2}F^2.$$

Thus for the Cartan space (M, K) *we always can locally associate a Finsler space* (M, F) *which will be called the* $\mathcal{L}$*-dual of a Cartan space* $(M, C_{|U^*})$ *vice versa, we can associate, locally, a Cartan space to every Finsler space which will be called the* $\mathcal{L}$ *-dual of a Finsler space* $(M, F_{|U})$*.*

3 The $\mathcal{L}$-dual of a Finsler space with exponential metric

In this case we put $\alpha^2 = y_i y^i$, $b^i = a^{ij} b_j$, $\beta = b_i y^i$, $\beta^* = b^i p_i$, $F^2 = y_i p^i$, $p^i = a^{ij} p_j$, $\alpha^{*2} = p_i p^i = a^{ij} p_i p_j$. we have $F = \alpha e^{\beta/\alpha}$ and

$$\begin{aligned} p_i &= \frac{1}{2}\frac{\partial}{\partial y^i}F^2 = F\frac{\partial}{\partial y^i}F \\ &= F\left(\alpha^i_y e^{\beta/\alpha} + \alpha e^{\beta/\alpha}\frac{\alpha\beta^i_y - \beta\alpha^i_y}{\alpha^2}\right) \\ &= F\left(\frac{y_i}{\alpha}e^{\beta/\alpha} + \alpha e^{\beta/\alpha}\frac{\alpha b_i - \beta\frac{y_i}{\alpha}}{\alpha^2}\right) \\ &= F\left(\frac{y_i}{\alpha^2}F + F\frac{\alpha^2 b_i - \beta y_i}{\alpha^3}\right) \\ &= \frac{F^2}{\alpha^2}\left\{\left(1 - \frac{\beta}{\alpha}\right)y_i + \alpha b_i\right\}. \end{aligned} \tag{5}$$

Contracting (5) with p^i and b^i respectively, we get

$$\begin{aligned} \alpha^{*2} &= \frac{F^2}{\alpha^2}\left\{\left(1 - \frac{\beta}{\alpha}\right)y_i p^i + \alpha b_i p^i\right\} \\ &= \frac{F^2}{\alpha^2}\left\{\left(1 - \frac{\beta}{\alpha}\right)F^2 + \alpha\beta^*\right\}. \end{aligned} \tag{6}$$

and

$$\begin{aligned}\beta^* &= \frac{F^2}{\alpha^2}\left\{\left(1-\frac{\beta}{\alpha}\right)y_i b^i + \alpha b_i b^i\right\}\\ &= \frac{F^2}{\alpha^2}\left\{\left(1-\frac{\beta}{\alpha}\right)\beta + \alpha b^2\right\}.\end{aligned} \tag{7}$$

In [18], for a Finsler (α,β)-metric F on a Manifold M, one constructs a positive function $\phi = \phi(s)$ on $(-b_0; b_0)$ with $\phi(0) = 1$ and $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$, where $\alpha = \sqrt{a_{ij}y^i y^j}$ and $\beta = b_i y^i$ with $\|\beta\|_x < b_0, \forall x \in M$. The function ϕ satisfies $\phi(s) - s\phi^{'}(s) + (b^2 - s^2)\phi^{''}(s) > 0, (|s| \leq b_0)$.
This mertic is a (α,β)-metric with $\phi = e^s$.

Using Shen's notation [18], put $s = \frac{\beta}{\alpha}$ and $\phi(s) = \frac{F}{\alpha} = e^s$ in (6) and (7), we get

$$\begin{aligned}\alpha^{*2} &= \frac{F^2}{\alpha}\left\{\left(1-\frac{\beta}{\alpha}\right)\frac{F^2}{\alpha} + \beta^*\right\}\\ &= Fe^s\{(1-s)Fe^s + \beta\}\end{aligned} \tag{8}$$

and

$$\beta^* = Fe^s\left\{(1-s)s + b^2\right\} \tag{9}$$

Now, we have the following two theorems under two different cases:

Theorem 2 *Let* (M, F) *be a special Finsler space, where* F *is given by the equation* (1). *If* $b^2 = a_{ij}b^i b^j = 1$, *then the* $\mathcal{L}$*-dual of* (M, F) *is the space on* T^*M *having the fundamental function* $H(x, p)$ *given by the equations* (16).

Proof. From the equation (9), we get

$$F = \frac{\beta^*}{e^s\{(1-s)s+1\}} \tag{10}$$

and substituting F from the equation (10) in (8), we get

$$\alpha^{*2} = \frac{\beta^*}{\{(1-s)s+1\}}[(1-s)\frac{\beta^*}{\{(1-s)s+1\}} + \beta] \tag{11}$$

which implies that

$$\begin{aligned}&(1+s-s^2)^2 - \delta(2-s^2) = 0\\ \text{or}\quad & s^4 - 2s^3 + (-1+\delta)s^2 + 2s + 1 - 2\delta = 0,\end{aligned} \tag{12}$$

where

$$\delta = \frac{\beta^{*2}}{\alpha^{*2}}.$$

Using Mathematica for solving the above equation (12), we get

$$s = (1 \pm \gamma_i)/2, \quad i = 1, 2 \tag{13}$$

where

$$\begin{aligned}
m_1 &= (1-\delta)/3, \\
m_2 &= 25 - 26\delta + \delta^2, \\
m_3 &= 125 - 195\delta + 69\delta^2 + \delta^3, \\
m_4 &= \delta\sqrt{25 - 48\delta + 21\delta^2 + 2\delta^3}, \\
m_5 &= m_3 + 3\sqrt{3}m_4{}^{1/3}, \\
m_6 &= \frac{m_2}{3m_5}, \\
m_7 &= \sqrt{2 - \delta - m_1 + m_7 + \frac{m_5}{3}}, \\
m_8 &= \sqrt{3 - \delta + m_1 - m_5 + m_6 + \frac{8\delta}{m_7}}, \\
\gamma_1 &= m_7 + m_8, \\
\text{and} \quad \gamma_2 &= m_7 - m_8.
\end{aligned}$$

From (10) and (13), we get

$$F = \frac{\beta^*}{e^{(1\pm\gamma_i)/2}\left\{1 + \frac{1\pm\gamma_i}{2} - (\frac{1\pm\gamma_i}{2})^2\right\}}. \tag{14}$$

As we know that $H(x,p) = \frac{1}{2}F^2$, therefore, by using the equation (14), we get

$$H(x,p) = \frac{\beta^{*2}}{e^{(1\pm\gamma_i)}\left\{1 + \frac{1\pm\gamma_i}{2} - (\frac{1\pm\gamma_i}{2})^2\right\}^2}, \tag{15}$$

putting $\beta^* = b^j p_j$, in equation (15), we get

$$H(x,p) = \frac{(b^j p_j)^2}{e^{(1\pm\gamma_i)}\left\{1 + \frac{1\pm\gamma_i}{2} - (\frac{1\pm\gamma_i}{2})^2\right\}^2}. \tag{16}$$

□

Theorem 3 *Let* (M,F) *be a special Finsler space, where* F *is given by the equation* (1)*. If* $b^2 = a_{ij}b^i b^j \neq 1$, *then the* $\mathcal{L}$*-dual of* (M,F) *is the space on* T^*M *having the fundamental function* $H(x,p)$ *given by the equations* (23)*.*

Proof. From (9), we get

$$F = \frac{\beta^*}{e^s\{(1-s)s + b^2\}}. \tag{17}$$

Substituting F from the equation (17) in (8), we get

$$\alpha^{*2} = \frac{\beta^*}{\{(1-s)s + b^2\}}[(1-s)\frac{\beta^*}{\{(1-s)s + b^2\}} + \beta] \tag{18}$$

which implies that

$$(b^2 + s - s^2)^2 - \delta(1 + b^2 - s^2) = 0$$
$$\text{or}\quad s^4 - 2s^3 + (1 - 2b^2 + \delta)s^2 + 2b^2 s + b^4 - (1+b^2)\delta = 0, \tag{19}$$

where

$$\delta = \frac{\beta^{*2}}{\alpha^{*2}}.$$

Using Mathematica for solving the above equation (19), we get

$$s = (1 \pm \overline{\gamma}_i)/2, \quad i = 1, 2 \tag{20}$$

where

$$
\begin{aligned}
n_1 &= (1 - 2b^2 + \delta)/3, \\
n_2 &= 1 - 10\delta + \delta^2 + 8(1 - 2\delta)b^2 + 16b^4, \\
n_3 &= 2(1 - 15\delta + 39\delta^2 + \delta^3) + (24 - 168\delta + 60\delta^2)b^2 \\
&\quad + (96 - 192\delta)b^2 + 128b^6, \\
n_4 &= 432\delta^3(-1 + 7\delta + \delta^2) + 432(1 - 19\delta + \delta^2 + \delta^3)\delta^2 b^2 \\
&\quad + 432(8 - 28\delta - \delta^2)\delta^2 b^4 + 6912\delta^2 b^6, \\
n_5 &= 8(1 - 2b^2 - 3n_1), \\
n_6 &= n_1 + \frac{2^{1/3} n_2}{3(n_3 + \sqrt{n_4})}, \\
n_7 &= \sqrt{2b^2 - \delta + n_7}, \\
n_8 &= 1 + 2b^2 - \delta - n_7, \\
n_9 &= \frac{n_5}{4n_7}, \\
\overline{\gamma}_1 &= n_7 + \sqrt{n_8 - n_9} \\
\text{and} \quad \overline{\gamma}_2 &= n_7 - \sqrt{n_8 - n_9}.
\end{aligned}
$$

From (17) and (20), we get

$$F = \frac{\beta^*}{e^{(1 \pm \overline{\gamma}_i)/2} \left\{ b_2 + \dfrac{1 \pm \overline{\gamma}_i}{2} - (\dfrac{1 \pm \overline{\gamma}_i}{2})^2 \right\}}. \tag{21}$$

As we know that $H(x, p) = \frac{1}{2}F^2$, therefore by using (21), we get

$$H(x, p) = \frac{\beta^{*2}}{e^{(1 \pm \overline{\gamma}_i)} \left\{ 1 + \dfrac{1 \pm \overline{\gamma}_i}{2} - (\dfrac{1 \pm \overline{\gamma}_i}{2})^2 \right\}^2}, \tag{22}$$

putting $\beta^* = b^j p_j$, in equation (22), we get

$$H(x, p) = \frac{(b^j p_j)^2}{e^{(1 \pm \overline{\gamma}_i)} \left\{ 1 + \dfrac{1 \pm \overline{\gamma}_i}{2} - (\dfrac{1 \pm \overline{\gamma}_i}{2})^2 \right\}^2}. \tag{23}$$

□

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