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Uniqueness theorems related to weighted sharing of two sets

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Abstract. Using the notion of weighted sharing of sets, we study the uniqueness problem of meromorphic functions sharing two finite sets. Our results are inspired from an article due to J. F. Chen (Open Math., 15 (2017), 1244–1250).

1 Introduction, Definitions and Main results

In this paper, a meromorphic function means a function which is meromorphic in the entire complex plane $\mathbb{C}$. Throughout the paper, we adopt the standard notations of Nevanlinna value distribution theory as explained in [6] and [12]. We denote by $\mathcal{M}(\mathbb{C})$ the class of all meromorphic functions defined in $\mathbb{C}$ and by $\mathcal{M}_1(\mathbb{C})$ the class of meromorphic functions which have finitely many poles in $\mathbb{C}$. For convenience, we denote any set of positive real numbers of finite linear measure by E, not necessarily the same at each occurrence. For a nonconstant meromorphic function h, we denote by $S(r,h)$ any quantity satisfying $S(r,h) = o\{T(r,h)\}$ for $r \to \infty, r \notin E$. The order $\lambda(f)$ of $f \in \mathcal{M}(\mathbb{C})$ is defined as

$$\lambda(f) = \limsup_{r \longrightarrow \infty} \frac{\log T(r,f)}{\log r}.$$

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For a meromorphic function f and a set $S \subset \mathbb{C} \cup \{\infty\}$, we define $E_f(S)$ ($\overline{E}_f(S)$) to be the set of all a-points of f, where $a \in S$, together with their multiplicities (ignoring their multiplicities). We say that two functions f and g share the set S CM (IM) if $E_f(S) = E_g(S)$ ($\overline{E}_f(S) = \overline{E}_g(S)$).

The development of research works related to set sharing problems was broadly initiated due to the following question which was raised by F. Gross [5].

Question 1 *Can one find two finite sets $S_i (i = 1, 2)$ of $\mathbb{C} \cup \{\infty\}$ such that any two nonconstant entire functions f and g satisfying $E_f(S_i) = E_g(S_i)$ for $i = 1, 2$ must be identical?*

In 1994, H. X. Yi [14] proved the following theorem which gives an affirmative answer to Gross's question.

Theorem A *Let $S_1 = \{\omega \mid \omega^n - 1 = 0\}$ and $S_2 = \{a\}$, where $n \geq 5$ is an integer, $a \neq 0$ and $a^{2n} \neq 1$. If f and g are entire functions such that $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$, then $f \equiv g$.*

In [5], F. Gross also pointed that if the answer of Question 1 is affirmative, then it would be interesting to know how large the sets can be.

In 1998, H. X. Yi [15] proved the following theorem which deals with the above comment.

Theorem B *Let $S_1 = \{0\}$ and $S_2 = \{\omega \mid \omega^2(\omega + a) - b = 0\}$, where a and b are two nonzero constants such that the algebraic equation $\omega^2(\omega + a) - b = 0$ has no multiple roots. If f and g are two entire functions satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$, then $f \equiv g$.*

In this direction, a lot of research works have been devoted during the last two decades (see [4], [9], [10], [13]).

We recall the following recent result due to J. F. Chen [2].

Theorem C *Let k be a positive integer and let $S_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$, $S_2 = \{\beta_1, \beta_2\}$, where $\alpha_1, \alpha_2, \ldots, \alpha_k, \beta_1, \beta_2$ are $k+2$ distinct finite complex numbers satisfying*

$$(\beta_1 - \alpha_1)^2(\beta_1 - \alpha_2)^2 \cdots (\beta_1 - \alpha_k)^2 \neq (\beta_2 - \alpha_1)^2(\beta_2 - \alpha_2)^2 \cdots (\beta_2 - \alpha_k)^2.$$

If two nonconstant meromorphic functions f and g in $\mathcal{M}_1(\mathbb{C})$ share S_1 CM, S_2 IM, and if the order of f is neither an integer nor infinite, then $f \equiv g$.

In the same paper, the author also proved another result concerning unique range sets. Before stating the result, we present the definition of unique range sets.

Definition 1 *For a family of functions $\mathcal{G}$, the subsets $S_1, S_2, \ldots, S_q$ of $\mathbb{C}\cup\{\infty\}$ such that for any $f, g \in \mathcal{G}$, f and g share S_j CM for $j = 1, 2, \ldots, q$ imply $f \equiv g$, are called unique range sets (URS, in brief) for the functions in $\mathcal{G}$.*

Theorem D *Let k be a positive integer and let $S_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$, $S_2 = \{\beta_1, \beta_2\}$, where α_1, $\alpha_2, \ldots, \alpha_k$, β_1, β_2 are $k+2$ distinct finite complex numbers satisfying*

$$(\beta_1 - \alpha_1)^2(\beta_1 - \alpha_2)^2 \cdots (\beta_1 - \alpha_k)^2 \neq (\beta_2 - \alpha_1)^2(\beta_2 - \alpha_2)^2 \cdots (\beta_2 - \alpha_k)^2.$$

If the order of f is neither an integer nor infinite, then the sets S_1 and S_2 are the URS of meromorphic functions in $\mathcal{M}_1(\mathbb{C})$.

The condition $(\beta_1-\alpha_1)^2(\beta_1-\alpha_2)^2 \ldots (\beta_1-\alpha_k)^2 \neq (\beta_2-\alpha_1)^2(\beta_2-\alpha_2)^2 \ldots (\beta_2-\alpha_k)^2$ in Theorems C and D can not be dropped as shown by the following example.

Example 1 [2] *For a positive integer k, let $f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^{3n}}$, $g(z) = -f(z)$, $S_1 = \{-1,\ 1,\ -2,\ 2, \ldots,\ -k,\ k\}$, and $S_2 = \{-(k+1),\ k+1\}$. Then using the result of* [3, p. 288] *we deduce*

$$\lambda(f) = \frac{1}{\liminf\limits_{n\longrightarrow\infty} \dfrac{\log n^{3n}}{n \log n}} = \limsup_{n\longrightarrow\infty} \frac{n \log n}{\log n^{3n}} = \frac{1}{3}.$$

Clearly $f(z),\ g(z) \in \mathcal{M}_1(\mathbb{C})$, $f(z)$ and $g(z)$ share S_1, S_2 CM. But $f(z) \not\equiv g(z)$.

The assumption "nonconstant meromorphic functions f and g in $\mathcal{M}_1(\mathbb{C})$" in Theorems C and D cannot be relaxed to "nonconstant meromorphic functions f and g in $\mathcal{M}(\mathbb{C})$" as shown by the following example.

Example 2 [2] *For a positive integer k, let $f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^{3n}}$, $g(z) = \frac{1}{f(z)}$, $S_1 = \left\{2,\ \frac{1}{2},\ 3,\ \frac{1}{3}, \ldots,\ k,\ \frac{1}{k}\right\}$, $S_2 = \left\{k+1,\ \frac{1}{k+1}\right\}$. From Example 1 we note that $\lambda(f) = \frac{1}{3}$ and, therefore, using the result of* [3, p. 293] *we see that $g(z)$ has infinitely many poles in $\mathbb{C}$. Moreover, $f(z)$ and $g(z)$ share the sets S_1, S_2 CM. But $f(z) \not\equiv g(z)$.*

The following example given in [2] shows the necessity of the assumption in Theorems C and D that the order of f is neither an integer nor infinite.

Example 3 *For a positive integer* k, *let* $f(z) = e^z$ *(resp.* $f(z) = e^{e^z}$*),* $g(z) = \frac{1}{f(z)}$, $S_1 = \left\{2, \frac{1}{2}, 3, \frac{1}{3}, \ldots, k, \frac{1}{k}\right\}$, $S_2 = \left\{k+1, \frac{1}{k+1}\right\}$. *Then by Lemma 8 in section 2 we see that* $\lambda(f) = 1$ *(resp.* $\lambda(f) = \infty$*). Though all other conditions of Theorems C and D are satisfied,* $f(z) \not\equiv g(z)$.

However, the research on set sharing problem gained a new dimension when the idea of weighted sharing, introduced by I. Lahiri in 2001 (see [7], [8]), was incorporated. The necessary definitions are as follows:

Definition 2 *Let* k *be a nonnegative integer or infinity. For* $a \in \mathbb{C} \cup \{\infty\}$ *we denote by* $E_k(a; f)$ *the set of all* a*-points of* f*, where an* a*-point of multiplicity* m *is counted* m *times if* $m \leq k$ *and* $k+1$ *times if* $m > k$*. If* $E_k(a; f) = E_k(a; g)$*, we say that* f *and* g *share the value* a *with weight* k.

We write f and g share (a, k) to mean that f and g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for any integer p where $0 \leq p < k$. In particular, f and g share a CM (IM) if and only if f and g share (a, ∞) $((a, 0))$.

Definition 3 *Let* S *be a set of distinct elements of* $\mathbb{C} \cup \{\infty\}$ *and* k *be a nonnegative integer or infinity. We denote by* $E_f(S, k)$ *the set* $\cup_{a \in S} E_k(a; f)$*. We say that* f *and* g *share the set* S *with weight* k*, or simply* f *and* g *share* (S, k) *if* $E_f(S, k) = E_g(S, k)$.

Definition 4 *Let* k *be a positive integer and* $S_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$*, where* α_i*'s are nonzero complex constants. Suppose that*

$$P(z) = \frac{z^k - (\sum \alpha_i) z^{k-1} + \ \ldots + (-1)^{k-1} (\sum \alpha_{i_1} \alpha_{i_2} \ldots \alpha_{i_{k-1}}) z}{(-1)^{k+1} \alpha_1 \alpha_2 \ldots \alpha_k}, \tag{1}$$

where $\alpha_i \in S_1$ *for* $i = 1, 2, \ldots, k$*. Let* m_1 *be the number of simple zeros of* $P(z)$ *and* m_2 *be the number of multiple zeros of* $P(z)$*. Then we define* $\Gamma_1 := m_1 + m_2$ *and* $\Gamma_2 := m_1 + 2m_2$.

Regarding Theorem C, one may ask the following question:

Question 2 *Is the conclusion of Theorem C still true if* f *and* g *share* $(S_1, 2)$ *and* S_2 *IM instead of sharing* S_1 *CM and* S_2 *IM?*

In this paper, we try to find possible answers to the above question and prove the following theorems:

Theorem 1 *Let* $f,\ g \in \mathcal{M}_1(\mathbb{C})$ *and* $S_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$, $S_2 = \{\beta_1, \beta_2\}$, *where* $\alpha_1,\ \alpha_2, \ldots, \alpha_k,\ \beta_1,\ \beta_2$ *are* $k+2$ *distinct nonzero complex constants satisfying* $k > 2\Gamma_2$. *If* $f,\ g$ *share* $(S_1, 2)$ *and* S_2 *IM, then* $f \equiv g$, *provided*

$$(\beta_1 - \alpha_1)^2(\beta_1 - \alpha_2)^2 \cdots (\beta_1 - \alpha_k)^2 \neq (\beta_2 - \alpha_1)^2(\beta_2 - \alpha_2)^2 \cdots (\beta_2 - \alpha_k)^2$$

and f *is of non-integer finite order.*

Theorem 2 *Let* S_1 *and* S_2 *be stated as in Theorem 1 with* $k > 2\Gamma_2$. *If* $\mathcal{M}_2(\mathbb{C})$ *denote the subclass of meromorphic functions of non-integer finite order in* $\mathcal{M}_1(\mathbb{C})$, *then the sets* S_1 *and* S_2 *are the URS of meromorphic functions in* $\mathcal{M}_2(\mathbb{C})$, *provided*

$$(\beta_1 - \alpha_1)^2(\beta_1 - \alpha_2)^2 \cdots (\beta_1 - \alpha_k)^2 \neq (\beta_2 - \alpha_1)^2(\beta_2 - \alpha_2)^2 \cdots (\beta_2 - \alpha_k)^2.$$

We now state some more definitions (see [7], [8]).

Definition 5 *For* $a \in \mathbb{C} \cup \{\infty\}$, *we denote by* $\overline{N}(r, a; f| = k)$ *the reduced counting function of the* a*-points of* f *whose multiplicities are exactly* k. *In particular,* $\overline{N}(r, a; f| = 1)$ *or* $N(r, a; f| = 1)$ *is the counting function of the simple* a*-points of* f.

Definition 6 *For a positive integer* m *we denote by* $N(r, a; f| \leq m)$ $(N(r, a; f| \geq m))$ *the counting function of those* a*-points of* f *whose multiplicities are not greater (less) than* m, *where each* a*-point is counted according to its multiplicity.* $\overline{N}(r, a; f| \leq m)$ *and* $\overline{N}(r, a; f| \geq m)$ *are the corresponding reduced counting functions.*

Definition 7 *We denote by* $N_2(r, a; f)$ *the sum* $\overline{N}(r, a; f) + \overline{N}(r, a; f| \geq 2)$.

Definition 8 *Let* f *and* g *be two nonconstant meromorphic functions such that* f *and* g *share* $(a, 2)$ *for* $a \in \mathbb{C} \cup \{\infty\}$. *Let* z_0 *be an* a*-point of* f *with multiplicity* p *and an* a*-point of* g *with multiplicity* q. *We denote by* $\overline{N}_L(r, a; f)$ $(\overline{N}_L(r, a; g))$ *the reduced counting function of those* a*-points of* f *and* g *where* $p > q \geq 3$ $(q > p \geq 3)$. *Also we denote by* $\overline{N}_E^{(3}(r, a; f)$ *the counting function of those* a*-points of* f *and* g *where* $p = q \geq 3$. *Clearly* $\overline{N}_E^{(3}(r, a; f) = \overline{N}_E^{(3}(r, a; g)$.

Definition 9 *Let* $f,\ g$ *share the value* a *IM. We denote by* $\overline{N}_*(r, a; f, g)$ *the reduced counting function of those* a*-points of* f *whose multiplicities differ from the multiplicities of the corresponding* a*-points of* g.

Clearly $\overline{N}_*(r, a; f, g) = \overline{N}_*(r, a; g, f)$ and $\overline{N}_*(r, a; f, g) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g)$.

2 Lemmas

In this section, we present some lemmas which will be needed in the sequel. We denote by H the following function:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right),$$

where F and G are two meromorphic functions in $\mathcal{M}_1(\mathbb{C})$.

Lemma 1 [7] *If* F, G *share* $(1,1)$ *and* $H \not\equiv 0$, *then*

$$N(r,1;F|=1) \leq N(r,\infty;H) + S(r,F) + S(r,G).$$

Lemma 2 *Let* $F, G \in \mathcal{M}_1(\mathbb{C})$. *If* F, G *share* $(1,0)$ *and* $H \not\equiv 0$, *then*

$$\begin{aligned} N(r,\infty,H) \quad \leq \quad & \overline{N}(r,0;F|\geq 2) + \overline{N}(r,0;G|\geq 2) + \overline{N}_*(r,1;F,G) \\ & + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') + S(r,F) + S(r,G), \end{aligned}$$

where $\overline{N}_0(r,0;F')$ *is the reduced counting function of those zeros of* F' *which are not the zeros of* $F(F-1)$. $\overline{N}_0(r,0;G')$ *is defined similarly.*

Proof. Noting that $\overline{N}_*(r,\infty;F,G) = S(r,F)+S(r,G)$, this lemma can be proved in a similar manner as in Lemma 4 of [9]. □

Lemma 3 [1] *Let* F *and* G *be two nonconstant meromorphic functions sharing* $(1,2)$. *Then*

$$\begin{aligned} 2\overline{N}_L(r,1;F) + 3\overline{N}_L(r,1;G) + 2\overline{N}_E^{(3}(r,1;F) + \overline{N}(r,1;F|=2) \\ \leq N(r,1;G) - \overline{N}(r,1;G). \end{aligned}$$

Lemma 4 [11] *Let* f *be a nonconstant meromorphic function and* $P(f) = a_0 + a_1 f + a_2 f^2 + \ldots + a_n f^n$, *where* a_0, a_1, $a_2, \ldots, a_n$ *are constants and* $a_n \neq 0$. *Then* $T(r,P(f)) = nT(r,f) + O(1)$.

Lemma 5 [15] *If* $H \equiv 0$, *then* $T(r,G) = T(r,F) + O(1)$. *If, in addition,*

$$\limsup_{r\to\infty, r\notin E} \frac{\overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}(r,0;G) + \overline{N}(r,\infty;G)}{T(r)} < 1,$$

where $T(r) = \max\{T(r,F), T(r,G)\}$ *then either* $F \equiv G$ *or* $F.G \equiv 1$.

Remark 1 *We observe that the above lemma holds for* $F, G \in \mathcal{M}(\mathbb{C})$*. As our discussion is restricted in* $\mathcal{M}_1(\mathbb{C})$*, we may drop the terms* $\overline{N}(r,\infty;F)$ *and* $\overline{N}(r,\infty;G)$ *while using this result.*

Lemma 6 *Let* $F, G \in \mathcal{M}_1(\mathbb{C})$*. If* F *and* G *share* $(1,2)$ *and* $H \not\equiv 0$*, then*

(i) $T(r,F) \le N_2(r,0;F) + N_2(r,0;G) - m(r,1;G) - \overline{N}_E^{(3}(r,1;F) - \overline{N}_L(r,1;G) + S(r,F) + S(r,G)$;

(ii) $T(r,G) \le N_2(r,0;G) + N_2(r,0;F) - m(r,1;F) - \overline{N}_E^{(3}(r,1;G) - \overline{N}_L(r,1;F) + S(r,F) + S(r,G)$.

Proof. The proof of this lemma flows in the line of the proof of Lemma 2.13 in [1]. As we are dealing with functions of class $\mathcal{M}_1(\mathbb{C})$, we insist in presenting the proof for the sake of completeness.

From the second fundamental theorem of Nevanlinna , we have

$$T(r,F) \le \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}(r,1;F) - N_0(r,0;F') + S(r,F);$$

that is,

$$T(r,F) \le \overline{N}(r,0;F) + \overline{N}(r,1;F) - N_0(r,0;F') + S(r,F). \tag{2}$$

Similarly,

$$T(r,G) \le \overline{N}(r,0;G) + \overline{N}(r,1;G) - N_0(r,0;G') + S(r,G). \tag{3}$$

Combining (2) and (3), we obtain

$$\begin{aligned} T(r,F) + T(r,G) \le\ & \overline{N}(r,0;F) + \overline{N}(r,0;G) + \overline{N}(r,1;F) + \overline{N}(r,1;G) \\ & - N_0(r,0;F') - N_0(r,0;G') + S(r,F) + S(r,G). \end{aligned} \tag{4}$$

We also see that

$$\begin{aligned} \overline{N}(r,1;F) + \overline{N}(r,1:G) \le\ & N(r,1;F|=1) + \overline{N}(r,1;F|=2) + \overline{N}_E^{(3}(r,1;F) \\ & + \overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) + \overline{N}(r,1;G). \end{aligned} \tag{5}$$

Using Lemma 1 and Lemma 2 in (5), we obtain that

$$\begin{aligned} \overline{N}(r,1;F) + \overline{N}(r,1;G) \le\ & \overline{N}(r,0;F| \ge 2) + \overline{N}(r,0;G| \ge 2) + 2\overline{N}_L(r,1;F) \\ & + 2\overline{N}_L(r,1;G) + \overline{N}(r,1;F|=2) + \overline{N}_E^{(3}(r,1;F) \\ & + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') + \overline{N}(r,1;G) \\ & + S(r,F) + S(r,G). \end{aligned}$$

Substituting the value of $\overline{N}(r,1;G)$ from Lemma 3, we obtain

$$\begin{aligned}\overline{N}(r,1;F)+\overline{N}(r,1;G) \leq\ & \overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)+2\overline{N}_L(r,1;F)\\ &+2\overline{N}_L(r,1;G)+\overline{N}(r,1;F|=2)+\overline{N}_E^{(3}(r,1;F)\\ &+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+N(r,1;G)\\ &-2\overline{N}_L(r,1;F)-3\overline{N}_L(r,1;G)-2\overline{N}_E^{(3}(r,1;F)\\ &-\overline{N}(r,1;F|=2)+S(r,F)+S(r,G)\\ \leq\ & \overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)-\overline{N}_L(r,1;G)\\ &-\overline{N}_E^{(3}(r,1;F)+T(r,G)-m(r,1;G)+\overline{N}_0(r,0;F')\\ &+\overline{N}_0(r,0;G')+S(r,F)+S(r,G). \end{aligned} \tag{6}$$

Noting the fact that $N_2(r,a;f)=\overline{N}(r,a;f)+\overline{N}(r,a;f|\geq 2)$, the lemma follows from (4) and (6). □

Lemma 7 *Let f, $g \in \mathcal{M}_1(\mathbb{C})$. If f, g share the set $\{\beta_1,\beta_2\}$ IM, then $\lambda(f)=\lambda(g)$.*

Proof. Proof of this lemma can be extracted from the first part of the proof of Theorem 1.3 in [2] (see p. 1247). □

Lemma 8 (see [12, p. 65]) *Let h be an entire function and $f(z)=e^{h(z)}$. Then*

(i) *if $h(z)$ is a polynomial of $\deg h$, then $\lambda(f)=\deg h$;*

(ii) *if $h(z)$ is a transcendental entire function, then $\lambda(f)=\infty$.*

Lemma 9 (see [12, p. 115]) *Let a_1, a_2 and a_3 be three distinct complex numbers in $\mathbb{C}\cup\{\infty\}$. If two nonconstant meromorphic functions f and g share a_1, a_2 and a_3 CM, and if the order of f and g is neither an integer nor infinity, then $f\equiv g$.*

3 Proof of the Theorems

Proof. [Proof of Theorem 1] Let $F=P(f)$ and $G=P(g)$ where $P(z)$ is defined as in (1). Clearly F, G share $(1,2)$ as f, g share $(S_1,2)$. From Lemma 4, we obtain

$$T(r,F)=kT(r,f)+S(r,f); \tag{7}$$

$$T(r,G) = kT(r,g) + S(r,g). \tag{8}$$

Let $H \not\equiv 0$. By Lemma 6, we have

$$\begin{aligned} T(r,F) &\le N_2(r,0;F) + N_2(r,0;G) + S(r,F) + S(r,G) \\ &= N_2(r,0;P(f)) + N_2(r,0;P(g)) + S(r,f) + S(r,g) \\ &\le \Gamma_2\overline{N}(r,0;f) + \Gamma_2\overline{N}(r,0;g) + S(r,f) + S(r,g) \\ &\le \Gamma_2\{T(r,f) + T(r,g)\} + S(r,f) + S(r,g). \end{aligned} \tag{9}$$

Similarly,

$$T(r,G) \le \Gamma_2\{T(r,f) + T(r,g)\} + S(r,f) + S(r,g). \tag{10}$$

From (7)-(10), we obtain

$$k\{T(r,f) + T(r,g)\} \le 2\Gamma_2\{T(r,f) + T(r,g)\} + S(r,f) + S(r,g),$$

which is a contradiction as $k > 2\Gamma_2$. Hence $H \equiv 0$.

Let $T(r) = \max\{T(r,F), T(r,G)\}$. Now,

$$\begin{aligned} \overline{N}(r,0;F) + \overline{N}(r,0;G) &\le \Gamma_1\overline{N}(r,0;f) + \Gamma_1\overline{N}(r,0;g) \\ &\le \Gamma_1\{T(r,f) + T(r,g)\} + S(r,f) + S(r,g) \\ &= \frac{\Gamma_1}{k}\{T(r,F) + T(r,G)\} + S(r,F) + S(r,G) \\ &\le \frac{2\Gamma_1}{k}T(r) + o\{T(r)\}. \end{aligned} \tag{11}$$

As $k > 2\Gamma_2 \ge 2\Gamma_1$, from Lemma 5 and (11), we obtain either $F \equiv G$ or $F.G \equiv 1$.

If possible, let $F.G \equiv 1$. Then $P(f).P(g) \equiv 1$. As $g \in \mathcal{M}_1(\mathbb{C})$, we have $P(g) \in \mathcal{M}_1(\mathbb{C})$. Hence $P(f)$ has at most finitely many zeros. Therefore $P(f) = \mu_1(z)e^{\phi_1(z)}$, where $\mu_1(z)$ is a rational function and $\phi_1(z)$ is an entire function, which is a contradiction by Lemma 8 as the order of f is neither an integer not infinity. Similarly if we consider the case when $P(g)$ has at most finitely many zeros, we arrive at a contradiction as $\lambda(g) = \lambda(f)$, by Lemma 7. Hence the case $F.G \equiv 1$ can not occur.

If $F \equiv G$, we have $P(f) \equiv P(g)$, which gives

$$\frac{(f(z)-\alpha_1)(f(z)-\alpha_2)\ldots(f(z)-\alpha_k)}{(g(z)-\alpha_1)(g(z)-\alpha_2)\ldots(g(z)-\alpha_k)} \equiv 1. \tag{12}$$

From (12) and the assumption

$$(\beta_1-\alpha_1)^2(\beta_1-\alpha_2)^2\cdots(\beta_1-\alpha_k)^2 \ne (\beta_2-\alpha_1)^2(\beta_2-\alpha_2)^2\cdots(\beta_2-\alpha_k)^2,$$

we obtain that $f(z) = \beta_1$ if and only if $g(z) = \beta_1$ since f and g share S_2 IM. Similarly, we see that $f(z) = \beta_2$ if and only if $g(z) = \beta_2$. Consequently, we have f and g share β_1 and β_2 IM. Again, from (12) we see that f and g share β_1, β_2 and ∞ CM. Noting that the order of f is neither an integer nor infinity, the conclusion follows from Lemma 7 and Lemma 9. □

Proof. [Proof of Theorem 2] If f, g share S_1 and S_2 CM, then f, g certainly share $(S_1, 2)$ and S_2 IM, which satisfies the conditions of Theorem 1 and hence the conclusion follows. Here we omit the details. □

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