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# On real valued $\omega$-continuous functions

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**Abstract.** The aim of this paper is to introduce and study upper and lower $\omega$-continuous functions. Some characterizations and several properties concerning upper (resp. lower) $\omega$-continuous functions are obtained.

## 1 Introduction

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologist worldwide. Indeed a significant theme in General Topology and Real analysis concerns the various modified forms of continuity, separation axioms etc. by utilizing generalized open sets. Recently, as generalization of closed sets, the notion of $\omega$-closed sets were introduced and studied by Hdeib [4]. Several characterizations and properties of $\omega$-closed sets were provided in [1, 2, 3, 4, 5]. Various types of functions play a significant role in the theory of classical point set topology. A great number of papers dealing with such functions have appeared, and a good many of them

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have been extended to the setting of multifunction. The purpose of this paper is to define upper and lower $\omega$-continuous functions. Also, some characterizations and several properties concerning upper (lower) $\omega$-continuous functions are obtained.

## 2 Preliminaries

Throughout this paper, spaces $(X,\tau)$ and $(Y,\sigma)$ (or simply $X$ and $Y$) always mean topological spaces in which no separation axioms are assumed unless explicitly stated. Let $A$ be a subset of a space $X$. For a subset $A$ of $(X,\tau)$, $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ denote the closure of $A$ with respect to $\tau$ and the interior of $A$ with respect to $\tau$, respectively. A point $x \in X$ is called a condensation point of $A$ if for each $U \in \tau$ with $x \in U$, the set $U \cap A$ is uncountable. $A$ is said to be $\omega$-closed [4] if it contains all its condensation points. The complement of an $\omega$-closed set is said to be an $\omega$-open set. It is well known that a subset $W$ of a space $(X,\tau)$ is $\omega$-open if and only if for each $x \in W$, there exists $U \in \tau$ such that $x \in U$ and $U\backslash W$ is countable. The intersection (resp. union) of all $\omega$-closed (resp. $\omega$-open) set containing (resp. contained in) $A \subset X$ is called the $\omega$-closure (resp. $\omega$-interior) of $A$ and is denoted by $\omega\,\mathrm{Cl}(A)$ (resp. $\omega\,\mathrm{Int}(A)$). The family of all $\omega$-open, $\omega$-closed sets of $(X,\tau)$ is, respectively denoted by $\omega O(X)$, $\omega C(X)$. We set $\omega O(X,x) = \{A : A \in \omega O(X) \text{ and } x \in A\}$ and $\omega C(X,x) = \{A : A \in \omega C(X) \text{ and } x \in A\}$. The $\omega$-$\theta$-closure [3] of $A$, denoted by $\omega\,\mathrm{Cl}_\theta(A)$, is defined to be the set of all $x \in X$ such that $A \cap \omega\,\mathrm{Cl}(U) \neq \varnothing$ for every $U \in \omega O(X,x)$. A subset $A$ is called $\omega$-$\theta$-closed [3] if and only if $A = \omega\,\mathrm{Cl}_\theta(A)$. The complement of $\omega$-$\theta$-closed set is called $\omega$-$\theta$-open. A subset $A$ is called $\omega$-regular if and only if it is $\omega$-$\theta$-open and $\omega$-$\theta$-closed. The family of all $\omega$-regular sets of $(X,\tau)$ is denoted by $\omega R(X)$. We set $\omega R(x) = \{A : A \in \omega R(X)$ and $x \in A\}$. A topological space $X$ is said to be $\omega$-closed if every cover of $X$ by $\omega$-open sets has a finite subcover whose $\omega$-closures cover $X$. Finally we recall that a function $f : (X,\tau) \to (Y,\sigma)$ is $\omega$-continuous at the point $x \in X$ if for each open set $V$ of $Y$ containing $f(x)$ there exists an $\omega$-open set $U$ in $X$ containing $x$ such that $f(U) \subset V$. If $f$ has the property at each point $x \in X$, then it is said to be $\omega$-continuous [5].

## 3 On upper and lower $\omega$-continuous functions

**Definition 1** *A function $f : X \to \mathbb{R}$ is said to be:*

(i) *lower (resp. upper) $\omega$-continuous at $x_1$ if to each $\alpha > 0$, there exists an*

$\omega$*-open set* $U_{x_1}$ *such that* $f(x) > f(x_1) - \alpha$ (*resp.* $f(x) < f(x_1) + \alpha$) *for all* $x \in U_{x_1}$*;*

(ii) *lower* (*resp. upper*) $\omega$*-continuous if it is respectively so at each point of* $X$.

**Example 1** *Consider* $X = \mathbb{R}$ *with topology* $\tau = \{\emptyset, \mathbb{R}\}$*, then* $\tau_\omega = \{\emptyset, \mathbb{R}, \mathbb{R} \setminus \mathbb{Q}\} \cup \{(\mathbb{R} \setminus \mathbb{Q}) \cup A :$ *where* $A$ *is a subset of* $\mathbb{Q}\}$*. Define* $f : X \to \mathbb{R}$ *as follows:* $f = \chi_{\mathbb{R}\setminus\mathbb{Q}}$. $f$ *is lower* $\omega$*-continuous but is not upper* $\omega$*-continuous. In the same form if we define* $g : X \to \mathbb{R}$ *as follows:* $g = \chi_{\mathbb{Q}}$*,* $g$ *is upper* $\omega$*-continuous but is not lower* $\omega$*-continuous.*

**Theorem 1** *A function* $f : X \to \mathbb{R}$ *is lower* $\omega$*-continuous if and only if for each* $\alpha \in \mathbb{R}$*, the set* $\{x \in X : f(x) \leq \alpha\}$ *is* $\omega$*-closed.*

**Proof.** Since the family of sets $T = \{\mathbb{R}, \emptyset\} \cup \{(\alpha, \infty) : \alpha \in \mathbb{R}\}$ forms a topology on $\mathbb{R}$, $f$ is lower $\omega$-continuous if and only if $f$ is $\omega$-continuous from $X$ into the topological space $(\mathbb{R}, T)$. But $(-\infty, \alpha]$ is a closed set in $(\mathbb{R}, T)$ and hence $f^{-1}((-\infty, \alpha])$ is $\omega$-closed in $X$. But $f^{-1}((-\infty, \alpha]) = \{x \in X : f(x) \leq \alpha\}$. Therefore, $\{x \in X : f(x) \leq \alpha\}$ is $\omega$-closed. □

**Corollary 1** *A subset* $A$ *of* $X$ *is* $\omega$*-open if and only if the characteristic function* $\chi_A$ *is lower* $\omega$*-continuous.*

Similarly for upper $\omega$-continuity, we have the following characterization.

**Theorem 2** *A function* $f : X \to \mathbb{R}$ *is upper* $\omega$*-continuous if and only if for each* $\alpha \in \mathbb{R}$*, the set* $\{x \in X : f(x) \geq \alpha\}$ *is* $\omega$*-closed.*

**Corollary 2** *A subset* $A$ *of* $X$ *is* $\omega$*-closed if and only if the characteristic function* $\chi_A$ *is upper* $\omega$*-continuous.*

**Theorem 3** *Let* $\{f_\alpha : \alpha \in \Lambda\}$ *be a family of lower* $\omega$*-continuous functions from* $X$ *into* $\mathbb{R}$*, then the function* $M(x) = \sup_{\alpha\in\Lambda} f_\alpha(x)$ (*if it exists*) *is lower* $\omega$*-continuous.*

**Proof.** Let $\lambda \in \mathbb{R}$ and $M(x) < \lambda$. Then $f_\alpha(x) < \lambda$, for all $\alpha \in \Lambda$. Now $\{x \in X : M(x) \leq \lambda\} = \bigcap_{\alpha\in\Lambda} \{x \in X : f_\alpha(x) \leq \lambda\}$. But each $f_\alpha$ being lower $\omega$-continuous, by Theorem 1, each set $\{x \in X : f_\alpha(x) \leq \lambda\}$ is $\omega$-closed in $X$. Since any intersection of $\omega$-closed sets is $\omega$-closed, $M$ is lower $\omega$-continuous. □

**Theorem 4** *Let $\Lambda$ be a finite index set and $\{f_\alpha : \alpha \in \Lambda\}$ be a family of lower $\omega$-continuous functions from $X$ into $\mathbb{R}$, then the function $m(x) = \min_{\alpha\in\Lambda}\{f_\alpha(x)\}$ (if it exists) is lower $\omega$-continuous.*

**Proof.** It is enough to prove the case, when $m(x) = \min\{f_1(x), f_2(x)\}$. Let $\lambda \in \mathbb{R}$ and $x_0 \in X$, since $f_1, f_2$ are lower $\omega$-continuous from $X$ into $\mathbb{R}$, there exists $\omega$-open sets $U_1(x_0)$ (resp. $U_2(x_0)$) such that $f_1(x) > f_1(x_0) + \lambda$ for all $x \in U_1(x_0)$ (resp. $f_2(x) > f_2(x_0) + \lambda$ for all $x \in U_2(x_0)$ ). It follows that for all $x \in U_1(x_0) \cap U_2(x_0)$, we obtain that $m(x) > m(x_0) + \lambda$ for all $x \in U_1(x_0) \cap U_2(x_0)$. In consequence, the result follows. □

**Remark 1** *If $\Lambda$ be an infinite index set and $\{f_\alpha : \alpha \in \Lambda\}$ be a family of lower $\omega$-continuous functions from $X$ into $\mathbb{R}$. Then the function $m(x) = \inf_{\alpha\in\Lambda}$ $\{f_\alpha(x)\}$ (if it exists) may not be lower $\omega$-continuous.*

**Example 2** *For each natural number $n$, define $f_n = \chi_{(-\frac{1}{n},\frac{1}{n})}$ then $m(x) = \chi_{\{0\}}$, is not lower $\omega$-continuous.*

**Theorem 5** *Let $\{f_\alpha : \alpha \in \Lambda\}$ be a family of upper $\omega$-continuous function from $X$ into $\mathbb{R}$, then the function $g(x) = \inf_{\alpha\in\Lambda}\{f_\alpha(x)\}$ (if it exists) is upper $\omega$-continuous.*

**Proof.** Similar to the proof of Theorem 3. □

**Theorem 6** *Let $\Lambda$ be a finite index set and $\{f_\alpha : \alpha \in \Lambda\}$ be a family of upper $\omega$-continuous functions from $X$ into $\mathbb{R}$, then the function $M(x) = \max_{\alpha\in\Lambda}\{f_\alpha(x)\}$ (if it exists) is upper $\omega$-continuous.*

**Proof.** Similar to the proof of Theorem 4. □

**Remark 2** *If $\Lambda$ be an infinite index set and $\{f_\alpha : \alpha \in \Lambda\}$ be a family of upper $\omega$-continuous functions from $X$ into $\mathbb{R}$. Then the function $m(x) = \sup_{\alpha\in\Lambda}$ $\{f_\alpha(x)\}$ (if it exists) may not be upper $\omega$-continuous.*

**Example 3** *Similar to Example 2.*

**Theorem 7** *Let $X$ be an $\omega$-closed space and let $f : X \to \mathbb{R}$ be a lower $\omega$-continuous function. Then $f$ assumes the value $m = \inf_{x\in X}\{f(x)\}$.*

**Proof.** Let $\alpha \in \mathbb{R}$ be such that $\alpha > m$. Then $f$ being the lower $\omega$-continuous, the set $T_\alpha = \{x \in X : f(x) \leq \alpha\}$ is a nonempty (by the property of infimum) $\omega$-closed set in $X$. The family $\{T_\alpha : \alpha \in \mathbb{R} \text{ and } \alpha > m\}$ is a collection of nonempty $\omega$-closed sets with finite intersection property in the $\omega$-closed space $X$; hence it has nonempty intersection. Let $x^* \in \bigcap_{\alpha > m} T_\alpha$. Then $f(x^*) = m$. □

**Theorem 8** *If $X$ is $\omega$-closed, then any upper $\omega$-continuous function $f : X \to \mathbb{R}$ attains the value $M = \sup_{x\in X}\{f(x)\}$.*

**Proof.** Similar to Theorem 7. □

**Remark 3** *If a real valued function $f : X \to \mathbb{R}$ from an $\omega$-closed space is lower $\omega$-continuous as well as upper $\omega$-continuous, then it is bounded and attains its bounds.*

**Definition 2** *Let $f : X \to Y$ be a function, where $X$ is a topological space and $Y$ is a poset. Then*

(i) *$f$ is said to be lower (resp. upper) $\omega$-continuous if $f^{-1}(\{y \in Y : y \leq y_0\})$ (resp. $f^{-1}(\{y \in Y : y \geq y_0\})$) is $\omega$-closed in $X$ for each $y_0 \in Y$.*

(ii) *a partial order relation $\leq$ on a topological space $X$ is said to be lower (resp. upper) compatible if the set $\{x \in X : x \leq x_0\}$ (resp. $\{x \in X : x \geq x_0\}$) is $\omega$-closed for each $x_0 \in X$.*

**Theorem 9** *A topological space $X$ is $\omega$-closed if and only if $X$ has a maximal element with respect to each upper compatible partial order on $X$.*

**Proof.** Suppose that $X$ is not $\omega$-closed. Then there exists a net $\{x_\lambda : \lambda \in \Lambda\}$ which has no $\omega$-accumulation point, where $\Lambda$ is a well-ordered index set. We define the set $A_\alpha = X \backslash \omega\,\mathrm{Cl}_\theta(\{x_\beta : \beta > \alpha\})$. We claim that for each $x \in X$, $x \in A_\alpha$ for some $\alpha$. In fact, $x$ is contained in some $\omega$-regular set $R$ such that $R \cap \{x_\beta : \beta \geq \lambda\} = \emptyset$ for some $\beta$. Consider $\mathcal{R} = \{R \in \omega R(X) : R \cap \{x_\beta : \beta \geq \lambda\} = \emptyset$ for some $\beta\}$. Let $\lambda_R$ be the smallest index such that $R \cap \{x_\beta : \beta \geq \lambda_R\} = \emptyset$; let $\lambda_x$ be the smallest element of $M = \{\lambda_R : R \in \mathcal{R}\}$. We define the relation $\leq$ on $X$ as follows: $x \leq y$ if and only if $A_{\lambda x} \subset A_{\lambda y}$, that is, if and only if $X \backslash \omega\,\mathrm{Cl}(\{x_\beta : \beta \geq \lambda_x\}) \subset X \backslash \omega\,\mathrm{Cl}(\{x_\beta : \beta \geq \lambda_y\})$, that is, if and only if $\omega\,\mathrm{Cl}(\{x_\beta : \beta \geq \lambda_y\}) \subset \omega\,\mathrm{Cl}(\{x_\beta : \beta \geq \lambda_x\})$, that is, if and only if $\lambda_x \leq \lambda_y$. Clearly $\leq$ is a partial order relation on $X$. We claim that $\lambda_x$ is the first element of $M$ for which $x \in A_{\lambda x}$. In fact if $\alpha < \lambda_x$ and $x \in A_\alpha$, then $x \notin \omega\,\mathrm{Cl}(\{x_\beta : \beta \geq \alpha\})$. Then there exists $R \in \omega R(X)$ such that $R \cap \{x_\beta : \beta \geq \alpha\} = \emptyset$, a contradiction. It is obvious that for the corresponding $\lambda_x$ there exists an $R_{\lambda x} \in \mathcal{R}$ such that $R_{\lambda x} \cap \{x_\beta : \beta \geq \lambda_x\} = \emptyset$ and for any $\alpha < \lambda_x$, $R_{\lambda x} \cap \{x_\beta : \beta \geq \lambda_x\} \neq \emptyset$. Also, $R_{\lambda x} \cap \{x_\beta : \beta \geq \lambda_x\} = \emptyset$. Then $R_{\lambda x} \cap \omega\,\mathrm{Cl}(\{x_\beta : \beta \geq \lambda_x\}) = \emptyset$, that is $R_{\lambda_x} \subset X \backslash \omega\,\mathrm{Cl}(\{x_\beta : \beta \geq \lambda_x\}) = A_{\lambda x}$ and this happens for every $x \in X$. To show $\leq$ is upper compatible, it is sufficient to show that $\{x \in X : x \geq x_0\}$ is $\omega$-closed for every $x_0 \in X$. If possible, for some $x_0 \in X$, $\{x \in X : x \geq x_0\}$ is not $\omega$-closed, that is, there exists $y \in \omega\,\mathrm{Cl}(\{x \in X : x \geq x_0\})$ such that $y < x_0$, $R_{\lambda_y}$ is an $\omega$-regular set containing $y$ such that $x \in R_{\lambda_y}$ with $x > y$, that is, $\lambda_x > \lambda_y$, that is $x \in X \backslash \omega\,\mathrm{Cl}(\{x_\beta : \beta \geq \lambda_y\}) = A_{\lambda y}$. But $\lambda_x$ is the first index such that $x \in A_{\lambda_x}$ and thus we arrive at a contradiction. Hence, $\leq$ is upper compatible. Further, $(X, \leq)$ has no maximal element; in fact, if there be any, say $x_0$, then for some fixed $\omega$, $\omega\,\mathrm{Cl}(\{x_\beta : \beta \geq \lambda\}) \subset \omega\,\mathrm{Cl}(\{x_\beta : \beta \geq \alpha\})$ for every $\alpha \in M$, that is, $x_\lambda \in \omega\,\mathrm{Cl}(\{x_\beta : \beta \geq \alpha\})$, for all $\alpha \in M$, a contradiction. Conversely, let $S$ be a linearly ordered subset of the topological upper compatible poset $X$. We denote by $S_x$ the set $\{y \in X : y \geq x\}$. As the partial order on $X$ is upper compatible, each $S_x$ is $\omega$-closed. Since $S$ is a linearly ordered subset of $X$, $\{S_x : x \in X\}$ has finite intersection property. Then $\bigcap_{x \in S} S_x \neq \emptyset$. Let $x^* \in \bigcap_{x \in S} S_x$. Then $x^* \geq x$, for all $x \in S$. Therefore, by Zorn's lemma $X$ has a maximal element. □

**Theorem 10** *A topological space $X$ is $\omega$-closed if and only if $X$ has a maximal element with respect to each lower compatible partial order on $X$.*

**Proof.** Similar to Theorem 9. □

**Theorem 11** *A topological space $X$ is $\omega$-closed if and only if each upper $\omega$-continuous function from $X$ into a poset assumes a maximal value.*

**Proof.** Suppose that $X$ is not $\omega$-closed, then there exists a net $\{x_\lambda : \lambda \in M\}$ with no $\omega$-accumulation point, where $M$ is a well-ordered set. We assumes that the topology on $M$ is the order topology. Now, for each $\beta \in M$, $A_\beta = \omega\,\mathrm{Cl}(\{x_\lambda : \lambda \geq \beta\})$. We define a function $f : X \to M$ as follows: $f(x) = \beta_x$, where $\beta_x$ is the first element of the $\beta$'s for which $x \notin A_\beta$. This is well defined because from the fact that $M$ is well-ordered, obviously, $f(x)$ has no maximal element. We define the relation $\leq$ on $X$ as follows: $x \leq y$ if and only if $f(x) \leq f(y)$. Clearly, $\leq$ is a partial order relation on $X$. Now, for each $x \in X$, $S_x = f^{-1}(\{z \in Y : z \geq f(x)\}) = \{y \in X : y \geq x\}$. As $f$ is $\omega$-continuous, each $S_x$ is $\omega$-closed and hence $\leq$ is an upper compatible partial order relation on $X$. Then $X$ being an

$\omega$-closed space, by Theorem 9 it has a maximal element $x^*$. Therefore, $f(x^*)$ is the maximal element of $f(X)$. □

**Theorem 12** *A topological space* $X$ *is* $\omega$*-closed if and only if each lower* $\omega$*-continuous function from* $X$ *into a poset assumes a minimum value.*

**Proof.** Similar to Theorem 11. □

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