Integrals of polylogarithmic functions with negative argument

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Abstract. The connection between polylogarithmic functions and Euler sums is well known. In this paper we explore the representation and many connections between integrals of products of polylogarithmic functions and Euler sums. We shall consider mainly, polylogarithmic functions with negative arguments, thereby producing new results and extending the work of Freitas. Many examples of integrals of products of polylogarithmic functions in terms of Riemann zeta values and Dirichlet values will be given.

1 Introduction and preliminaries

It is well known that integrals of products of polylogarithmic functions can be associated with Euler sums, see [16]. In this paper we investigate the representations of integrals of the type

$$\int_0^1 x^m \operatorname{Li}_t(-x) \operatorname{Li}_q(-x) \, dx,$$

for $m \geq -2$, and for integers q and t. For $m = -2, -1, 0$ we give explicit representations of the integral in terms of Euler sums and for $m \geq 0$ we give a recurrence relation for the integral in question. We also mention two specific

integrals with a different argument in the polylogarithm. Some examples are highlighted, almost none of which are amenable to a computer mathematical package. This work extends the results given by [16], who examined a similar integral with positive arguments of the polylogarithm. Devoto and Duke [14] also list many identities of lower order polylogarithmic integrals and their relations to Euler sums. Some other important sources of information on polylogarithm functions are the works of [19] and [20]. In [3] and [12] the authors explore the algorithmic and analytic properties of generalized harmonic Euler sums systematically, in order to compute the massive Feynman integrals which arise in quantum field theories and in certain combinatorial problems. Identities involving harmonic sums can arise from their quasi-shuffle algebra or from other properties, such as relations to the Mellin transform

$$M[f(z)](N) = \int_0^1 dz\ z^N f(z),$$

where the basic functions $f(z)$ typically involve polylogarithms and harmonic sums of lower weight. Applying the latter type of relations, the author in [6], expresses all harmonic sums of the above type with weight $w = 6$, in terms of Mellin transforms and combinations of functions and constants of lower weight. In another interesting and related paper [17], the authors prove several identities containing infinite sums of values of the Roger's dilogarithm function. defined on $x \in [0.1]$, by

$$L_R(x) = \begin{cases} \text{Li}_2(x) + \frac{1}{2}\ln x \ln(1-x); & 0 < x < 1 \\ 0 & ; \quad x = 0 \\ \zeta(2) & ; \quad x = 1 \end{cases}.$$

The Lerch transcendent,

$$\Phi(z, t, a) = \sum_{m=0}^{\infty} \frac{z^m}{(m+a)^t}$$

is defined for $|z| < 1$ and $\Re(a) > 0$ and satisfies the recurrence

$$\Phi(z, t, a) = z\ \Phi(z, t, a+1) + a^{-t}.$$

The Lerch transcendent generalizes the Hurwitz zeta function at $z = 1$,

$$\Phi(1, t, a) = \sum_{m=0}^{\infty} \frac{1}{(m+a)^t}$$

and the polylogarithm, or de-Jonquière's function, when $a = 1$,

$$L_{i_t}(z) := \sum_{m=1}^{\infty} \frac{z^m}{m^t}, \ t \in \mathbb{C} \text{ when } |z| < 1; \ \Re(t) > 1 \text{ when } |z| = 1.$$

Let

$$H_n = \sum_{r=1}^{n} \frac{1}{r} = \int_0^1 \frac{1-t^n}{1-t} \, dt = \gamma + \psi(n+1) = \sum_{j=1}^{\infty} \frac{n}{j(j+n)}, \qquad H_0 := 0$$

be the nth harmonic number, where γ denotes the Euler-Mascheroni constant, $H_n^{(m)} = \sum_{r=1}^{n} \frac{1}{r^m}$ is the mth order harmonic number and $\psi(z)$ is the digamma (or psi) function defined by

$$\psi(z) := \frac{d}{dz} \{\log \Gamma(z)\} = \frac{\Gamma'(z)}{\Gamma(z)} \text{ and } \psi(1+z) = \psi(z) + \frac{1}{z},$$

moreover,

$$\psi(z) = -\gamma + \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+z} \right).$$

More generally a non-linear Euler sum may be expressed as,

$$\sum_{n \geq 1} \frac{(\pm 1)^n}{n^p} \left(\prod_{j=1}^{t} \left(H_n^{(\alpha_j)} \right)^{q_j} \prod_{k=1}^{r} \left(J_n^{(\beta_k)} \right)^{m_k} \right)$$

where $p \geq 2, \ t, r, q_j, \alpha_j, m_k, \beta_k$ are positive integers and

$$\left(H_n^{(\alpha)} \right)^q = \left(\sum_{j=1}^{n} \frac{1}{j^\alpha} \right)^q, \ \left(J_n^{(\beta)} \right)^m = \left(\sum_{j=1}^{n} \frac{(-1)^{j+1}}{j^\beta} \right)^m.$$

If, for a positive integer

$$\lambda = \sum_{j=1}^{t} \alpha_j q_j + \sum_{j=1}^{r} \beta_j m_j + p,$$

then we call it a λ-order Euler sum. The polygamma function

$$\psi^{(k)}(z) = \frac{d^k}{dz^k} \{\psi(z)\} = -(-1)^{k+1} k! \sum_{r=0}^{\infty} \frac{1}{(r+z)^{k+1}}$$

and has the recurrence

$$\psi^{(k)}(z+1) = \psi^{(k)}(z) + \frac{(-1)^k k!}{z^{k+1}}.$$

The connection of the polygamma function with harmonic numbers is,

$$H_z^{(\alpha+1)} = \zeta(\alpha+1) + \frac{(-1)^\alpha}{\alpha!}\psi^{(\alpha)}(z+1), \ z \neq \{-1, -2, -3, ...\}. \tag{1}$$

and the multiplication formula is

$$\psi^{(k)}(pz) = \delta_{m,0}\ln p + \frac{1}{p^{k+1}}\sum_{j=0}^{p-1}\psi^{(k)}(z+\frac{j}{p}) \tag{2}$$

for p a positive integer and $\delta_{p,k}$ is the Kronecker delta. We define the alternating zeta function (or Dirichlet eta function) $\eta(z)$ as

$$\eta(z) := \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^z} = \left(1 - 2^{1-z}\right)\zeta(z) \tag{3}$$

where $\eta(1) = \ln 2$. If we put

$$S(p,q) := \sum_{n=1}^{\infty}\frac{(-1)^{n+1} H_n^{(p)}}{n^q},$$

in the case where p and q are both positive integers and $p + q$ is an odd integer, Flajolet and Salvy [15] gave the identity:

$$2S(p,q) = (1 - (-1)^p)\zeta(p)\eta(q) + 2(-1)^p \sum_{i+2k=q}\binom{p+i-1}{p-1}\zeta(p+i)\ \eta(2k)$$

$$+ \eta(p+q) - 2\sum_{j+2k=p}\binom{q+j-1}{q-1}(-1)^j\eta(q+j)\ \eta(2k), \tag{4}$$

where $\eta(0) = \frac{1}{2}$, $\eta(1) = \ln 2$, $\zeta(1) = 0$, and $\zeta(0) = -\frac{1}{2}$ in accordance with the analytic continuation of the Riemann zeta function. We also know, from the work of [11] that for odd weight $(p + q)$ we have

$$BW(p.q) = \sum_{n=1}^{\infty}\frac{H_n^{(p)}}{n^q} = (-1)^p \sum_{j=1}^{\left[\frac{p}{2}\right]}\binom{p+q-2j-1}{p-1}\zeta(p+q-2j)\zeta(2j) \tag{5}$$

$$+\frac{1}{2}\left(1+(-1)^{p+1}\right)\zeta\left(p\right)\zeta\left(q\right)+(-1)^{p}\sum_{j=1}^{\left[\frac{p}{2}\right]}\binom{p+q-2j-1}{q-1}\zeta\left(p+q-2j\right)\zeta\left(2j\right)$$

$$+\frac{\zeta\left(p+q\right)}{2}\left(1+(-1)^{p+1}\binom{p+q-1}{p}+(-1)^{p+1}\binom{p+q-1}{q}\right),$$

where $[z]$ is the integer part of z. It appears that some isolated cases of $BW(p.q)$, for even weight $(p+q)$, can be expressed in zeta terms, but in general, almost certainly, for even weight $(p+q)$, no general closed form expression exits for $BW(p.q)$. (at least at the time of writing this paper). Two examples with even weight are

$$\sum_{n=1}^{\infty}\frac{H_{n}^{(2)}}{n^{4}}=\zeta^{2}\left(3\right)-\frac{1}{3}\zeta\left(6\right),\quad\sum_{n=1}^{\infty}\frac{H_{n}^{(4)}}{n^{4}}=\frac{13}{12}\zeta\left(8\right).$$

The work in this paper extends the results of [16] and later [25], in which they gave identities of products of polylogarithmic functions with positive argument in terms of zeta functions. Other works including, [1], [4], [8], [10], [13], [18], [22], [23], [24], cite many identities of polylogarithmic integrals and Euler sums, but none of these examine the negative argument case. The following result was obtained by Freitas, [16].

Lemma 1 *For* q *and* t *positive integers*

$$\int_{0}^{1}\frac{Li_{t}\left(x\right)\,Li_{q}\left(x\right)}{x}dx=\sum_{j=1}^{q-1}(-1)^{j+1}\zeta\left(t+j\right)\zeta\left(q-j+1\right)+(-1)^{q+1}EU\left(t+q\right)$$

where $EU\left(m\right)$ *is Euler's identity given in the next lemma.*

The following lemma will be useful in the development of the main theorem.

Lemma 2 *The following identities hold: for* $m\in\mathbb{N}$*. Euler's identity states*

$$EU\left(m\right)=\sum_{n=1}^{\infty}\frac{H_{n}}{n^{m}}=(m+2)\zeta\left(m+1\right)-\sum_{j=1}^{m-2}\zeta\left(m-j\right)\zeta\left(j+1\right).\qquad(6)$$

For p a positive even integer,

$$
HE\left(p\right) = \sum_{n=1}^{\infty} \frac{H_n}{\left(2n+1\right)^p} = \frac{p}{2}\left(\zeta\left(p+1\right)+\eta\left(p+1\right)\right) - \left(\zeta\left(p\right)+\eta\left(p\right)\right)\ln 2
$$

$$
-\frac{1}{2}\sum_{j=1}^{\frac{p}{2}-1}\left(\zeta\left(p+1-2j\right)+\eta\left(p+1-2j\right)\right)\left(\zeta\left(2j\right)+\eta\left(2j\right)\right). \tag{7}
$$

For p a positive odd integer,

$$
HO\left(p\right) = \sum_{n=1}^{\infty} \frac{H_n}{\left(2n+1\right)^p} = \frac{p}{4}\left(\zeta\left(p+1\right)+\eta\left(p+1\right)\right) - \left(\zeta\left(p\right)+\eta\left(p\right)\right)\ln 2
$$

$$
-\frac{1}{4}\left(\frac{1+\left(-1\right)^{\frac{p-1}{2}}}{2}\right)\left(\zeta\left(\frac{p+1}{2}\right)+\eta\left(\frac{p+1}{2}\right)\right) \tag{8}
$$

$$
-\frac{1}{2}\sum_{j=1}^{b}\left(\zeta\left(p-2j\right)+\eta\left(p-2j\right)\right)\left(\zeta\left(2j+1\right)+\eta\left(2j+1\right)\right)
$$

where $\eta\left(z\right)$ is the Dirichlet eta function, $b = \left[\frac{p-1}{4}\right] - \left(\frac{1+\left(-1\right)^{\frac{p-1}{2}}}{2}\right)$ and $[z]$ is the greatest integer less than z. For p and t positive integers we have

$$
F\left(p,t\right) = \sum_{n=1}^{\infty} \frac{\left(-1\right)^{n+1}}{n^p\left(n+1\right)^t}
$$

$$
= \sum_{r=1}^{p}\left(-1\right)^{p-r}\binom{p+t-r-1}{p-r}\eta\left(r\right) \tag{9}
$$

$$
+ \sum_{s=1}^{t}\left(-1\right)^{p+1}\binom{p+t-s-1}{t-s}\left(1-\eta\left(s\right)\right),
$$

$$
G\left(p,t\right) = \sum_{n=1}^{\infty} \frac{1}{n^p\left(n+1\right)^t} = \left(-1\right)^{p+1}\binom{p+t-1}{p}
$$

$$
+ \sum_{r=2}^{p}\left(-1\right)^{p-r}\binom{p+t-r-1}{p-r}\zeta\left(r\right) \tag{10}
$$

$$
+ \sum_{s=1}^{t}\left(-1\right)^{p}\binom{p+t-s-1}{t-s}\zeta\left(s\right),
$$

and

$$
\begin{aligned}
\mathsf{HG}(p,t) = \sum_{n=1}^{\infty} \frac{H_n}{n^p(n+1)^t} &= (-1)^{p+1}\binom{p+t-2}{p-1}\zeta(2) \\
&+ \sum_{r=2}^{p}(-1)^{p-r}\binom{p+t-r-1}{p-r}\mathsf{EU}(r) \\
&+ \sum_{s=2}^{t}(-1)^{p}\binom{p+t-s-1}{t-s}(\mathsf{EU}(s)-\zeta(s+1)).
\end{aligned} \tag{11}
$$

Proof. The identity (6) is the Euler relation and by manipulation we arrive at (7) and (8). The results (7) and (8) are closely related to those given by Nakamura and Tasaka [21]. For the proof of (9) we notice that

$$
\begin{aligned}
\frac{1}{n^p(n+1)^t} &= \sum_{r=1}^{p}(-1)^{p-r}\binom{p+t-r-1}{p-r}\frac{1}{n^r} \\
&+ \sum_{s=1}^{t}(-1)^{p}\binom{p+t-s-1}{t-s}\frac{1}{(n+1)^s}
\end{aligned}
$$

therefore, summing over the integers n,

$$
\begin{aligned}
\mathsf{F}(p,t) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^p(n+1)^t} &= \sum_{r=1}^{p}(-1)^{p-r}\binom{p+t-r-1}{p-r}\eta(r) \\
&+ \sum_{s=1}^{t}(-1)^{p}\binom{p+t-s-1}{t-s}(1-\eta(s))
\end{aligned}
$$

and hence (9) follows. Consider,

$$
\begin{aligned}
\frac{1}{n^p(n+1)^t} &= \frac{(-1)^{p+1}}{n(n+1)}\binom{p+t-2}{p-1}+\sum_{r=2}^{p}(-1)^{p-r}\binom{p+t-r-1}{p-r}\frac{1}{n^r} \\
&+ \sum_{s=2}^{t}(-1)^{p}\binom{p+t-s-1}{t-s}\frac{1}{(n+1)^s},
\end{aligned}
$$

and summing over the integers n produces the result (10). The proof of (11) follows by summing $\sum_{n=1}^{\infty}\frac{H_n}{n^p(n+1)^t}$ in partial fraction form. An example, from (8)

$$
\sum_{n=1}^{\infty}\frac{H_n}{(2n+1)^9} = \frac{9207}{2048}\zeta(10) - \frac{961}{1024}\zeta^2(5) - \frac{889}{512}\zeta(7)\zeta(3) - \frac{511}{256}\zeta(9)\ln 2
$$

and from (7),

$$\sum_{n=1}^{\infty} \frac{H_n}{(2n+1)^8} = \frac{511}{64} \zeta(9) - \frac{381}{256} \zeta(7) \zeta(2) - \frac{441}{256} \zeta(6) \zeta(3)$$

$$- \frac{465}{256} \zeta(5) \zeta(4) - \frac{255}{128} \zeta(8) \ln 2.$$

$$\square$$

2 Summation identity

We now prove the following theorems.

Theorem 1 *For positive integers* q *and* t, *the integral of the product of two polylogarithmic functions with negative arguments*

$$I_0(q, t) = \int_0^1 Li_t(-x) \ Li_q(-x) \, dx = \int_{-1}^0 Li_t(x) \ Li_q(x) \, dx$$

$$= \sum_{j=1}^{q-1} (-1)^{j+1} \eta(q-j+1) F(t, j)$$

$$+ (-1)^q (F(t, q+1) - (F(t, q) - G(t, q)) \ln 2) + (-1)^q W_n(q, t) \tag{12}$$

where the sum

$$W_n(q, t) = \sum_{n=1}^{\infty} H_n \left(\frac{1}{(2n)^t (2n+1)^q} - \frac{1}{n^t (n+1)^q} + \frac{1}{(2n+1)^t (2n+2)^q} \right) \tag{13}$$

is obtained from (6), (7), (8) *and the terms* $F(\cdot, \cdot)$, $G(\cdot, \cdot)$ *are obtained from* (9) *and* (10) *respectively.*

Proof. By the definition of the polylogarithmic function we have

$$I_0(q, t) = \int_0^1 Li_t(-x) \ Li_q(-x) \, dx = \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} \frac{(-1)^{n+r}}{n^t r^q (n+r+1)}$$

$$= \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} \frac{(-1)^{n+r}}{n^t} \left(\frac{(-1)^q}{(n+r+1)(n+1)^q} + \sum_{j=1}^{q} \frac{(-1)^{j+1}}{(n+1)^j \ r^{q-j+1}} \right)$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+r}}{n^t} \left(\frac{(-1)^{q+1}}{(n+1)^q} \left(\frac{1}{2} H_{\frac{n+1}{2}} - \frac{1}{2} H_{\frac{n}{2}} \right) + \sum_{j=1}^{q} \frac{(-1)^{j+1} \eta(q-j+1)}{(n+1)^j} \right)$$

$$= \sum_{j=1}^{q-1} (-1)^{j+1} \eta(q-j+1) \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^t (n+1)^j} + (-1)^q \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^t (n+1)^{q+1}}$$

$$+ (-1)^q \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^t (n+1)^q} \left(\frac{1}{2} H_{\frac{n+1}{2}} - \frac{1}{2} H_{\frac{n}{2}} - \ln 2 \right).$$

Now we utilize the double argument identity (2) together with (9) we obtain

$$I_0(q,t) = \sum_{j=1}^{q-1} (-1)^{j+1} \eta(q-j+1) F(t,j) + (-1)^q F(t,q+1)$$

$$+ (-1)^q \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^t (n+1)^q} \left(H_n - H_{\frac{n}{2}} - 2\ln 2 \right),$$

we can use the alternating harmonic number sum identity (4) to simplify the last sum, however we shall simplify further as follows.

$$I_0(q,t) = \sum_{j=1}^{q-1} (-1)^{j+1} \eta(q-j+1) F(t,j) + (-1)^q F(t,q+1)$$

$$+ (-1)^q \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^t (n+1)^q} \left((-1)^{n+1} \left(H_{\left[\frac{n}{2}\right]} - H_n \right) - (1 + (-1)^n) \ln 2 \right)$$

where $[z]$ is the integer part of z. Now

$$I_0(q,t) = \sum_{j=1}^{q-1} (-1)^{j+1} \eta(q-j+1) F(t,j) + (-1)^q F(t,q+1)$$

$$- (-1)^q (F(t,q) - G(t,q)) \ln 2 + (-1)^q W_n(q,t)$$

where

$$W_n(q,t) = \sum_{n=1}^{\infty} H_n \left(\frac{1}{(2n)^t (2n+1)^q} - \frac{1}{n^t (n+1)^q} + \frac{1}{(2n+1)^t (2n+2)^q} \right)$$

and the infinite positive harmonic number sums are easily obtainable from (6), (7), (8), hence the identity (12) is achieved. □

The next theorem investigates the integral of the product of polylogarithmic functions divided by a linear function.

Theorem 2 *Let* (t, q) *be positive integers, then for* $t + q$ *an odd integer*

$$I_1(t, q) = \int_0^1 \frac{Li_t(-x) \; Li_q(-x)}{x} \, dx = -\int_{-1}^0 \frac{Li_t(x) \; Li_q(x)}{x} \, dx$$

$$= \sum_{j=1}^{q-1} (-1)^{j+1} \eta(t+j) \eta(q-j+1)$$

$$+ (-1)^{q+1} (\zeta(t+q) + \eta(t+q)) \ln 2$$

$$+ (-1)^{q+1} (2^{-t-q} - 1) EU(q+t) + (-1)^{q+1} HO(q+t).$$

(14)

For $t + q$ *an even integer*

$$I_1(t, q) = \sum_{j=1}^{q-1} (-1)^{j+1} \eta(t+j) \eta(q-j+1) + (-1)^{q+1}(\zeta(t+q)$$

$$+ \eta(t+q)) \ln 2 + (-1)^{q+1} (2^{-t-q} - 1) EU(q+t)$$

$$+ (-1)^{q+1} HE(q+t).$$

(15)

Proof. Consider

$$I_1(t, q) = \int_0^1 \frac{Li_t(-x) \; Li_q(-x)}{x} \, dx = \sum_{n \geq 1} \frac{(-1)^n}{n^t} \int_0^1 x^{n-1} \, Li_q(-x) \, dx,$$

and successively integrating by parts leads to

$$I_1(t, q) = \sum_{n \geq 1} \frac{(-1)^n}{n^{t+j}} \sum_{j=1}^{q-1} \eta(q-j+1) + \sum_{n \geq 1} \frac{(-1)^{n+q+1}}{n^{t+q-1}} \int_0^1 x^{n-1} \, Li_1(-x) \, dx.$$

Evaluating the inner integral,

$$\int_0^1 x^{n-1} \, Li_1(-x) \, dx = -\int_0^1 x^{n-1} \ln(1+x) \, dx = \frac{1}{n} \left(\frac{1}{2} H_{\frac{n}{2}} - \frac{1}{2} H_{\frac{n-1}{2}} - \ln 2 \right),$$

so that

$$I_1(t, q) = \sum_{n \geq 1} \frac{(-1)^n}{n^{t+j}} \sum_{j=1}^{q-1} (-1)^j \eta(q - j + 1)$$

$$+ \sum_{n \geq 1} \frac{(-1)^{n+q+1}}{n^{t+q}} \left(\frac{1}{2} H_{\frac{n}{2}} - \frac{1}{2} H_{\frac{n-1}{2}} - \ln 2 \right)$$

$$= \sum_{j=1}^{q-1} (-1)^{j+1} \eta(q - j + 1) \eta(t + j)$$

$$+ \sum_{n \geq 1} \frac{(-1)^{n+q+1}}{n^{t+q}} \left(\frac{1}{2} H_{\frac{n}{2}} - \frac{1}{2} H_{\frac{n-1}{2}} - \ln 2 \right).$$

If we now utilize the multiplication formula (2) we can write

$$I_1(t, q) = \sum_{j=1}^{q-1} (-1)^{j+1} \eta(q - j + 1) \eta(t + j) + (-1)^{q+1} \sum_{n \geq 1} \frac{(-1)^{n+1}}{n^{t+q}} \left(H_n - H_{\frac{n}{2}} \right).$$

Now consider the harmonic number sum

$$\sum_{n \geq 1} \frac{(-1)^{n+1}}{n^{t+q}} \left(H_n - H_{\frac{n}{2}} \right) = \sum_{n \geq 1} \frac{(-1)^{n+1}}{n^{t+q}} \left(\begin{array}{c} (1 - (-1)^n) \ln 2 \\ + (-1)^{n+1} \left(H_{\left[\frac{n}{2} \right]} - H_n \right) \end{array} \right)$$

$$= \sum_{n \geq 1} \frac{(-1)^{n+1}}{n^{t+q}} \left((1 - (-1)^n) \ln 2 + (-1)^{n+1} \sum_{j=1}^{n} \frac{(-1)^j}{j} \right)$$

$$= \sum_{n \geq 1} \frac{(-1)^{n+1}}{n^{t+q}} (1 - (-1)^n) \ln 2 + \sum_{n \geq 1} \left(\frac{1}{2^{t+q}} - 1 \right) \frac{H_n}{n^{t+q}} + \sum_{n \geq 1} \frac{H_n}{(2n+1)^{t+q}}$$

$$= (\zeta(t + q) + \eta(t + q)) \ln 2 + \sum_{n \geq 1} \left(\frac{1}{2^{t+q}} - 1 \right) \frac{H_n}{n^{t+q}} + \sum_{n \geq 1} \frac{H_n}{(2n+1)^{t+q}}$$

where $[z]$ is the integer part of z. Hence

$$I_1(t,q) = \sum_{j=1}^{q-1} (-1)^{j+1} \eta(q-j+1)\, \eta(t+j) + (-1)^{q+1}(\zeta(t+q) + \eta(t+q))\ln 2$$

$$+ (-1)^{q+1} \sum_{n\geq 1} \left(\frac{1}{2^{t+q}} - 1\right)\frac{H_n}{n^{t+q}} + (-1)^{q+1} \sum_{n\geq 1} \frac{H_n}{(2n+1)^{t+q}}$$

$$= \sum_{j=1}^{q-1} (-1)^{j+1} \eta(q-j+1)\, \eta(t+j) + (-1)^{q+1}(\zeta(t+q) + \eta(t+q))\ln 2$$

$$+ (-1)^{q+1}\left(\frac{1}{2^{t+q}} - 1\right) EU(q+t)$$

$$+ (-1)^{q+1} \begin{cases} HO(q+t), & \text{for } t+q \text{ odd} \\[2mm] HE(q+t), & \text{for } t+q \text{ even} \end{cases},$$

hence (14) and (15) follow.   □

Remark 1 *It is interesting to note that, for* $m \in \mathbb{R}$,

$$\int_0^1 \frac{Li_t(-x^m)\; Li_q(-x^m)}{x}\,dx = \frac{1}{m}\int_0^1 \frac{Li_t(-x)\; Li_q(-x)}{x}\,dx$$

The next theorem investigates the integral of the product of polylogarithmic functions divided by a quadratic factor.

Theorem 3 *For positive integers* q *and* t, *the integral of the product of two polylogarithmic functions with negative arguments*

$$I_2(t,q) = \int_0^1 \frac{Li_t(-x)\; Li_q(-x)}{x^2}\,dx = \int_{-1}^0 \frac{Li_t(x)\; Li_q(x)}{x^2}\,dx$$

$$= \eta(q+1) + \sum_{j=1}^{q-1} (-1)^j \eta(q-j+1)\, F(j,t)$$

$$+ (-1)^q (F(q,t) + G(q,t))\ln 2 + (-1)^q W_n(t,q)$$

(16)

where the sum,

$$W_n(t,q) = \sum_{n=1}^{\infty} H_n \left(\frac{1}{(2n)^q(2n+1)^t} - \frac{1}{n^q(n+1)^t} + \frac{1}{(2n+1)^q(2n+2)^t}\right)$$

(17)

is obtained from (6), (7), (8) *and the terms* $F(\cdot, \cdot)$, $G(\cdot, \cdot)$ *are obtained from* (9) *and* (10) *respectively.*

Proof. Following the same process as in Theorem 2, we have,

$$I_2(t, q) = \int_0^1 \frac{\mathrm{Li}_t(-x) \ \mathrm{Li}_q(-x)}{x^2} dx = \sum_{n \geq 1} \frac{(-1)^n}{n^t} \int_0^1 x^{n-2} \ \mathrm{Li}_q(-x) \ dx$$

$$= -\int_0^1 x^{-1} \ \mathrm{Li}_q(-x) \ dx + \sum_{n \geq 2} \frac{(-1)^n}{n^t} \int_0^1 x^{n-2} \ \mathrm{Li}_q(-x) \ dx,$$

and re ordering the summation index n, produces

$$I_2(t, q) = \eta(q + 1) + \sum_{n \geq 1} \frac{(-1)^{n+1}}{(n+1)^t} \int_0^1 x^{n-1} \ \mathrm{Li}_q(-x) \ dx.$$

Integrating by parts, we have,

$$I_2(t, q) = \eta(q + 1) + \sum_{n \geq 1} \frac{(-1)^{n+1}}{(n+1)^t} \left(\begin{array}{c} \sum_{j=1}^{q-1} \frac{(-1)^j \eta(q-j+1)}{n^j} \\ + \frac{(-1)^{q+1}}{n^{q-1}} \int_0^1 x^{n-1} \ \mathrm{Li}_1(-x) \ dx \end{array} \right)$$

$$= \eta(q + 1) + \sum_{j=1}^{q-1} (-1)^j \eta(q - j + 1) \sum_{n \geq 1} \frac{(-1)^{n+1}}{n^j (n+1)^t}$$

$$+ \sum_{n \geq 1} \frac{(-1)^{n+q}}{n^q (n+1)^t} \left(\frac{1}{2} H_{\frac{n}{2}} - \frac{1}{2} H_{\frac{n-1}{2}} - \ln 2 \right).$$

Using the multiplication Theorem (2) and following the same steps as in Theorem 2, we have

$$I_2(t, q) = \eta(q + 1) + \sum_{j=1}^{q-1} (-1)^j \eta(q - j + 1) F(j, t)$$

$$+ (-1)^q (F(q, t) + G(q, t)) \ln 2 + (-1)^q W_n(t, q),$$

and the proof of Theorem 3 is finalized. □

The following recurrence relation holds for the reduction of the integral of the product of polylogarithmic functions multiplied by the power of its argument.

Lemma 3 *For* $(q, t) \in \mathbb{N}$ *and* $m \geq 0$, *let*

$$J(m, q, t) = \int_0^1 x^m \, Li_t(-x) \, Li_q(-x) \, dx = (-1)^m \int_{-1}^0 x^m \, Li_t(x) \, Li_q(x) \, dx$$

then

$$(m + 1) J(m, q, t) = \eta(q) \eta(t) - J(m, q, t - 1) - J(m, q - 1, t).$$

For $q = 1$,

$$(m + 1) J(m, 1, t) = \eta(t) + mJ(m - 1, 1, t) + J(m - 1, 1, t - 1) - J(m, 1, t - 1)$$
$$- mK(m, t) - K(m, t - 1)$$

where

$$K(m, t) = \int_0^1 x^m \, Li_t(-x) \, dx.$$

Proof. The proof of the lemma follows in a straight forward manner after integration by parts.   □

We list some examples of the results of the integrals in Theorems 1, 2 and 3.

Example 1

$$I_0(3, 3) = \int_0^1 (Li_3(-x))^2 \, dx = \frac{9}{16} \zeta^2(3) + \frac{5}{8} \zeta(4) - \frac{3}{4} \zeta(2) \zeta(3)$$

$$+ (3\zeta(3) - 6\zeta(2) - 40) \ln 2 + 4\zeta(2) + 12 \ln^2 2 + 20.$$

$$I_0(3, 4) = \int_0^1 Li_3(-x) \, Li_4(-x) \, dx = \frac{3}{4} \eta(4) + \zeta(3) \, 2\zeta(5) - \frac{49}{64} \zeta(6) - \frac{9}{16} \zeta^2(3)$$

$$+ \frac{5}{4} \zeta(3) - \frac{15}{2} \zeta(2) + \left(10\zeta(2) - 6\zeta(3) + \frac{7}{4} \zeta(4) + 70\right) \ln 2$$

$$- \frac{3}{2} \zeta(4) - \frac{17}{16} \zeta(5) + \frac{3}{2} \zeta(2) \zeta(3) - 20 \ln^2 2 - 35.$$

$$I_1(2m, 2m + 1) = \int_0^1 \frac{Li_{2m}(-x) \, Li_{2m+1}(-x)}{x} \, dx = \frac{1}{2} \eta^2(2m + 1)$$

for $m \in \mathbb{N}$.

$$I_1(4,7) = \int_0^1 \frac{Li_4(-x) \; Li_7(-x)}{x} \, dx = \eta(5)\,\eta(7) - \frac{1}{2}\eta^2(6) \, .$$

$$I_2(3,4) = \int_0^1 \frac{Li_3(-x) \; Li_4(-x)}{x^2} \, dx = 2\zeta(5) - \frac{49}{64}\zeta(6) + \frac{23}{8}\zeta(4) - \frac{9}{16}\zeta^2(3)$$

$$+ 10\zeta(3) - \left(10\zeta(2) + 6\zeta(3) + \frac{7}{4}\zeta(4)\right)\ln 2$$

$$+ 10\zeta(2) - \frac{3}{2}\zeta(2)\,\zeta(3) - 20\ln^2 2 - \frac{21}{32}\zeta(3)\,\zeta(4) \, ,$$

$$I_2(3,3) = \int_0^1 \left(\frac{Li_3(-x)}{x}\right)^2 \, dx = \frac{9}{8}\zeta(4) - \frac{9}{16}\zeta^2(3) + 6\zeta(3)$$

$$+ 6\zeta(2) - \frac{3}{4}\zeta(2)\,\zeta(3) - (6\zeta(2) + 3\zeta(3))\ln 2 - 12\ln^2 2 \, .$$

These results build on the work of [16] and [25] where they explored integrals of polylogarithmic functions with positive arguments only. Freitas gives many particular examples of identities for $\int_0^1 \frac{Li_q(x) \; Li_t(x)}{x^2} \, dx$, but no explicit identity of the form (16) is given. Therefore in the interest of presenting a complete record we list the following theorem.

Theorem 4 *For positive integers* q *and* t, *the integral of the product of two polylogarithmic functions with positive arguments,*

$$P(q,t) = \int_0^1 \frac{Li_q(x) \; Li_t(x)}{x^2} \, dx = (-1)^q \, \mathsf{HG}(q,t)$$

$$+ \sum_{j=1}^{q-1} (-1)^{j+1} \, \zeta(t+j) \, \mathsf{G}(j,t) \, ,$$

where $\mathsf{G}(\cdot,\cdot)$ *and* $\mathsf{HG}(\cdot,\cdot)$ *are given by* (10) *and* (11) *respectively.*

Proof. The proof follows the same technique as that used in Theorem 3. □

Example 2

$$P(4,5) = \int_0^1 \frac{Li_4(x) \ Li_5(x)}{x^2} dx = 70\zeta(2) - 35\zeta(3) - \frac{114}{5}\zeta(4) - 10\zeta(5)$$

$$- \zeta(4)\,\zeta(5) - \frac{31}{4}\zeta(6) - \frac{5}{2}\zeta^2(3) - 5\zeta(2)\,\zeta(3) - 3\zeta(3)\,\zeta(4)$$

$$- \zeta(2)\,\zeta(5) - \frac{7}{6}\zeta(8) - \zeta(3)\,\zeta(5),$$

$$\int_0^1 \frac{Li_4(x^3) \ Li_4(x^3)}{x} dx = \frac{2}{3}\zeta(4)\,\zeta(5) + \frac{2}{3}\zeta(2)\,\zeta(7) - \frac{5}{3}\zeta(9).$$

It is interesting to note the degenerate case, that is when $t = 0$, of theorems 1, 2 and 3. The following results are noted.

Remark 2 *For* $t = 0$, $Li_0(-x) = -\frac{x}{1+x}$, *hence*

$$I_0(q,0) = \int_0^1 Li_q(-x) \ Li_0(-x)\,dx = (-1)^q (1 - \eta(q+1))$$

$$+ \sum_{j=1}^{q-1} (-1)^{j+1}\,\eta(q-j+1)(1-\eta(j)) - (-1)^q (2 - \zeta(q) - \eta(q))\ln 2$$

$$+ (-1)^q \left(\frac{1}{2^q} - 1\right)(EU(q) - \zeta(q+1)) + (-1)^q \begin{cases} HO(q), \text{ for q odd} \\ \\ HE(q), \text{ for q even} \end{cases}.$$

$$I_1(q,0) = \int_0^1 \frac{Li_q(-x) \ Li_0(-x)}{x} dx = \sum_{j=1}^{q-1} (-1)^{j+1}\,\eta(q-j+1)\,\eta(j)$$

$$+ (-1)^{q+1} \left(\frac{1}{2^q} - 1\right) EU(q) + (-1)^{q+1} (\zeta(q) + \eta(q))\ln 2$$

$$+ (-1)^{q+1} \begin{cases} HO(q), \quad \text{for q odd} \\ \\ HE(q), \quad \text{for q even} \end{cases}.$$

$$I_2(q,0) = \int_0^1 \frac{Li_q(-x)\ Li_0(-x)}{x^2}dx = \sum_{j=1}^{q-1}(-1)^j\,\eta(q-j+1)\,\eta(j)$$

$$+\,\eta(q+1)+(-1)^q\left(\frac{1}{2^q}-1\right)EU(q)+(-1)^q\left(\zeta(q)+\eta(q)\right)\ln 2$$

$$+(-1)^q\begin{cases} HO(q), & \text{for } q \text{ odd}\\[2mm] HE(q), & \text{for } q \text{ even}\end{cases}.$$

Here we notice that

$$I_2(q,0) = \eta(q+1) - I_1(q,0).$$

There are some special cases of polylogarithmic integrals which are worthy of a mention and we list two in the following corollary.

Corollary 1 *Let* $q, t \in \mathbb{N}$ *then,*

$$S1(q,t) = \int_0^1 \frac{Li_q\left(-\frac{1}{x}\right)\ Li_t(-x)}{x}dx = \sum_{j=1}^{q-1}\eta(t+j)\,\eta(q-j+1)$$

$$+\,\eta(q+t+1)+(\eta(q+t)+\zeta(q+t))\ln 2 \qquad (18)$$

$$+\left(\frac{1}{2^{q+t}}-1\right)EU(q+t)+\begin{cases} HE(q+t), & \text{for } q+t \text{ even}\\[2mm] HO(q+t), & \text{for } q+t \text{ odd}\end{cases}.$$

$$S2(q) = \int_0^1 \frac{Li_2(1-x)\ Li_q(x)}{x}dx = \zeta(2)\,\zeta(q+1) - BW(2,q+1), \qquad (19)$$

where $BW(2, q+1)$ *is given by* (5).

Proof. If we follow the same procedure as in theorem 2, we obtain

$$S1(q,t) = \int_0^1 \frac{Li_q\left(-\frac{1}{x}\right)\ Li_t(-x)}{x}dx = \sum_{j=1}^{q-1}\eta(q-j+1)\,\eta(t+j)+\eta(q+t+1)$$

$$+\sum_{n\geq 1}\frac{(-1)^{n+1}}{n^{t+q}}\left(H_n - H_{\frac{n}{2}}\right).$$

simplifying as in Theorem 2, we arrive at the identity (18).

From Euler's reflection formula we now that

$$\text{Li}_2\,(1-x) + \text{Li}_2\,(x) + \ln x \ln (1-x) = \zeta\,(2)$$

so that

$$S2\,(q) = \int_0^1 \frac{\left(-\text{Li}_2\,(x) - \ln x \ln (1-x) + \zeta\,(2)\right)\,\text{Li}_q\,(x)}{x}\,dx.$$

Integrating term by term as in theorem 2, we obtain (19)   □

Example 3 *Some examples of the corollary follow.*

$$S1\,(2,5) = \int_0^1 \frac{\text{Li}_2\left(-\frac{1}{x}\right)\,\text{Li}_5\,(-x)}{x}\,dx = \frac{2345}{768}\zeta\,(8) - \eta\,(3)\,\eta\,(5)\,,$$

$$S1\,(q,q) = \int_0^1 \frac{\text{Li}_q\left(-\frac{1}{x}\right)\,\text{Li}_q\,(-x)}{x}\,dx = q\zeta\,(2q+1)\,,$$

$$S1\,(9,5) = \int_0^1 \frac{\text{Li}_9\left(-\frac{1}{x}\right)\,\text{Li}_5\,(-x)}{x}\,dx = 7\zeta\,(15) - \eta\,(6)\,\eta\,(9) - \eta\,(7)\,\eta\,(8)\,.$$

$$S2\,(3) = \int_0^1 \frac{\text{Li}_2\,(1-x)\,\text{Li}_3\,(x)}{x}\,dx = \frac{25}{12}\zeta\,(6) - \zeta^2\,(3)$$

$$S2\,(8) = \int_0^1 \frac{\text{Li}_2\,(1-x)\,\text{Li}_8\,(x)}{x}\,dx$$

$$= 27\zeta\,(11) - 8\zeta\,(2)\,\zeta\,(9) - 6\zeta\,(4)\,\zeta\,(7) - 4\zeta\,(6)\,\zeta\,(5) - 2\zeta\,(8)\,\zeta\,(3)\,.$$

Summary In this paper we have developed new Euler sum identities (7) and (8) of general weight $p+1$ for $p \in \mathbb{N}$. Moreover, we have developed the new identities (16) and (18). In a series of papers [2], [5], [6], the authors explore linear combinations of associated harmonic polylogarithms and nested

harmonic numbers. The multiple zeta value data mine, computed by Blumlein et. al. [7], is an invaluable tool for the evaluation of harmonic numbers. Values with weights of twelve, for alternating sums and weights above twenty for non-alternating sums are presented.

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