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Some sufficient conditions for certain class of meromorphic multivalent functions involving Cho-Kwon-Srivastava operator

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Abstract. Making use of a meromorphic analogue of the Cho-Kwon-Srivastava operator for normalized analytic functions, we introduce below a new class of meromorphic multivalent function in the punctured unit disk and obtain certain sufficient conditions for functions to belong to this class. Some consequences of the main result are also mentioned.

1 Introduction and motivation

Let $\sum_p$ denote the class of functions of the form:

$$f(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} a_{k-p} z^{k-p} \quad (p \in \mathbb{N} := \{1, 2, 3, \cdots\}) \tag{1}$$

which are analytic in the punctured unit disk:

$$\mathbb{U}^* := \{z : z \in \mathbb{C}, 0 < |z| < 1\} = \mathbb{U} \setminus \{0\},$$

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having the only pole of order p at origin. In particular for $p = 1$, we write $\sum_1 := \sum$.

For functions $f \in \sum_p$ given by (1) and $g \in \sum_p$ given by

$$g(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} b_{k-p} z^{k-p} \quad (z \in \mathbb{U}^*), \tag{2}$$

we define $f * g$ by

$$(f * g)(z) := \frac{z^p f(z) * z^p g(z)}{z^p} =: \frac{1}{z^p} + \sum_{k=1}^{\infty} a_{k-p} b_{k-p} z^{k-p} = (g * f)(z) \quad (z \in \mathbb{U}^*), \tag{3}$$

where $*$ denotes the usual Hadamard product(or convolution) of analytic functions.

Let $\sum_p^*(\alpha)$, $\sum_p^k(\alpha)$ and $\sum_p^c(\alpha)$ be the subclasses of the class $\sum_p$ consists of meromorphic multivalent functions which are respectively starlike, convex and close-to-convex funcions of order $\alpha \ \ (0 \leq \alpha < p)$.

Analytically, a function $f \in \sum_p$ is said to be in the class $\sum_p^*(\alpha)$ if and only if

$$\Re\left[-\frac{zf'(z)}{f(z)}\right] > \alpha \quad (z \in \mathbb{U}^*). \tag{4}$$

Similarly, a function $f \in \sum_p$ is said to be in the class $\sum_p^k(\alpha)$ if and only if

$$\Re\left[-1 - \frac{zf''(z)}{f'(z)}\right] > \alpha \quad (z \in \mathbb{U}^*). \tag{5}$$

Furthermore, a function $f \in \sum_p^c(\alpha)$ if and only if f is of the form (1) and satisfies

$$\Re\left[-\frac{f'(z)}{z^{-p-1}}\right] > \alpha \quad (z \in \mathbb{U}^*). \tag{6}$$

We observe that $\sum_1^*(\alpha) := \sum^*(\alpha)$, $\sum_1^k(\alpha) := \sum_k(\alpha)$, $\sum_1^c(\alpha) := \sum_c(\alpha)$ where $\sum^*(\alpha)$, $\sum_k(\alpha)$ and $\sum_c(\alpha)$ are subclasses of $\sum$ consisting of meromorphic univalent functions which are respectively starlike, convex and close-to-convex of order $\alpha \ \ (0 \leq \alpha < 1)$. For recent expository work on meromorphic functions see([5, 7, 11, 14, 16]).

For the purpose of defining transform, Liu and Srivastava [7] studies meromorphic analogue of the Carlson-Shaffer operator [1] by introducing the function $\phi_p(a, c; z)$ given by

$$\phi_p(a, c; z) := \frac{{}_2F_1(a, 1; c; z)}{z^p} =: \frac{1}{z^p} + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k} z^{k-p}, \tag{7}$$
$$(z \in \mathbb{U}^*; a \in \mathbb{C}, c \in \mathbb{C} \setminus \mathbb{Z}_0^-, \mathbb{Z}_0^- := \{0, 1, 2, \cdots\})$$

where ${}_2F_1(a, 1; c; z)$ is the Gauss hypergeometric series and $(\lambda)_n$ is the *Pochhammer symbol* (or shifted factorial) given by

$$(\lambda)_n := \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n = 0) \\ \lambda(\lambda + 1)...(\lambda + n - 1) & (n \in \mathbb{N}). \end{cases}$$

Recently, Mishra et al. [9] (also see [10]) defined the function $\phi_p^{\dagger}(a, c; z)$, the generalized multiplicative inverse of $\phi_p(a, c; z)$ given by the relation

$$\phi_p(a, c; z) * \phi_p^{\dagger}(a, c; z) = \frac{1}{z^p(1 - z)^{\lambda+p}} \quad (a, c \in \mathbb{C} \setminus \mathbb{Z}_0^-, \lambda > -p, z \in \mathbb{U}^*). \tag{8}$$

If $\lambda = -p + 1$, then $\phi_p^{\dagger}(a, c; z)$ is the inverse of $\phi_p(a, c; z)$ with respect to the Hadamard product $*$. Using this function $\phi_p(a, c; z)$, they considered an operator $\mathcal{L}_p^{\lambda}(a, c) : \sum_p \longrightarrow \sum_p$ as follows:

$$\begin{aligned} \mathcal{L}_p^{\lambda}(a, c)f(z) :&= \phi_p^{\dagger}(a, c; z) * f(z) = \frac{{}_2F_1(\lambda + p, c; a; z)}{z^p} \\ &= \frac{1}{z^p} + \sum_{k=1}^{\infty} \frac{(\lambda + p)_k (c)_k}{(a)_k (1)_k} a_{k-p} z^{k-p} \quad (z \in \mathbb{U}^*). \end{aligned} \tag{9}$$

The holomorphic version of the function $\phi_p^{\dagger}(a, c; z)$ is given by the relation:

$$z^p {}_2F_1(a, 1; c; z) * \phi_p^{\dagger}(a, c; z) := \frac{z^p}{(1 - z)^{\lambda+p}} \quad (a, c \in \mathbb{C} \setminus \mathbb{Z}_0^-, \lambda > -p; z \in \mathbb{U}),$$

and the associated transform $\mathcal{L}_p^{\lambda}(a, c)f(z) = \phi_p^{\dagger}(a; c; z) * f(z)$ were studied by Cho et al. [2]. The transform $\mathcal{L}_p^{\lambda}(a, c)$ is popularly known as the Cho-Kwon-Srivastava operator in literature (see, for details [4, 12, 15]).

Recently, Prajapat [13] (also see [3]) introduced a class of analytic and multivalent function $\mathbb{B}(p, n, \mu, \alpha)$ and investigated some sufficient conditions for this class. Furthermore, Goyal and Prajapat [5] introduced the class $\mathcal{T}_p(\lambda, \mu, \alpha)$

by making use of an extended derivative operator of Ruscheweyh type and investigated some sufficient conditions for a certain function to belong to this class.

Motivated by the aforementioned work, in this paper the authors introduce a new class $\mathcal{T}_p^{\lambda,\alpha}(\mu, a, c)$ by making use of a meromorphic analogue of Cho-Kwon-Srivastava operator $\mathcal{L}_p^{\lambda}(a, c)$ for normalized multivalent analytic function as follows:

Definition 1 *A function* $f \in \sum_p$ *is said to be in the class* $\mathcal{T}_p^{\lambda,\alpha}(\mu, a, c)$ *if it satisfies the following condition:*

$$\left| \frac{z^{p+1}\left(\mathcal{L}_p^{\lambda}(a, c)f(z)\right)'}{\left(z^p\mathcal{L}_p^{\lambda}(a, c)f(z)\right)^{\mu-1}} + p \right| < p - \alpha \tag{10}$$
$$(z \in \mathbb{U}^*,\ p \in \mathbb{N},\ \lambda > -p,\ \mu \geq 0,\ 0 \leq \alpha < p,\ a,\ c \in \mathbb{C} \setminus \mathbb{Z}_0^-)$$

The condition (10) implies that

$$\Re\left\{ -\frac{z^{p+1}(\mathcal{L}_p^{\lambda}(a, c)f(z))'}{\left(z^p\mathcal{L}_p^{\lambda}(a, c)f(z)\right)^{\mu-1}} \right\} > \alpha \tag{11}$$

It is clear from the above definition that
$\mathcal{T}_p^{-p+1,\alpha}(2, a, a) = \sum_p^*(\alpha)$ and $\mathcal{T}_p^{-p+1,\alpha}(1, a, a) = \sum_p^c(\alpha)$.

In the present paper, we obtain certain sufficient conditions for functions f to be in the class $\mathcal{T}_p^{\lambda,\alpha}(\mu, a, c)$.

We need the following lemma for our investigation.

Lemma 1 (*see* [6, 8]) *Let the function* $w(z)$ *be non-constant and regular in* $\mathbb{U}$ *such that* $w(0) = 0$*. If* $|w(z)|$ *attains its maximum value on the circle* $|z| = r < 1$ *at a point* $z_0 \in \mathbb{U}$*, then*

$$z_0 w'(z_0) = k w(z_0),$$

where k *is real and* $k \geq 1$*.*

2 Main results

Unless otherwise stated, we mention throughout the sequel that

$$p \in \mathbb{N},\ \mu \geq 0,\ \lambda > -p,\ a,\ c \in \mathbb{C} \setminus \mathbb{Z}_0^-,\ 0 \leq \alpha < p.$$

Theorem 1 *If* $f \in \sum_p$ *given by* (1) *satisfies anyone of the following inequalities:*

$$\left| -\frac{z^{p+1}\left(\mathcal{L}_p^{\lambda}(a,c)f(z)\right)'}{\left(z^p\mathcal{L}_p^{\lambda}(a,c)f(z)\right)^{\mu-1}}\left[1+p+\frac{z\left(\mathcal{L}_p^{\lambda}(a,c)f(z)\right)''}{(\mathcal{L}_p^{\lambda}(a,c)f(z))'} -(\mu-1)\left\{p+\frac{z\left(\mathcal{L}_p^{\lambda}(a,c)f(z)\right)'}{\mathcal{L}_p^{\lambda}(a,c)f(z)}\right\}\right]\right| < p-\alpha, \tag{12}$$

$$\left|\frac{1+p+\frac{z\left(\mathcal{L}_p^{\lambda}(a,c)f(z)\right)''}{(\mathcal{L}_p^{\lambda}(a,c)f(z))'}-(\mu-1)\left\{p+\frac{z\left(\mathcal{L}_p^{\lambda}(a,c)f(z)\right)'}{\mathcal{L}_p^{\lambda}(a,c)f(z)}\right\}}{-\frac{z^{p+1}\left(\mathcal{L}_p^{\lambda}(a,c)f(z)\right)'}{\left(z^p\mathcal{L}_p^{\lambda}(a,c)f(z)\right)^{\mu-1}}}\right| < \frac{p-\alpha}{(2p-\alpha)^2}, \tag{13}$$

$$\left|\frac{1+p+\frac{z\left(\mathcal{L}_p^{\lambda}(a,c)f(z)\right)''}{(\mathcal{L}_p^{\lambda}(a,c)f(z))'}-(\mu-1)\left\{p+\frac{z\left(\mathcal{L}_p^{\lambda}(a,c)f(z)\right)'}{\mathcal{L}_p^{\lambda}(a,c)f(z)}\right\}}{-\left[\frac{z^{p+1}\left(\mathcal{L}_p^{\lambda}(a,c)f(z)\right)'}{\left(z^p\mathcal{L}_p^{\lambda}(a,c)f(z)\right)^{\mu-1}}+p\right]}\right| < \frac{1}{2p-\alpha}, \tag{14}$$

and

$$\Re\left[\frac{z^{p+1}\left(\mathcal{L}_p^{\lambda}(a,c)f(z)\right)'}{\left(z^p\mathcal{L}_p^{\lambda}(a,c)f(z)\right)^{\mu-1}}\left\{\frac{1+p+\frac{z\left(\mathcal{L}_p^{\lambda}(a,c)f(z)\right)''}{(\mathcal{L}_p^{\lambda}(a,c)f(z))'}-(\mu-1)\left(p+\frac{z\left(\mathcal{L}_p^{\lambda}(a,c)f(z)\right)'}{\mathcal{L}_p^{\lambda}(a,c)f(z)}\right)}{\frac{z^{p+1}\left(\mathcal{L}_p^{\lambda}(a,c)f(z)\right)'}{\left(z^p\mathcal{L}_p^{\lambda}(a,c)f(z)\right)^{\mu-1}}+p}\right\}\right] < 1, \tag{15}$$

then $f \in \mathcal{T}_p^{\lambda,\alpha}(\mu, a, c)$.

Proof. Let $f(z) \in \sum_p$ be given by (1). Define the function $w(z)$ by

$$-\frac{z^{p+1}\left(\mathcal{L}_p^{\lambda}(a,c)f(z)\right)'}{\left(z^p\mathcal{L}_p^{\lambda}(a,c)f(z)\right)^{\mu-1}} = p+(p-\alpha)w(z). \tag{16}$$

Clearly $w(z)$ is analytic in $\mathbb{U}$ with $w(0) = 0$. Taking logarithmic differentiation on both sides of (16) with respect to z, we obtain

$$1+p+\frac{z\left(\mathcal{L}_p^{\lambda}(a,c)f(z)\right)''}{(\mathcal{L}_p^{\lambda}(a,c)f(z))'}-(\mu-1)\left\{p+\frac{z\left(\mathcal{L}_p^{\lambda}(a,c)f(z)\right)'}{\mathcal{L}_p^{\lambda}(a,c)f(z)}\right\} = \frac{(p-\alpha)zw'(z)}{p+(p-\alpha)w(z)}. \tag{17}$$

From (16) and (17), we have

$$\begin{aligned}\phi_1(z) = &-\frac{z^{p+1}\left(\mathcal{L}_p^\lambda(a,c)f(z)\right)'}{\left(z^p\mathcal{L}_p^\lambda(a,c)f(z)\right)^{\mu-1}}\left[1+p+\frac{z\left(\mathcal{L}_p^\lambda(a,c)f(z)\right)''}{(\mathcal{L}_p^\lambda(a,c)f(z))'}\right.\\ &\left.-(\mu-1)\left\{p+\frac{z\left(\mathcal{L}_p^\lambda(a,c)f(z)\right)'}{\mathcal{L}_p^\lambda(a,c)f(z)}\right\}\right] = (p-\alpha)zw'(z),\end{aligned} \tag{18}$$

$$\begin{aligned}\phi_2(z) &= \frac{1+p+\frac{z\left(\mathcal{L}_p^\lambda(a,c)f(z)\right)''}{(\mathcal{L}_p^\lambda(a,c)f(z))'}-(\mu-1)\left\{p+\frac{z\left(\mathcal{L}_p^\lambda(a,c)f(z)\right)'}{\mathcal{L}_p^\lambda(a,c)f(z)}\right\}}{-\frac{z^{p+1}\left(\mathcal{L}_p^\lambda(a,c)f(z)\right)'}{\left(z^p\mathcal{L}_p^\lambda(a,c)f(z)\right)^{\mu-1}}}\\ &= \frac{(p-\alpha)zw'(z)}{[p+(p-\alpha)w(z)]^2},\end{aligned} \tag{19}$$

$$\begin{aligned}\phi_3(z) &= \frac{1+p+\frac{z\left(\mathcal{L}_p^\lambda(a,c)f(z)\right)''}{(\mathcal{L}_p^\lambda(a,c)f(z))'}-(\mu-1)\left\{p+\frac{z\left(\mathcal{L}_p^\lambda(a,c)f(z)\right)'}{\mathcal{L}_p^\lambda(a,c)f(z)}\right\}}{-\left[\frac{z^{p+1}\left(\mathcal{L}_p^\lambda(a,c)f(z)\right)'}{\left(z^p\mathcal{L}_p^\lambda(a,c)f(z)\right)^{\mu-1}}+p\right]}\\ &= \frac{zw'(z)}{w(z)[p+(p-\alpha)w(z)]},\end{aligned} \tag{20}$$

and

$$\begin{aligned}\phi_4(z) &= \frac{z^{p+1}\left(\mathcal{L}_p^\lambda(a,c)f(z)\right)'}{\left(z^p\mathcal{L}_p^\lambda(a,c)f(z)\right)^{\mu-1}}\frac{1+p+\frac{z\left(\mathcal{L}_p^\lambda(a,c)f(z)\right)''}{(\mathcal{L}_p^\lambda(a,c)f(z))'}-(\mu-1)\left\{p+\frac{z\left(\mathcal{L}_p^\lambda(a,c)f(z)\right)'}{\mathcal{L}_p^\lambda(a,c)f(z)}\right\}}{\left[\frac{z^{p+1}\left(\mathcal{L}_p^\lambda(a,c)f(z)\right)'}{\left(z^p\mathcal{L}_p^\lambda(a,c)f(z)\right)^{\mu-1}}+p\right]}\\ &= \frac{zw'(z)}{w(z)}.\end{aligned} \tag{21}$$

Now we claim that $|w(z)| < 1$ in $\mathbb{U}$. For otherwise there exists a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z|<|z_0|}|w(z)| = |w(z_0)| = 1. \tag{22}$$

Then from Lemma 1 we find that

$$z_0w'(z_0) = kw(z_0) \quad (k \geq 1). \tag{23}$$

Therefore, letting $w(z_0) = e^{i\theta}$ in each of the equation (18) to (21), we obtain

$$|\phi_1(z_0)| = |(p-\alpha)z_0w'(z_0)| = |(p-\alpha)ke^{i\theta}| \geq (p-\alpha), \tag{24}$$

$$|\phi_2(z_0)| = \left|\frac{(p-\alpha)z_0 w'(z_0)}{[p+(p-\alpha)w(z_0)]^2}\right| = \frac{|(p-\alpha)ke^{i\theta}|}{|p+(p-\alpha)e^{i\theta}|^2} \geq \frac{(p-\alpha)}{(2p-\alpha)^2}, \tag{25}$$

$$\begin{aligned} |\phi_3(z_0)| &= \left|\frac{z_0 w'(z_0)}{w(z_0)[p+(p-\alpha)w(z_0)]}\right| \\ &= \left|\frac{k}{[p+(p-\alpha)e^{i\theta}]}\right| \geq \frac{1}{2p-\alpha}, \end{aligned} \tag{26}$$

$$\Re\{\phi_4(z_0)\} = \Re\left\{\frac{z_0 w'(z_0)}{w(z_0)}\right\} = k \geq 1, \tag{27}$$

which contradicts our assumption (12) to (15), respectively. Therefore, $|w(z)| < 1$ holds true for all $z \in \mathbb{U}$. Then (16) we have

$$\left|\frac{z^{p+1}(\mathcal{L}_p^{\lambda}(a,c)f(z))'}{\left(z^p \mathcal{L}_p^{\lambda}(a,c)f(z)\right)^{\mu-1}} + p\right| = |(p-\alpha)w(z)| < (p-\alpha)$$

which implies that

$$f \in \mathcal{T}_p^{\lambda,\alpha}(\mu, a, c).$$

□

3 Consequences of main result

Putting $a = c,\ \lambda = -p+1,\ \mu = 1$ in Theorem 1, we get the following result:

Corollary 1 *Let the function* $f(z)$ *defined by* (1) *belong to the class* $\sum_p$. *If* $f(z)$ *satisfies any one of the following inequalities:*

$$\left|-\frac{f'(z)}{z^{-p-1}}\left(1+p+\frac{zf''(z)}{f'(z)}\right)\right| < p-\alpha,$$

$$\left|\frac{1+p+\frac{zf''(z)}{f'(z)}}{-\frac{f'(z)}{z^{-p-1}}}\right| < \frac{p-\alpha}{(2p-\alpha)^2},$$

$$\left|\frac{1+p+\frac{zf''(z)}{f'(z)}}{-\frac{f'(z)}{z^{-p-1}}-p}\right| < \frac{1}{2p-\alpha},$$

and

$$\Re\left\{\frac{-\frac{f'(z)}{z^{-p-1}}\left(1+p+\frac{zf''(z)}{f'(z)}\right)}{-\frac{f'(z)}{z^{-p-1}}-p}\right\} < 1,$$

then $f(z) \in \sum_p^c(\alpha)$.

Letting $p = 1$ in Corollary 1 we obtain the following result.

Corollary 2 *If* $f(z) \in \sum$ *satisfies any one of the following inequalities:*

$$\left|-\frac{f'(z)}{z^{-2}}\left(2+\frac{zf''(z)}{f'(z)}\right)\right| < 1-\alpha,$$

$$\left|\frac{2+\frac{zf''(z)}{f'(z)}}{-\frac{f'(z)}{z^{-2}}}\right| < \frac{1-\alpha}{(2-\alpha)^2},$$

$$\left|\frac{2+\frac{zf''(z)}{f'(z)}}{-\frac{f'(z)}{z^{-2}}-1}\right| < \frac{1}{2-\alpha},$$

and

$$\Re\left\{-\frac{f'(z)}{z^{-2}}\left(\frac{2+\frac{zf''(z)}{f'(z)}}{-\frac{f'(z)}{z^{-2}}-1}\right)\right\} < 1,$$

then $f(z) \in \sum_c(\alpha)$.

Further in the special case when $\alpha = 0$, Corollary 2 reduces to Corllary 3 stated below:

Corollary 3 *If* $f(z) \in \sum$ *satisfies anyone of the following inequalities:*

$$\left|-z^2f'(z)\left(2+\frac{zf''(z)}{f'(z)}\right)\right| < 1,$$

$$\left|-\frac{2+\frac{zf''(z)}{f'(z)}}{z^2f'(z)}\right| < \frac{1}{4},$$

$$\left|-\frac{2+\frac{zf''(z)}{f'(z)})}{z^2f'(z)+1}\right| < \frac{1}{2},$$

and

$$\Re\left\{\left(2+\frac{zf''(z)}{f'(z)}\right)\frac{z^2f'(z)}{z^2f'(z)+1}\right\} < 1,$$

then $f(z) \in \sum_c \quad (\equiv \sum_c(0))$.

Letting $a = c,\ \lambda = -p+1, \mu = 2$ in Theorem 12, we obtain the following:

Corollary 4 *If* $f \in \sum_p$ *given by (1) satisfies anyone of the following inequalities:*

$$\left|-\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\right| < p-\alpha,$$

$$\left|-\frac{f(z)}{zf'(z)}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\right| < \frac{p-\alpha}{(2p-\alpha)^2},$$

$$\left|1+\frac{\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}}{-\frac{zf'(z)}{f(z)}-p}\right| < \frac{1}{2p-\alpha},$$

and

$$\Re\left\{\frac{zf'(z)}{f(z)}\left(\frac{1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}}{\frac{zf'(z)}{f(z)}+p}\right)\right\} < 1,$$

then $f(z) \in \sum_p^*(\alpha)$.

By putting $p = 1$ in Corollary 4, we have

Corollary 5 *If* $f \in \sum$ *satisfies anyone of the following inequalities:*

$$\left|-\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\right| < 1-\alpha,$$

$$\left|-\frac{f(z)}{zf'(z)}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\right| < \frac{1-\alpha}{(2-\alpha)^2},$$

$$\left|\frac{1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}}{-\frac{zf'(z)}{f(z)}-1}\right| < \frac{1}{2-\alpha},$$

and

$$\Re\left\{\frac{zf'(z)}{f(z)}\left(\frac{1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}}{\frac{zf'(z)}{f(z)}+1}\right)\right\} < 1,$$

then $f(z) \in \sum^*(\alpha)$.

On further setting $\alpha = 0$ in Corollary 5, we get:

Corollary 6 *If* $f(z) \in \sum$ *satisfies any one of the following inequalities:*

$$\left|-\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\right| < 1,$$

$$\left|-\frac{f(z)}{zf'(z)}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\right|<\frac{1}{4},$$

$$\left|\frac{1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}}{-\frac{zf'(z)}{f(z)}-1}\right|<\frac{1}{2},$$

$$\Re\left\{\frac{zf'(z)}{f(z)}\left(\frac{1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}}{\frac{zf'(z)}{f(z)}+1}\right)\right\}<1,$$

then $f(z)\in\sum^*$.

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