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Scaling functions on the spectrum

Abdullah
Department of Mathematics,
ZHDC, University of Delhi, India
email: abd.zhc.du@gmail.com

Firdous A. Shah
Department of Mathematics,
University of Kashmir, India
email: fashah79@gmail.com

Abstract. A generalization of Mallat's classic theory of multiresolution analysis based on the theory of spectral pairs was considered by Gabardo and Nashed [4] for which the translation set $\Lambda = \{0, r/N\} + 2\,\mathbb{Z}$ is no longer a discrete subgroup of $\mathbb{R}$ but a spectrum associated with a certain one-dimensional spectral pair. In this short communication, we characterize the scaling functions associated with such a nonuniform multiresolution analysis by means of some fundamental equations in the Fourier domain.

1 Introduction

Multiresolution analysis (MRA) is an important mathematical tool since it provides a natural framework for understanding and constructing discrete wavelet systems. The concept of an MRA structure has been extended in various setups in recent years. More precisely, they have been generalized to different dimensionalities, to lattices different from $\mathbb{Z}^d$, allowing the subspaces of MRA to be generated by Riesz basis instead of orthonormal basis, admitting a finite number of scaling functions, replacing the dilation factor 2 by an integer $M \geq 2$ or by an expansive matrix $A \in GL_d(\mathbb{R})$ as long as $A \subset A\mathbb{Z}^d$ (see [1]). All these concepts were developed on regular lattices, that is the translation set is always a group. Recently, Gabardo and Nashed [3, 4] considered a generalization of Mallat's classical MRA [6] based on the theory of

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spectral pairs, in which the translation set $\Lambda = \{0, r/N\} + 2\mathbb{Z}$, where $N \geq 1$ is an integer, $1 \leq r \leq 2N-1, r$ is an odd integer relatively prime to N, acting on the scaling function related with an MRA to generate the core subspace V_0 is no longer a group, but a union of two lattices, which is associated with a famous open conjecture of Fuglede on spectral pairs [2]. They call it *nonuniform multiresolution analysis* (NUMRA). By an NUMRA, we mean a sequence of embedded closed subspaces $\{V_j : j \in \mathbb{Z}\}$ of the Hilbert space $L^2(\mathbb{R})$ that satisfies the following conditions:

(a) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;

(b) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$;

(c) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;

(d) $f(x) \in V_j$ if and only if $f(2Nx) \in V_{j+1}$ for all $j \in \mathbb{Z}$;

(e) there exists a function $\phi \in V_0$ such that $\{\phi(x-\lambda)\}_{\lambda\in\Lambda}$ is an orthonormal basis for V_0.

It is worth noticing that, when $N = 1$, one recovers the standard definition of one dimensional MRA with dyadic dilation 2. When, $N > 1$, the dilation factor of $2N$ ensures that $2N\Lambda \subset \mathbb{Z} \subset \Lambda$.

If ϕ is a scaling function of an NUMRA, then by condition (e) we can express this function in terms of the orthonormal basis $\{\phi(x-\lambda) : \lambda \in \Lambda\}$ as

$$\phi(x) = \sum_{\lambda\in\Lambda} h_\lambda \phi\big(2Nx - \lambda\big). \tag{1}$$

where the convergence is in $L^2(\mathbb{R})$ and $\{h_\lambda\}_{\lambda\in\Lambda} \in l^2$. Refinement equation (1) can be rewritten in the Fourier domain as

$$\hat{\phi}(\xi) = m_0\left(\frac{\xi}{2N}\right)\hat{\phi}\left(\frac{\xi}{2N}\right) \tag{2}$$

where m_0 is the low pass filter associated with the scaling function ϕ and is of the form

$$m_0(\xi) = m_0^1(\xi) + e^{-2\pi i\xi r/N} m_0^2(\xi). \tag{3}$$

One of the fundamental problems in the study of wavelet theory is to find conditions on the scaling functions so that they can generate an MRA for $L^2(\mathbb{R})$. Our main purpose in this short communication is to characterize those functions that are scaling functions for an NUMRA of $L^2(\mathbb{R})$.

To achieve our goal, we need the following technical results obtained in [4, 5, 7] that will be used in sequel.

Theorem 1 [4] *Let* $\{V_j : j \in \mathbb{Z}\}$ *be a sequence of closed subspaces of* $L^2(\mathbb{R})$ *satisfying conditions (a), (d) and (e). Then,* $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$.

Theorem 2 [5] *Let* $\{V_j : j \in \mathbb{Z}\}$ *be a sequence of closed subspaces of* $L^2(\mathbb{R})$ *satisfying conditions (a), (d) and (e). Assume that the function* ϕ *of condition (e) is such that* $\hat{\phi}$ *is continuous at* $\xi = 0$. *Then the following two conditions are equivalent:*

(i) $\lim_{j\to\infty} \left|\hat{\phi}\big((2N)^{-j}\xi\big)\right| = 1 \quad \text{a.e. } \xi \in \mathbb{R};$

(ii) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$.

Proposition 1 [7] *Let* N *be a positive integer, and* $r \in \{1, 3, \ldots, 2N-1\}$ *be an odd integer. Let* $\phi \in L^2(\mathbb{R})$ *with* $\|\phi\|_2 = 1$. *Then,*

(i) *For each fixed odd* r, *the family* $\{\phi(x-\lambda) : \lambda \in \Lambda\}$ *is an orthonormal system in* $L^2(\mathbb{R})$ *if and only if*

$$\sum_{p\in\mathbb{Z}} \left|\hat{\phi}\left(\xi + \frac{p}{2}\right)\right|^2 = 2, \quad \text{for a.e. } \xi \in \mathbb{R} \quad \text{and} \tag{4}$$

$$\sum_{p\in\mathbb{Z}} e^{-i\pi rp/N} \left|\hat{\phi}\left(\xi + \frac{p}{2}\right)\right|^2 = 0, \quad \text{for a.e. } \xi \in \mathbb{R}. \tag{5}$$

(ii) *The collection* $\{\phi(x-\lambda) : \lambda \in \Lambda\}$ *is an orthonormal system for every odd integer* $r \in \{1, 3, \ldots, 2N-1\}$ *if and only if*

$$\sum_{\beta\in\Gamma_N} \left|\hat{\phi}(\xi - \beta)\right|^2 = 1, \ \text{for a.e. } \xi \in \mathbb{R}, \tag{6}$$

where $\Gamma_N = \{nN + j/2 : n \in \mathbb{Z},\ j = 0, 1, 2, \ldots, N-1\}$.

2 Characterization of scaling functions on the spectrum

In this section we will characterize those functions that are scaling functions for an NUMRA of $L^2(\mathbb{R})$ by means of some basic equations in the Fourier domain.

Before formulating our main result, let us clarify what we mean when we say that a function is a scaling function for an NUMRA. Given a function $\phi \in L^2(\mathbb{R})$, we define the closed subspaces $\{V_j : j \in \mathbb{Z}\}$ of $L^2(\mathbb{R})$ as follows:

$$V_0 = \overline{\text{span}}\{\phi(x-\lambda) : \lambda \in \Lambda\}, \text{ and } V_j = \left\{f : f\big((2N)^{-j}x\big) \in V_0\right\},\ j \in \mathbb{Z}\backslash\{0\}.$$

We say that $\phi \in L^2(\mathbb{R})$ is a scaling function for an NUMRA of $L^2(\mathbb{R})$ if the sequence of closed subspaces $\{V_j : j \in \mathbb{Z}\}$ as defined above forms an NUMRA for $L^2(\mathbb{R})$.

Theorem 3 *A function $\phi \in L^2(\mathbb{R})$ is a scaling function for an NUMRA of $L^2(\mathbb{R})$ if and only if*

$$\sum_{\beta \in \Gamma_N} \left|\hat{\phi}(\xi - \beta)\right|^2 = 1, \ \mathrm{texta.e} \tag{7}$$

$$\lim_{j \to \infty} \left|\hat{\phi}\left((2N)^{-j}\xi\right)\right| = 1 \ a.e. \ \xi \in \mathbb{R} \tag{8}$$

and there exists a periodic function m_0 of the form (3) such that

$$\hat{\phi}(\xi) = m_0\left(\frac{\xi}{2N}\right)\hat{\phi}\left(\frac{\xi}{2N}\right), \quad \text{a.e. } \xi \in \mathbb{R}. \tag{9}$$

Proof. Suppose ϕ is a scaling function for an NUMRA. Then, $\{\phi(x - \lambda) : \lambda \in \Lambda\}$ forms an orthonormal system in $L^2(\mathbb{R})$ which is equivalent to equation (7) by Proposition 1. Equality (9) follows from equations (2) and (3). Since $\{V_j : j \in \mathbb{Z}\}$ is an NUMRA for $L^2(\mathbb{R})$, we have $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$. Therefore, from Theorem 2, we infer that

$$\lim_{j \to \infty} \int_{\Gamma_N} \left|\hat{\phi}\left((2N)^{-j}\xi\right)\right|^2 d\xi = 1.$$

Since $m_0(\xi)$ is of the form (3), so it is easy to compute the following two conditions in terms of the 1/2-periodic functions m_0^1, m_0^2 as

$$\sum_{p=0}^{2N-1} \left\{\left|m_0^1\left(\xi + \frac{p}{4N}\right)\right|^2 + \left|m_0^2\left(\xi + \frac{p}{4N}\right)\right|^2\right\} = 1, \quad \text{and} \tag{10}$$

$$\sum_{p=0}^{2N-1} e^{-i\pi r p/N}\left\{\left|m_0^1\left(\xi + \frac{p}{4N}\right)\right|^2 + \left|m_0^2\left(\xi + \frac{p}{4N}\right)\right|^2\right\} = 0. \tag{11}$$

If we take $M_0(\xi) = \left|m_0^1(\xi)\right|^2 + \left|m_0^2(\xi)\right|^2$, then clearly $M_0\left(\xi + \frac{1}{4}\right) = M_0(\xi)$ and

$$M_0(\xi) = \frac{|m_0(\xi + N/2)|^2 + |m_0(\xi)|^2}{2}. \tag{12}$$

Subsequently, Eqs. (10) and (11) takes the form

$$\sum_{p=0}^{2N-1} M_0\left(\xi+\frac{p}{4N}\right)=1, \quad \text{and} \quad \sum_{p=0}^{2N-1} e^{-i\pi r p/N} M_0\left(\xi+\frac{p}{4N}\right)=0.$$

Hence, $M_0(\xi)\leq 1$, a.e. $\xi\in\mathbb{R}$, which together with (12) implies $|m_0(\xi)|\leq 1$ a.e. $\xi\in\mathbb{R}$. This inequality along with equality (9) shows that $\left|\hat{\phi}((2N)^{-j}\xi)\right|$ is non-decreasing for a.e. $\xi\in\mathbb{R}$ as $j\to\infty$. Let

$$\Phi(\xi)=\lim_{j\to\infty}\left|\hat{\phi}((2N)^{-j}\xi)\right|. \tag{13}$$

Since $\left|\hat{\phi}(\xi)\right|\leq 1$ a.e, therefore, Lebesgue's dominated convergence theorem implies that

$$\int_{\Gamma_N}\Phi(\xi)\,d\xi=1.$$

We now prove the converse. Assume that (7), (8) and (9) are satisfied. The orthonormality of the system $\{\phi(x-\lambda):\lambda\in\Lambda\}$ follows immediately from (7). This fact alongwith the definition of V_0 gives us (e) of the definition of an NUMRA. Moreover, the definition of the subspaces V_j also shows that $f(x)\in V_j$ holds if and only if $f\big(2Nx\big)\in V_{j+1}$ which is (d) of the definition of an NUMRA. Thus, we say that if $(2N)^{-j/2}f((2N)^{-j}x)\in V_0$, then there exists a sequence $\{h_\lambda\}_{\lambda\in\Lambda}$ satisfying $\sum_{\lambda\in\Lambda}h_\lambda<\infty$ such that

$$f\left((2N)^{-j}x\right)=(2N)^{j/2}\sum_{\lambda\in\Lambda}h_\lambda\phi(x-\lambda). \tag{14}$$

Taking Fourier transform on both sides of (14), we obtain

$$\hat{f}\left((2N)^{j}\xi\right)=\mu_j(\xi)\hat{\phi}(\xi) \tag{15}$$

where $\mu_j(\xi)=\sum_{\lambda\in\Lambda}h_\lambda e^{-2\pi i\lambda\xi}$. Since $\Lambda=\{0,r/N\}+2\mathbb{Z}$, we can rewrite $\mu_j(\xi)$ as

$$\mu_j(\xi)=\mu_j^1(\xi)+e^{-2\pi i\xi r/N}\mu_j^2(\xi) \tag{16}$$

where μ_j^1 and μ_j^2 are locally L^2, $1/2$-periodic functions. Now, for each $j\in\mathbb{Z}$, we claim that

$$V_j=\left\{f:\hat{f}\left((2N)^{j}\xi\right)=\mu_j(\xi)\hat{\phi}(\xi)\text{ for some periodic function }\mu_j(\xi)\right\}. \tag{17}$$

To prove the inclusion $V_j \subset V_{j+1}$, it is enough to show that $V_0 \subset V_1$. Assume that $f \in V_0$, then by equation (17), it follows that there exists a locally L^2 function say μ_0 such that $\hat{f}(\xi) = \mu_0(\xi)\hat{\phi}(\xi)$, where $\mu_0(\xi) = \mu_0^1(\xi) + e^{-2\pi i \xi r/N}\mu_0^2(\xi)$. Using (9), we obtain

$$\hat{f}(2N\xi) = \mu_0(2N\xi)\hat{\phi}(2N\xi) = \mu_0(2N\xi)m_0(\xi)\hat{\phi}(\xi).$$

Moreover, $\mu_0(2N\xi)m_0(\xi)$ can be further expressed in the form

$$\eta_1(\xi) + e^{-2\pi i \xi r/N}\eta_2(\xi),$$

where

$$\eta_1(\xi) = \left\{\mu_0^1(2N\xi) + e^{-4\pi i \xi r}\mu_0^2(2N\xi)\right\} m_0^1(\xi)$$
$$\eta_2(\xi) = \left\{\mu_0^1(2N\xi) + e^{-4\pi i \xi r}\mu_0^2(2N\xi)\right\} m_0^2(\xi).$$

Using the fact that $|m_0(\xi)| \le 1$ for a.e. $\xi \in \Gamma_N$, we have

$$\int_{\Gamma_N} \left|\mu_0(2N\xi)\right|^2 \left|m_0(\xi)\right|^2 d\xi \le \int_{\Gamma_N} \left|\mu_0(2N\xi)\right|^2 d\xi < \infty,$$

which implies that $f \in V_1$. We have already seen that separation property (c) of an NUMRA follows from (a), (d) and (e). Now it remains to prove density property (b) of an NUMRA, that is; $L^2(\mathbb{R}) = \overline{\cup_{j\in\mathbb{Z}} V_j}$. To prove this, we assume that P_j be the orthogonal projection onto the closed subspace V_j of $L^2(\mathbb{R})$, then it suffices to show that

$$\left\|P_j f - f\right\|_2^2 = \|f\|_2^2 - \left\langle P_j(f), f\right\rangle_2 \to 0 \quad \text{as} \quad j \to \infty.$$

Since $\left\{(2N)^{j/2}\phi\left((2N)^j x - \lambda\right)\right\}_{\lambda\in\Lambda}$ is an orthonormal basis for V_j. Therefore, for any compactly supported function f, we have

$$\left\langle P_j f, f\right\rangle_2 = \int_{\mathbb{R}} \left|\hat{\phi}\left((2N)^{-j}\xi\right)\right|^2 \left|\hat{f}(\xi)\right|^2 d\xi. \tag{18}$$

Implementing condition (8), it follows that the right hand side of (18) converges to $\|f\|_2^2$ as $j \to \infty$. This completes the proof of Theorem 3. □

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