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Certain classes of analytic functions defined by fractional q-calculus operator

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Abstract. In this paper, the concept of fractional q- calculus and generalized Al-Oboudi differential operator defining certain classes of analytic functions in the open disc are used. The results investigated for these classes of functions include the coefficient estimates, inclusion relations, extreme points and some more properties.

1 Introduction

Let $\mathcal{A}$ denote the class of all analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

defined in the unit disc $\mathcal{U} = \{z : |z| < 1\}$.

Let $\mathcal{T}$ denote the subclass of $\mathcal{A}$ in $\mathcal{U}$, consisting of analytic functions whose non-zero coefficients from the second onwards are negative. That is, an analytic function $f \in \mathcal{T}$ if it has a Taylor expansion of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \quad (a_k \geq 0) \tag{2}$$

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which are analytic in the open disc $\mathcal{U}$.

The q-shifted factorial is defined for $\alpha,\ q \in \mathbb{C}$ as a product of n factors by

$$(\alpha, q)_n = \begin{cases} 1, & n{=}0; \\ (1-\alpha)(1-\alpha q)\cdots(1-\alpha q^{n-1}), & n \in \mathbb{N}, \end{cases} \tag{3}$$

and in terms of the basic analogue of the gamma function

$$(q^{\alpha}; q)_n = \frac{\Gamma_q(\alpha+n)(1-q)^n}{\Gamma_q(\alpha)},\ (n > 0), \tag{4}$$

where the q-gamma functions [4, 5] is defined by

$$\Gamma_q(x) = \frac{(q;q)_\infty (1-q)^{1-x}}{(q^x;q)_\infty}\ \ (0 < q < 1). \tag{5}$$

Note that, if $|q| < 1$, the q-shifted factorial (3) remains meaningful for $n = \infty$ as a convergent infinite product

$$(\alpha; q)_\infty = \prod_{m=0}^{\infty}(1-\alpha q^m).$$

Now recall the following q-analogue definitions given by Gasper and Rahman [4]. The recurrence relation for q-gamma function is given by

$$\Gamma_q(x+1) = [x]_q \Gamma_q(x), \text{ where, } [x]_q = \frac{(1-q^x)}{(1-q)}, \tag{6}$$

and called q-analogue of x.

Jackson's q-derivative and q-integral of a function f defined on a subset of $\mathbb{C}$ are, respectively, given by (see Gasper and Rahman [4])

$$D_q f(z) = \frac{f(z) - f(zq)}{z(1-q)},\ \ (z \neq 0,\ q \neq 0). \tag{7}$$

$$\int_0^z f(t) d_q(t) = z(1-z) \sum_{m=0}^{\infty} q^m f(zq^m). \tag{8}$$

In view of the relation

$$\lim_{q \to 1^-} \frac{(q^{\alpha}; q)_n}{(1-q)^n} = (\alpha)_n, \tag{9}$$

we observe that the q-shifted fractional (2) reduces to the familiar Pochhammer symbol $(\alpha)_n$, where $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n+1)$.

Now recall the definition of the fractional q-calculus operators of a complex-valued function $f(z)$, which were recently studied by Purohit and Raina [7].

Definition 1 *(fractional q-integral operator). The fractional q-integral operator $I^{\delta}_{q,z}$ of a function $f(z)$ of order $\delta\,(\delta>0)$ is defined by*

$$I^{\delta}_{q,z}=D^{-\delta}_{q,z}f(z)=\frac{1}{\Gamma_q(\delta)}\int_0^z(z-tq)_{1-\delta}f(t)d_qt, \tag{10}$$

where $f(z)$ is a analytic in a simply connected region in the z-plane containing the origin. Here, the term $(z-tq)_{\delta-1}$ is a q-binomial function defined by

$$(z-tq)_{\delta-1}=z^{\delta-1}\prod_{m=0}^{\infty}\left[\frac{1-(\frac{tq}{z})q^m}{1-(\frac{tq}{z})q^{\delta}+m-1}\right] \tag{11}$$

$$=z^{\delta}\,{}_1\phi_0\left[q^{-\delta+1};-;q,\frac{tq^{\delta}}{z}\right].$$

According to Gasper and Rahman [4], the series ${}_1\phi_0[\delta;-;q,z]$ is single-valued when $|\arg(z)|<\pi$. Therefore, the function $(z-tq)_{\delta-1}$ in (11) is single-valued when $|\arg(\frac{-tq^{\delta}}{z}|<\pi$, $|tq^{\frac{\delta}{z}}|<1$, and $|\arg(z)|<\pi$.

Definition 2 *(fractional q-derivative operator). The fractional q-derivative operator $D^{\delta}_{q,z}$ of a $f(z)$ of order $\delta(0\le\delta<1)$ is defined by*

$$D^{\delta}_{q,z}f(z)=D_{q,z}I^{1-\delta}_{q,z}f(z)=\frac{1}{\Gamma_q(1-\delta)}D_q\int_0^z(z-tq)_{-\delta}f(t)d_qt, \tag{12}$$

where $f(z)$ is suitably constrained and the multiplicity of $(z-tq)_{-\delta}$ is removed as in Definition 1 above.

Definition 3 *(extended fractional q-derivative operator). Under the hypotheses of Definition 2, the fractional q-derivative for the function $f(z)$ of order δ is defined by*

$$D^{\delta}_{q,z}f(z)=D^{n}_{q,z}I^{n-\delta}_{q,z}f(z), \tag{13}$$

where, $n-1\le\delta<n$, $n\in\mathbb{N}_0$.

The fractional q-defferintegral operator is defined by $\Omega^{\delta}_{q,z}f(z)$ for the function $f(z)$ of the form (1),

$$\Omega^{\delta}_{q}f(z)=\Gamma_q(2-\delta)z^{\delta}D^{\delta}_{q,z}f(z)=z+\sum_{k=2}^{\infty}\frac{\Gamma_q(k+1)\Gamma_q(2-\delta)}{\Gamma_q(k+1-\delta)}a_kz^k, \tag{14}$$

where $D^{\delta}_{q,z}$ in (14), represents, respectively, a fractional q-integral of $f(z)$ of order δ when $-\infty<\delta<0$ and a fractional q-derivative of $f(z)$ of order δ when $0<\delta<2$.

A linear multiplier fractional q-differintegral operator is defined as

$$\begin{aligned}
&\mathcal{D}_{q,\lambda}^{\delta,0}f(z) = f(z),\\
&\mathcal{D}_{q,\lambda}^{\delta,1}f(z) = (1-\lambda)\Omega_q^{\delta}f(z) + \lambda z D_q(\Omega_q^{\delta}f(z)),\\
&\mathcal{D}_{q,\lambda}^{\delta,2}f(z) = \mathcal{D}_{q,\lambda}^{\delta,1}(\mathcal{D}_{q,\lambda}^{\delta,1}f(z))\\
&\vdots
\end{aligned}$$

$$\mathcal{D}_{q,\lambda}^{\delta,n}f(z) = \mathcal{D}_{q,\lambda}^{\delta,1}(\mathcal{D}_{q,\lambda}^{\delta,n-1}f(z)). \tag{15}$$

We note that if $f \in \mathcal{A}$ is given by (1), then by (15), we have

$$\mathcal{D}_{q,\lambda}^{\delta,n}f(z) = z + \sum_{k=2}^{\infty} B(k,\ \delta,\ \lambda, n,\ q)a_k z^k, \tag{16}$$

where

$$B(k,\ \delta,\ \lambda, n,\ q) = \left(\frac{\Gamma_q(2-\delta)\Gamma_q(k+1)}{\Gamma_q(k+1-\delta)}[([k]_q-1)\lambda+1]\right)^n. \tag{17}$$

It can be seen that, by specializing the parameters, the operator $\mathcal{D}_{q,\lambda}^{\delta,n}$ reduces tomany known and new integral and differential operators. In particular, when $\delta = 0$, and $q \to 1^-$ the operator $\mathcal{D}_{q,\lambda}^{\delta,n}$ reduces to the operator introduced by AL-Oboudi [1] and if $\delta = 0$, $\lambda = 1$ and $q \to 1^-$ and it reduces to the operator introduced by Sălăgean [9].

Now using above differential operator, we define the following subclass of $\mathcal{T}$.

Let $\mathcal{T}_q^n(\alpha, \beta, \delta, \lambda)$ be the subclass of $\mathcal{T}$ consisting of functions which satisfy the conditions

$$\Re\left\{\frac{zD_q(\mathcal{D}_{q,\lambda}^{\delta,n}f)}{\beta z D_q(\mathcal{D}_{q,\lambda}^{\delta,n}f) + (1-\beta)\mathcal{D}_{q,\lambda}^{\delta,n}f}\right\} > \alpha, \tag{18}$$

for some $\alpha,\ \beta\ (0 \le \alpha, \beta < 1)$, $\delta \le 2$, $\lambda > 0$ and $n \in \mathbf{N}_0$.

In particular, if $\delta = 0$, and $q \to 1^-$ we get the classes studied by Ravikumar, Dileep and Latha [8] and if $\delta = 0$, and $q \to 1^-$ and different parametric of values n we get the classes studied by Mostafa [6], Altintas and Owa [2].

2 Main results

Theorem 1 *A function* $f(z)$ *defined by* (2) *is in the class* $\mathcal{T}_q^n(\alpha,\beta,\delta,\lambda)$ *if and only if*

$$\sum_{k=2}^{\infty} B(k,\ \delta,\ \lambda, n,\ q)a_k[(1-\alpha\beta)[k]_q+\alpha\beta-\alpha]<1-\alpha, \tag{19}$$

where, $B(k,\ \delta,\ \lambda, n,\ q)$ *is defined in* (17), $\alpha,\ \beta\quad (0\leq\alpha,\beta<1), \lambda>0$ *and* $n\in\mathbf{N}_0$.

Proof. Suppose $f\in\mathcal{T}_q^n(\alpha,\beta,\delta,\lambda)$. Then

$$\Re\left\{\frac{zD_q(\mathcal{D}_{q,\lambda}^{\delta,n}f)}{\beta zD_q(\mathcal{D}_{q,\lambda}^{\delta,n}f)+(1-\beta)\mathcal{D}_{q,\lambda}^{\delta,n}f}\right\}>\alpha,$$

$$\Re\left\{\frac{z-\sum\limits_{k=2}^{\infty}B(k,\delta,\lambda,n,\ q)[k]_q a_k z^k}{\beta[z-\sum\limits_{k=2}^{\infty}B(k,\delta,\lambda,n,\ q)[k]_q a_k z^k]+(1-\beta)[z-\sum\limits_{k=2}^{\infty}B(k,\delta,\lambda,n,\ q)a_k z^k]}\right\}>\alpha,$$

$$\Re\left\{\frac{z-\sum\limits_{k=2}^{\infty}B(k,\ \delta,\ \lambda,n,\ q)[k]_q a_k z^k}{z-\sum\limits_{k=2}^{\infty}B(k,\ \delta,\ \lambda,n,\ q)a_k z^k[\beta([k]_q-1)+1]}\right\}>\alpha.$$

Letting $z\to 1$, we get,

$$1-\sum_{k=2}^{\infty}B(k,\ \delta,\ \lambda,n,\ q)[k]_q a_k>\alpha\left\{1-\sum_{k=2}^{\infty}B(k,\ \delta,\ \lambda,n,\ q)a_k[\beta([k]_q-1)+1]\right\}.$$

Equivalenty we have,

$$\sum_{k=2}^{\infty}B(k,\ \delta,\ \lambda,n,\ q)[k]_q a_k-\alpha\left\{\sum_{k=2}^{\infty}B(k,\ \delta,\ \lambda,n,\ q)a_k[\beta([k]_q-1)+1]\right\}<(1-\alpha)$$

which implies

$$\sum_{k=2}^{\infty}B(k,\ \delta,\ \lambda,n,\ q)a_k[(1-\alpha\beta)[k]_q+\alpha\beta-\alpha]<(1-\alpha).$$

Conversely, assume that (2.1) is be true. We have to show that (6) is satisfied or equivalently

$$\left| \left\{ \frac{zD_q(\mathcal{D}^{\delta,n}_{q,\lambda} f)}{\beta z D_q(\mathcal{D}^{\delta,n}_{q,\lambda} f) + (1-\beta)\mathcal{D}^{\delta,n}_{q,\lambda} f} \right\} - 1 \right| < 1 - \alpha.$$

But

$$\left| \left\{ \frac{z - \sum_{k=2}^{\infty} B(k,\delta,\lambda,n,q)[k]_q a_k z^k}{z - \sum_{k=2}^{\infty} B(k,\delta,\lambda,n,q) a_k z^k [\beta([k]_q - 1) + 1]} \right\} - 1 \right|$$

$$= \left| \frac{\sum_{k=2}^{\infty} B(k,\delta,\lambda,n,q) a_k([k]_q - 1)(\beta - 1) z^k}{z - \sum_{k=2}^{\infty} B(k,\delta,\lambda,n,q) a_k [\beta([k]_q - 1) + 1] z^k} \right|$$

$$\leq \frac{\sum_{k=2}^{\infty} B(k,\delta,\lambda,n,q) a_k([k]_q - 1)(\beta - 1)|z^k|}{|z| - \sum_{k=2}^{\infty} B(k,\delta,\lambda,n,q) a_k [\beta([k]_q - 1) + 1]|z^k|}$$

$$\leq \frac{\sum_{k=2}^{\infty} B(k,\delta,\lambda,n,q) a_k([k]_q - 1)(\beta - 1)}{1 - \sum_{k=2}^{\infty} B(k,\delta,\lambda,n,q) a_k [\beta([k]_q - 1) + 1]}.$$

The last expression is bounded above by $1 - \alpha$ if

$$\sum_{k=2}^{\infty} B(k,\delta,\lambda,n,q) a_k([k]_q - 1)(\beta - 1)$$
$$\leq (1-\alpha)\Big(1 - \sum_{k=2}^{\infty} B(k,\delta,\lambda,n,q) a_k [\beta([k]_q - 1) + 1]\Big)$$

or

$$\sum_{k=2}^{\infty} B(k,\delta,\lambda,n,q) a_k [(1-\alpha\beta)[k]_q + \alpha\beta - \alpha] < 1 - \alpha,$$

which is true by hypothesis. This completes the assertion of Theorem 1. $\square$

Corollary 2 *If* $f \in \mathcal{T}_q^n(\alpha, \beta, \delta, \lambda)$, *then*

$$|a_k| \le \frac{1-\alpha}{B(k,\ \delta,\ \lambda, n,\ q)[(1-\alpha\beta)[k]_q + \alpha\beta - \alpha]}.$$

Theorem 3 *Let* $0 \le \alpha < 1, \quad 0 \le \beta_1 \le \beta_2 < 1, \quad n \in \mathbb{N}_0,$ *then* $\mathcal{T}_q^n(\alpha, \beta_1, \delta, \lambda) \subset \mathcal{T}_q^n(\alpha, \beta_2, \delta, \lambda)$.

Proof. For $f(z) \in \mathcal{T}_q^n(\alpha, \beta_2, \delta, \lambda)$. We have,

$$\sum_{k=2}^{\infty} B(k,\ \delta,\ \lambda, n,\ q) a_k [(1-\alpha\beta_2)[k]_q + \alpha\beta_2 - \alpha]$$
$$\le \sum_{k=2}^{\infty} B(k,\ \delta,\ \lambda, n,\ q) a_k [(1-\alpha\beta_1)[k]_q + \alpha\beta_1 - \alpha] < 1-\alpha.$$

Hence $f(z) \in \mathcal{T}_q^n(\alpha, \beta_1, \delta, \lambda)$. $\square$

Theorem 4 *Let* $f(z) \in \mathcal{T}_q^n(\alpha, \beta, \delta, \lambda)$. *Define* $f_1(z) = z$ *and*

$$f_k(z) = z + \frac{1-\alpha}{B(k,\ \delta,\ \lambda, n,\ q)[(1-\alpha\beta)[k]_q + \alpha\beta - \alpha]} z^k, \quad k = 2, 3, \cdots,$$

for some $\alpha,\ \beta\,(0 \le \beta < 1), n \in \mathbb{N}_0,\ \lambda > 0$ *and* $z \in \mathcal{U}$. *Then* $f(z) \in \mathcal{T}_q^n(\alpha, \beta, \delta, \lambda)$ *if and only if* $f(z)$ *can be expressed as* $f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z)$ *where* $\mu_k \ge 0$ *and* $\sum_{k=1}^{\infty} \mu_k = 1$.

Proof. If $f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z)$ with $\sum_{k=1}^{\infty} \mu_k = 1, \quad \mu_k \ge 0$, then

$$\sum_{k=2}^{\infty} \frac{B(k,\ \delta,\ \lambda, n,\ q)[(1-\alpha\beta)[k]_q + \alpha\beta - \alpha]\mu_k}{B(k,\ \delta,\ \lambda, n,\ q)[(1-\alpha\beta)[k]_q + \alpha\beta - \alpha]}(1-\alpha) \sum_{k=2}^{\infty} \mu_k (1-\alpha)$$
$$= (1-\mu_1)(1-\alpha) \le (1-\alpha).$$

Hence $f(z) \in \mathcal{T}_q^n(\alpha, \beta, \delta, \lambda)$.

Conversely, let $f(z) = z - \sum_{k=2}^{\infty} a_k z^k \in \mathcal{T}_q^n(\alpha, \beta, \delta, \lambda)$, define

$$\mu_k = \frac{B(k,\ \delta,\ \lambda, n,\ q)\,[(1-\alpha\beta)[k]_q + \alpha\beta - \alpha]\,|a_k|}{(1-\alpha)}, \quad k = 2, 3, \cdots,$$

and define $\mu_1 = 1 - \sum_{k=2}^{\infty} \mu_k$. From Theorem (1), $\sum_{k=2}^{\infty} \mu_k \leq 1$ and hence $\mu_1 \geq 0$.

Since $\mu_k f_k(z) = \mu_k f(z) + a_k z^k$, $\sum_{k=1}^{\infty} \mu_k f_k(z) = z - \sum_{k=2}^{\infty} a_k z^k = f(z)$. □

Theorem 5 *The class* $\mathcal{T}_q^n(\alpha, \beta, \delta, \lambda)$ *is closed under convex linear combination.*

Proof. Let $f(z)$, $g(z) \in \mathcal{T}_q^n(\alpha, \beta, \delta, \lambda)$ and let

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = z - \sum_{k=2}^{\infty} b_k z^k.$$

For η such that $0 \leq \eta \leq 1$, it suffices to show that the function defined by $h(z) = (1-\eta) f(z) + \eta g(z), \quad z \in \mathcal{U}$ belongs to $\mathcal{T}_q^n(\alpha, \beta, \delta, \lambda)$. Now

$$h(z) = z - \sum_{k=2}^{\infty} [(1-\eta) a_k + \eta b_k] z^k.$$

Applying Theorem 1, to $f(z)$, $g(z) \in \mathcal{T}_q^n(\alpha, \beta, \delta, \lambda)$, we have

$$\begin{aligned}
&\sum_{k=2}^{\infty} B(k,\ \delta,\ \lambda, n,\ q)[(1-\alpha\beta)[k]_q + \alpha\beta - \alpha]\,[(1-\eta) a_k + \eta b_k] \\
&= (1-\eta) \sum_{k=2}^{\infty} B(k,\ \delta,\ \lambda, n,\ q)[(1-\alpha\beta)[k]_q + \alpha\beta - \alpha] a_k \\
&\quad + \eta \sum_{k=2}^{\infty} B(k,\ \delta,\ \lambda, n,\ q)[(1-\alpha\beta)[k]_q + \alpha\beta - \alpha] b_k \\
&\leq (1-\eta)(1-\alpha) + \eta(1-\alpha) = (1-\alpha).
\end{aligned}$$

This implies that $h(z) \in \mathcal{T}_q^n(\alpha, \beta, \delta, \lambda)$. □

Corollary 6 *If* $f_1(z)$, $f_2(z)$ *are in* $\mathcal{T}_q^n(\alpha,\beta,\delta,\lambda)$ *then the function defined by* $g(z)=\frac{1}{2}[f_1(z)+f_2(z)]$ *is also in* $\mathcal{T}_q^n(\alpha,\beta,\delta,\lambda)$.

Theorem 7 *Let for* $j=1,2,\cdots,k,\quad f_j(z)=z-\sum_{k=2}^{\infty}a_{k,j}z^k\in\mathcal{T}_q^n(\alpha,\beta,\delta,\lambda)$ *and* $0<\beta_j<1$ *such that* $\sum_{j=1}^{k}\beta_j=1$, *then the function* $F(z)$ *defined by* $F(z)=\sum_{j=1}^{k}\beta_j f_j(z)$ *is also in* $\mathcal{T}_q^n(\alpha,\beta,\delta,\lambda)$.

Proof. For each $j\in\{1,2,3,\cdots,k\}$ we obtain

$$\sum_{k=2}^{\infty}B(k,\,\delta,\,\lambda,n,\,q)[(1-\alpha\beta)[k]_q+\alpha\beta-\alpha]|a_k|<(1-\alpha).$$

$$F(z)=\sum_{j=1}^{k}\beta_j\left(z-\sum_{k=2}^{\infty}a_{k,j}z^k\right)=z-\sum_{k=2}^{\infty}\left(\sum_{j=1}^{k}\beta_j a_{k,j}\right)z^k$$

$$\cdot\sum_{k=2}^{\infty}B(k,\,\delta,\,\lambda,n,\,q)[(1-\alpha\beta)[k]_q+\alpha\beta-\alpha]\left[\sum_{j=1}^{k}\beta_j a_{k,j}\right]$$

$$=\sum_{j=1}^{k}\beta_j\left[\sum_{k=2}^{\infty}B(k,\,\delta,\,\lambda,n,\,q)[(1-\alpha\beta)[k]_q+\alpha\beta-\alpha]\right]$$

$$<\sum_{j=1}^{k}\beta_j(1-\alpha)<(1-\alpha).$$

Therefore $F(z)\in\mathcal{T}_q^n(\alpha,\beta,\delta,\lambda)$. □

Bernardi Libera's integral operator is defined as

$$L_\gamma f(z)=\frac{\gamma+1}{z^\gamma}\int_0^z t^{\gamma-1}f(t)dt,$$

which was studied by Bernardi in [3].

Theorem 8 *Let* $f(z)\in\mathcal{T}_q^n(\alpha,\beta,\delta,\lambda)$. *The q-analogous Bernardi's integral operator defined by* $L_{q,\gamma}f(z)=\frac{[\gamma+1]_q}{z^\gamma}\int_0^z t^{\gamma-1}f(t)d_qt$ *then* $L_{q,\gamma}f(z)\in\mathcal{T}_q^n(\alpha,\beta,\delta,\lambda)$.

Proof. We have

$$\begin{aligned}
L_{q,\gamma}f(z) &= \frac{[\gamma+1]_q}{z^\gamma} z(1-q)\sum_{j=0}^{\infty} q^j (zq^j)^{\gamma-1} f(zq^j) \\
&= [\gamma+1]_q(1-q)\sum_{j=0}^{\infty} q^{j\gamma} f(zq^j) \\
&= [\gamma+1]_q(1-q)\sum_{j=0}^{\infty} q^{j\gamma}\sum_{k=1}^{\infty} q^{jk} a_k z^k, \\
&= [\gamma+1]_q\sum_{j=0}^{\infty}\sum_{k=1}^{\infty}(1-q)q^{j(\gamma+k)} a_k z^k \\
&= z-\sum_{k=2}^{\infty}\frac{[\gamma+1]_q}{[\gamma+k]_q} a_k z^k.
\end{aligned}$$

Since $f \in \mathcal{T}_q^n(\alpha,\beta,\delta,\lambda)$ and since $\dfrac{[\gamma+1]_q}{[\gamma+k]_q} < 1$, we have

$$\sum_{k=2}^{\infty} B(k,\ \delta,\ \lambda, n,\ q)[(1-\alpha\beta)[k]_q+\alpha\beta-\alpha]\frac{[\gamma+1]_q}{[\gamma+k]_q} a_k < (1-\alpha). \qquad \square$$

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