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A Tauberian theorem for the statistical generalized Nörlund-Euler summability method

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Abstract. Let (p_n) and (q_n) be any two non-negative real sequences with

$$R_n := \sum_{k=0}^{n} p_k q_{n-k} \neq 0 \ \ (n \in \mathbb{N}).$$

With E_n^1- we will denote the Euler summability method. Let (x_n) be a sequence of real or complex numbers and set

$$N_{p,q}^n E_n^1 := \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} \frac{1}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} x_\nu$$

for $n \in \mathbb{N}$. In this paper, we present necessary and sufficient conditions under which the existence of the $st-$ limit of (x_n) follows from that of $st - N_{p,q}^n E_n^1-$ limit of (x_n). These conditions are one-sided or two-sided if (x_n) is a sequence of real or complex numbers, respectively.

1 Introduction

In what follows we give the concept of the summability method known as the generalized Nörlund summability method (N, p, q) (see [1]). Given two

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non-negative sequences (p_n) and (q_n), the convolution $(p \star q)$ is defined by

$$R_n := (p \star q)_n = \sum_{k=0}^{n} p_k q_{n-k} = \sum_{k=0}^{n} p_{n-k} q_k.$$

In this paper we suppose $R_n \to \infty$ as $n \to \infty$. With E_n^1- we will denote the Euler summability method. Let (x_n) be a sequence. When $(p \star q)_n \neq 0$ for all $n \in \mathbb{N}$, the generalized Nörlund-Euler transform of the sequence (x_n) is the sequence $N_{p,q}^n E_n^1$ obtained by putting

$$N_{p,q}^n E_n^1 = \frac{1}{(p \star q)_n} \sum_{k=0}^{n} p_k q_{n-k} \frac{1}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} x_\nu. \tag{1}$$

We say that the sequence (x_n) is generalized Nörlund-Euler summable to L determined by the sequences (p_n) and (q_n) or briefly summable $N_{p,q}^n E_n^1$ to L if

$$\lim_{n \to \infty} N_{p,q}^n E_n^1 = L. \tag{2}$$

Suppose throughout the paper we assume that the sequence $q = (q_n)$ satisfies the following conditions:

$$q_{t_n - k} \leq 2 q_{n-k}, k = 0, 1, 2, 3, \cdots, n; t > 1, \tag{3}$$

$$q_{n-k} \leq 2 q_{t_n - k}, k = 0, 1, 2, 3, \cdots, t_n; 0 < t < 1, \tag{4}$$

where $t_n = [t \cdot n]$.

If

$$\lim_{n \to \infty} x_n = L \tag{5}$$

implies (2), then the summation method generated by $N_{p,q}^n E_n^1$ is regular, and it is satisfied under certain conditions. However, the converse is not always true. We can show by the following example

Example 1 *Let us consider that* $x = (x_k) = (-1)^k$, *then we have*

$$\left| \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} \frac{1}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} (-1)^\nu \right| \leq \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} \frac{1}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} 1 \to 1 \text{ as } n \to \infty.$$

And as we know $x = (x_k)$, *is not convergent.*

Notice that (2) may imply (5) under a certain condition, which is called a Tauberian condition. Any theorem which states that convergence of a sequence follows from its $N^n_{p,q}E^1_n$ summability and some Tauberian condition is said to be a Tauberian theorem for the $N^n_{p,q}E^1_n$ summability method. The inclusion and Tauberian type theorems are proved in the papers [6, 7, 2, 3, 4, 5, 8], and some theorems of inclusion, Tauberian and convexity type for certain families of generalized Nörlund methods are obtained in [9].

2 Results

In this paper, we present necessary and sufficient conditions under which the existence of the limit $st - \lim_{n\to\infty} x_n = L$ follows from that of $st - \lim_{n\to\infty} N^n_{p,q}E^1_n = L$. These conditions are one-sided or two-sided if (x_n) is a sequence of real or complex numbers, respectively.

Definition 1 *A sequence* (x_n) *is statistically convergent to* L, *if for every* $\epsilon > 0$, *we have*

$$\lim_{n\to\infty} \frac{|\{k \le n : |x_k - L| \ge \epsilon\}|}{n} = 0,$$

where $|A|$, *stands for cardinality of the set.*

The theory of Tauberian is extensively studied by many authors ([1], [2], [3], [4], [7], [9]). In this section our aim is to find conditions (so-called Tauberian) under which the converse implication holds, for defined convergence. Exactly, we will prove under which conditions convergence of sequences (x_n), follows from $N^n_{p,q}E^1_n-$ convergence.

Definition 2 *A sequence* (x_n) *is weighted* $N^n_{p,q}E^1_n-$*statistically convergent to* L *if for every* $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{R_n} \left| \left\{ k \le R_n : \left| \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} \frac{1}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} x_\nu - L \right| \ge \epsilon \right\} \right| = 0.$$

And we say that the sequence (x_n) is statistically summable to L by the weighted summability method $N^n_{p,q}E^1_n$, if $st - \lim_n N^n_{p,q}E^1_n = L$. We denote by $N^n_{p,q}E^1_n(st)$ the set of all sequences which are statistically summable.

Theorem 1 *If sequence* $x = (x_n)$ *is* $N^n_{p,q}E^1_n$ *summable to* L, *then sequence* $x = (x_n)$ *is* $N^n_{p,q}E^1_n-$ *statistically convergent to* L. *But not conversely.*

Proof. The first part of the proof is obvious. To prove the second part we will show this example:

Example 2 *Let us consider that* $(p_k) = (1)$, $(q_{n-k}) = (2^k)$, *and define*

$$x_k = \begin{cases} \sqrt{2^k} & , \quad \textit{for} \quad k = 2^n \\ 0 & , \quad \textit{otherwise} \end{cases}$$

Under this conditions we get:

$$\frac{1}{2^n - 1}\left|\left\{k \leq 2^n - 1 : \left|\frac{1}{2^n - 1}\sum_{k=0}^{n} 1 \cdot \sum_{\nu=0}^{k}\binom{k}{\nu}x_\nu - 0\right| \geq \epsilon\right\}\right| \leq \frac{\sqrt{2^n}}{2^n - 1} \to 0.$$

Hence, it is $N_{p,q}^n E_n^1$ *statistically summable to 0. On the other hand, if we take in consideration that* $k = 2^n$, *we get*

$$\frac{1}{2^n - 1}\sum_{k=0}^{n} 1 \cdot \sum_{\nu=0}^{k}\binom{k}{\nu}x_\nu \to \infty, \quad \textit{as} \quad n \to \infty.$$

From last relation follows that $x = (x_n)$ *is not* $N_{p,q}^n E_n^1$ *summable to 0.*

□

Theorem 2 *Let us suppose that sequence* (x_n)*-statistically convergent to* L, *and*

$$\sup_{\substack{0 \leq \nu \leq k \\ 0 \leq k \leq n \\ n \in \mathbb{N}}} (\nu + k + n)|x_\nu - L| < \infty.$$

Then it converges $N_{p,q}^n E_n^1$*-statistically to* L. *Converse is not true.*

Proof. From fact that (x_n) converges statistically to L, we get

$$\lim_{n \to \infty}\frac{|\{k \leq n : |x_k - L| \geq \epsilon\}|}{n} = 0.$$

Let us denote by $B_\epsilon = \{k \leq n : |x_k - L| \geq \epsilon\}$ and $\overline{B_\epsilon} = \{k \leq n : |x_k - L| < \epsilon\}$. Then

$$\left|\frac{1}{R_n}\sum_{k=0}^{n} p_k q_{n-k}\frac{1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}x_\nu - L\right| = \left|\frac{1}{R_n}\sum_{k=0}^{n} p_k q_{n-k}\frac{1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}(x_\nu - L)\right| \leq$$

$$\frac{1}{R_n}\sum_{\substack{k=0\\k\in B_\epsilon}}^{n} p_k q_{n-k}\frac{1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}|x_\nu - L| + \frac{1}{R_n}\sum_{\substack{k=0\\k\in \overline{B}_\epsilon}}^{n} p_k q_{n-k}\frac{1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}|x_\nu - L| \leq$$

(from given conditions for sequence (x_ν), there exists a constant C such that $|x_\nu - L| \leq \frac{C}{\nu+k+n}$)

$$\frac{C}{R_n}\sum_{\substack{k=0\\k\in B_\epsilon}}^{n} p_k q_{n-k}\frac{1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}\frac{1}{\nu+k+n} + \frac{\epsilon}{R_n}\sum_{\substack{k=0\\k\in \overline{B}_\epsilon}}^{n} p_k q_{n-k}\frac{1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu} \leq$$

$$\frac{C}{R_n}\sum_{\substack{k=0\\k\in B_\epsilon}}^{n} p_k q_{n-k}\frac{1}{2^k}\frac{1}{k+n}\sum_{\nu=0}^{k}\binom{k}{\nu} + \frac{\epsilon}{R_n}\sum_{k=0}^{n} p_k q_{n-k}\frac{1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu} \leq$$

$$\leq \frac{C|B_\epsilon|}{n}\cdot\frac{\max_{0\leq k\leq n}\{p_k q_{n-k}\}}{R_n} + \epsilon \to 0 + \epsilon, \quad \text{as} \quad n \to \infty.$$

To show that converse is not true we will use into consideration this

Example 3 *Let us consider that* $(p_n) = 1$, $(q_{n-k}) = (2^k)$ *for* $n \in \mathbb{N}$ *and we define the sequence* $x = (x_n)$, *as follows:*

$$x_k = \begin{cases} 1 & , \quad \text{for} \quad k = m^2 - m, \cdots, m^2 - 1 \\ -m & , \quad \text{for} \quad k = m^2, m = 2, \cdots \\ 0 & , \quad \text{otherwise} \end{cases}.$$

Under this conditions, after some calculations we get:

$$\left|\frac{1}{2^n - 1}\sum_{k=0}^{n} 1\cdot\sum_{\nu=0}^{k}\binom{k}{\nu}x_\nu - 1\right| \leq \left|\frac{1}{2^n - 1}\sum_{k=0}^{n} 1\cdot\sum_{\nu=0}^{k}\binom{k}{\nu} - 1\right| = 0.$$

From last relation follows that $x = (x_n)$ *is* $N^n_{p,q}E^1_n-$ *summable to* 1. *Hence from Theorem 1,* (x_n) *is* $N^n_{p,q}E^1_n-$ *statistically convergent. On the other hand, the sequence* $(m^2; m = 2, 3\cdots,)$ *has natural density zero and it is clear that* $st-\liminf_n x_n = 0$ *and* $st-\limsup_n x_n = 1$. *Thus,* (x_k) *is not statistically convergent.*

□

Theorem 3 *If*

$$st - \liminf_{n} \frac{R_{t_n}}{R_n} > 1, t > 1 \tag{6}$$

where t_n, *denotes the integral parts of the* $[tn]$ *for every* $n \in \mathbb{N}$, *and let* (x_k) *be a sequence of real numbers which converges to* L, $N_{p,q}^{n}E_{n}^{1}-$ *statistically. Then* (x_k) *is* $st-$ *convergent to the same number* L *if and only if the following two conditions hold:*

$$\inf_{t>1}\limsup_{n}\frac{1}{R_n}\left|\left\{k\leq R_n : \frac{1}{R_{t_k}-R_k}\sum_{j=k+1}^{t_k} p_j q_{t_k-j}\frac{1}{2^j}\sum_{\nu=0}^{j}\binom{j}{\nu}(x_\nu - x_k)\leq -\epsilon\right\}\right| = 0 \tag{7}$$

and

$$\inf_{0<t<1}\limsup_{n}\frac{1}{R_n}\left|\left\{k\leq R_n : \frac{1}{R_k-R_{t_k}}\sum_{j=t_k+1}^{k} p_j q_{k-j}\frac{1}{2^j}\sum_{\nu=0}^{j}\binom{j}{\nu}(x_k - x_\nu)\leq -\epsilon\right\}\right| = 0. \tag{8}$$

Remark 1 *Let us suppose that* $st-\lim_k x_k = L$; (x_n) *is* $N_{p,q}^{n}E_{n}^{1}-$ *statistically convergent and relation* (6) *satisfies, then for every* $t>1$, *is valid the following relation:*

$$st-\lim_{k}\frac{1}{R_{t_k}-R_k}\sum_{j=k+1}^{t_k} p_j q_{t_k-j}\frac{1}{2^j}\sum_{\nu=0}^{j}\binom{j}{\nu}(x_\nu - x_k) = 0 \tag{9}$$

and in case where $0<t<1$,

$$st-\lim_{k}\frac{1}{R_k-R_{t_k}}\sum_{j=t_k+1}^{k} p_j q_{k-j}\frac{1}{2^j}\sum_{\nu=0}^{j}\binom{j}{\nu}(x_k - x_\nu) = 0. \tag{10}$$

In the next result, we will consider the case where $x=(x_n)$ is a sequence of complex numbers.

Theorem 4 *Let us suppose that relation* (6) *is satisfied. And* (x_n) *be a sequence of complex numbers, which is* $N_{p,q}^{n}E_{n}^{1}-$ *statistically convergent to* L. *Then* (x_n) *is* $st-$ *convergent to the same number* L *if and only if the following two conditions hold:*

$$\inf_{t>1}\limsup_{n}\frac{1}{R_n}\left|\left\{k\leq R_n : \left|\frac{1}{R_{t_k}-R_k}\sum_{j=k+1}^{t_k} p_j q_{t_k-j}\frac{1}{2^j}\sum_{\nu=0}^{j}\binom{j}{\nu}(x_\nu - x_k)\right|\geq \epsilon\right\}\right| = 0 \tag{11}$$

and

$$\inf_{0<t<1} \limsup_n \frac{1}{R_n} \left| \left\{ k \leq R_n : \left| \frac{1}{R_k - R_{t_k}} \sum_{j=t_k+1}^{k} p_j q_{k-j} \frac{1}{2^j} \sum_{\nu=0}^{j} \binom{j}{\nu} (x_k - x_\nu) \right| \geq \epsilon \right\} \right| = 0. \tag{12}$$

In what follows we will show some auxiliary lemmas which are needful in the sequel.

Lemma 1 *Condition given by relation* (6) *is equivalent to this one:*

$$st - \liminf_n \frac{R_n}{R_{t_n}} > 1, \quad 0 < t < 1. \tag{13}$$

Proof. Let us suppose that relation (6) is valid, $0 < t < 1$ and $m = t_n = [t \cdot n]$, $n \in \mathbb{N}$. Then it follows that

$$\frac{1}{t} > 1 \Rightarrow \frac{m}{t} = \frac{[t \cdot n]}{t} \leq n,$$

from above relation we obtain:

$$\frac{R_n}{R_{t_n}} \geq \frac{R_{\left[\frac{m}{t}\right]}}{R_{t_n}} \Rightarrow st - \liminf_n \frac{R_n}{R_{t_n}} \geq st - \liminf_n \frac{R_{\left[\frac{m}{t}\right]}}{R_{t_n}} > 1.$$

Conversely, let us suppose that relation (13) is valid. Let $t > 1$ be given number and let t_1 be chosen such that $1 < t_1 < t$. Set $m = t_n = [t \cdot n]$. From $0 < \frac{1}{t} < \frac{1}{t_1} < 1$, it follows that:

$$n \leq \frac{tn - 1}{t_1} < \frac{[tn]}{t_1} = \frac{m}{t_1},$$

provided $t_1 \leq t - \frac{1}{n}$, which is a case where if n is large enough. Under this conditions we have:

$$\frac{R_{t_n}}{R_n} \geq \frac{R_{t_n}}{R_{\left[\frac{m}{t_1}\right]}} \Rightarrow st_\lambda - \liminf_n \frac{R_{t_n}}{R_n} \geq st_\lambda - \liminf_n \frac{R_{t_n}}{R_{\left[\frac{m}{t_1}\right]}} > 1.$$

$\square$

Lemma 2 *Let us suppose that relation* (6) *is satisfied and let* $x = (x_k)$ *be a sequence of complex numbers which is* $N_{p,q}^n E_n^1-$*statistically convergent to* L. *Then for every* $t > 0$,

$$st - \lim_n N_{p,q}^{t_n} E_n^1 = L.$$

Proof. (I) Let us consider that $t > 1$. Then

$$\lim_{n\to\infty} \frac{1}{R_n}\sum_{k=0}^{n} p_k q_{n-k}\frac{1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}x_\nu = \lim_{n\to\infty}\frac{1}{R_{t_n}}\sum_{k=0}^{t_n} p_k q_{t_n-k}\frac{1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}x_\nu, \quad (14)$$

and for every $\epsilon > 0$ we have:

$$\{k \le R_{t_n} : |N_{p,q}^{t_n}E_n^1 - L| \ge \epsilon\} \subset \{k \le R_n : |N_{p,q}^{n}E_n^1 - L| \ge \epsilon\} \cup$$

$$\left\{k \le R_n : \frac{1}{R_n}\sum_{k=0}^{n} p_k q_{n-k}\frac{1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}x_\nu \ne \frac{1}{R_{t_n}}\sum_{k=0}^{t_n} p_k q_{t_n-k}\frac{1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}x_\nu\right\}.$$

Now proof of the lemma in this case follows from relation (14) and $st - \lim_n N_{p,q}^{n}E_n^1 = L$.

(II) In this case we have that $0 < t < 1$. For $t_n = [t \cdot n]$, for any natural number n, we can conclude that $N_{p,q}^{t_n}E_n^1$ does not appears more than $[1 + t^{-1}]$ times in the sequence $N_{p,q}^{n}E_n^1$. In fact if there exist integers k, l such that

$$n \le t \cdot k < t(k+1) < \cdots < t(k+l-1) < n+1 \le t(k+l),$$

then

$$n + t(l-1) \le t(k+l-1) < n+1 \Rightarrow l < 1 + \frac{1}{t}.$$

And we have this estimation

$$\begin{aligned}
&\frac{1}{R_n}\left|\left\{k \le R_n : |N_{p,q}^{t_n}E_n^1 - L| \ge \epsilon\right\}\right| \\
&\le \left(1 + \frac{1}{t}\right)\frac{1}{R_n}\left|\left\{k \le R_{t_n} : |N_{p,q}^{n}E_n^1 - L| \ge \epsilon\right\}\right| \\
&\le 2(1+t)\frac{1}{R_{t_n}}\left|\left\{k \le R_{t_n} : |N_{p,q}^{n}E_n^1 - L| \ge \epsilon\right\}\right|,
\end{aligned}$$

provided $\frac{1}{R_n}(\frac{t+1}{t}) \le 2(t+1)\frac{1}{R_{t_n}}$, which is the case where n is large enough. From last relation it follows: $st - \lim_n N_{p,q}^{t_n}E_n^1 = L$. □

Proposition 1 *Let us suppose that relation (6) is satisfied and let* $x = (x_k)$ *be a sequence of complex numbers which is* $N_{p,q}^{t_n}E_n^1-$*statistically convergent to* L. *Then for every* $t > 1$,

$$st - \lim_k \frac{1}{R_{t_k} - R_k}\sum_{j=k+1}^{t_k} p_j q_{t_k-j}\frac{1}{2^j}\sum_{\nu=0}^{j}\binom{j}{\nu}x_\nu = L; \quad (15)$$

and for every $0 < t < 1$,

$$st-\lim_{k} \frac{1}{R_k - R_{t_k}} \sum_{j=t_k+1}^{k} p_j q_{k-j} \frac{1}{2^j} \sum_{\nu=0}^{j} \binom{j}{\nu} x_\nu = L. \tag{16}$$

Proof. (I) Let us consider the case where $t > 1$. Then we obtain

$$\begin{aligned}
&\frac{1}{R_{t_n} - R_n} \sum_{k=n+1}^{t_n} p_k q_{t_n-k} \frac{1}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} (x_\nu - L) \\
&= \frac{R_{t_n}}{R_{t_n} - R_n} \frac{1}{R_{t_n}} \sum_{k=0}^{t_n} p_k q_{t_n-k} \frac{1}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} (x_\nu - L) \\
&\quad - \frac{R_n}{R_{t_n} - R_n} \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} \frac{1}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} (x_\nu - L) \\
&= \frac{R_{t_n}}{R_{t_n} - R_n} \frac{1}{R_{t_n}} \sum_{k=0}^{t_n} p_k q_{t_n-k} \frac{1}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} (x_\nu - L) \\
&\quad - \frac{R_{t_n}}{R_{t_n} - R_n} \frac{1}{R_{t_n}} \sum_{k=0}^{n} p_k (q_{t_n-k} + q_{n-k} - q_{n-k}) \frac{1}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} (x_\nu - L) \\
&= \frac{R_{t_n}}{R_{t_n} - R_n} \frac{1}{R_{t_n}} \sum_{k=0}^{t_n} p_k q_{t_n-k} \frac{1}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} (x_\nu - L) \\
&\quad - \frac{R_n}{R_{t_n} - R_n} \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} \frac{1}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} (x_\nu - L) \\
&\quad - \frac{R_n}{R_{t_n} - R_n} \frac{1}{R_n} \sum_{k=0}^{n} p_k (q_{t_n-k} - q_{n-k}) \frac{1}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} (x_\nu - L).
\end{aligned} \tag{17}$$

From

$$\limsup_{n} \frac{R_{t_n}}{R_{t_n} - R_n} < \infty, \tag{18}$$

definition of the sequence (q_n), Lemma 2 and relation (17), we get relation (15).

(II) In this case we have that $0 < t < 1$. Then

$$\frac{1}{R_n - R_{t_n}} \sum_{k=t_n+1}^{n} p_k q_{n-k} \frac{1}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} x_\nu = \frac{R_n}{R_n - R_{t_n}} \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} \frac{1}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} x_\nu -$$

$$\frac{R_{t_n}}{R_n - R_{t_n}} \frac{1}{R_{t_n}} \sum_{k=0}^{t_n} p_k q_{n-k} \frac{1}{2^k} \sum_{v=0}^{k} \binom{k}{v} x_v = \frac{R_{t_n}}{R_n - R_{t_n}} \frac{1}{R_{t_n}} \sum_{k=0}^{n} p_k q_{n-k} \frac{1}{2^k} \sum_{v=0}^{k} \binom{k}{v} x_v -$$

$$\frac{R_{t_n}}{R_n - R_{t_n}} \frac{1}{R_{t_n}} \sum_{k=0}^{t_n} p_k q_{t_n-k} \frac{1}{2^k} \sum_{v=0}^{k} \binom{k}{v} x_v - \frac{1}{R_n - R_{t_n}} \sum_{k=0}^{t_n} p_k (q_{n-k} - q_{t_n-k}) \frac{1}{2^k} \sum_{v=0}^{k} \binom{k}{v} x_v.$$

Now proof of the proposition is similar to the first part. □

3 Proofs of the theorems

Proof of Theorem 3. *Necessity.* Suppose that $\lim_{n\to\infty} x_n = L$, and (6) holds. Following Proposition 1, we have

$$st - \lim_{n\to\infty} \frac{1}{R_{\lambda_n} - R_n} \sum_{k=n+1}^{\lambda_n} p_k q_{\lambda_n - k} \frac{1}{2^k} \sum_{v=0}^{k} \binom{k}{v} (x_v - x_n) =$$

$$st - \lim_{n\to\infty} \frac{1}{R_{\lambda_n} - R_n} \sum_{k=n+1}^{\lambda_n} p_k q_{\lambda_n - k} \frac{1}{2^k} \sum_{v=0}^{k} \binom{k}{v} x_v - x_n = 0,$$

for every $\lambda > 1$. In case where $0 < \lambda < 1$, we find that

$$st - \lim_{n\to\infty} \frac{1}{R_n - R_{\lambda_n}} \sum_{k=\lambda_n+1}^{n} p_k q_{n-k} \frac{1}{2^k} \sum_{v=0}^{k} \binom{k}{v} (x_n - x_v) =$$

$$x_n - st - \lim_{n\to\infty} \frac{1}{R_n - R_{\lambda_n}} \sum_{k=\lambda_n+1}^{n} p_k q_{n-k} \frac{1}{2^k} \sum_{v=0}^{k} \binom{k}{v} x_v = 0.$$

Sufficiency. Assume that conditions (7) and (8) are satisfied. In what follows we will prove that $st - \lim_{n\to\infty} x_n = L$. Or equivalently, $st - \lim \left(N_{p,q}^n E_n^1 - x_n\right) = 0$. First we will consider the case where $t > 1$. We will start from this estimation:

$$\begin{aligned} x_n &- N_{p,q}^n E_n^1 \\ &= \frac{R_{t_n}}{R_{t_n} - R_n} \left[\frac{1}{R_{t_n}} \sum_{j=0}^{t_n} p_j q_{t_n - j} \frac{1}{2^j} \sum_{v=0}^{j} \binom{j}{v} x_v - \frac{1}{R_n} \sum_{j=0}^{n} p_j q_{n-j} \frac{1}{2^j} \sum_{v=0}^{j} \binom{j}{v} x_v \right] \\ &\quad - \frac{1}{R_{t_n} - R_n} \sum_{j=n+1}^{t_n} p_j q_{t_n - j} \frac{1}{2^j} \sum_{v=0}^{j} \binom{j}{v} (x_v - x_n). \end{aligned}$$

For any $\epsilon > 0$, we obtain:

$$\{k \leq R_n : x_k - N_{p,q}^n E_n^1 \geq \epsilon\} \subset \left\{k \leq R_n : \frac{R_{t_k}}{R_{t_k} - R_k}(N_{p,q}^{t_k} E_k^1 - N_{p,q}^k E_k^1) \geq \frac{\epsilon}{2}\right\} \cup$$

$$\left\{k \leq R_n : \frac{1}{R_{t_k} - R_k} \sum_{j=k+1}^{t_k} p_j q_{t_k - j} \frac{1}{2^j} \sum_{\nu=0}^{j} \binom{j}{\nu} (x_\nu - x_k) \leq -\frac{\epsilon}{2}\right\}.$$

From relation (7), it follows that for every $\gamma > 0$, exists a $t > 1$ such that

$$\limsup_n \frac{1}{R_n} \left| \left\{k \leq R_n : \frac{1}{R_{t_k} - R_k} \sum_{j=k+1}^{t_k} p_j q_{t_k - j} \frac{1}{2^j} \sum_{\nu=0}^{j} \binom{j}{\nu} (x_\nu - x_k) \leq -\epsilon\right\} \right| \leq \gamma.$$

By Lemma 2 and relation (18) we get

$$\limsup_n \frac{1}{R_n} \left| \left\{k \leq R_n : |R_{t_k}(R_{t_k} - R_k)^{-1}(N_{p,q}^{t_k} E_k^1 - N_{p,q}^k E_k^1)| \geq \frac{\epsilon}{2}\right\} \right| = 0.$$

Combining last three relations we have:

$$\limsup_n \frac{1}{R_n} \left| \left\{k \leq R_n : x_k - N_{p,q}^k E_k^1 \geq \epsilon\right\} \right| \leq \gamma,$$

and γ is arbitrary, we conclude that for every $\epsilon > 0$,

$$\limsup_n \frac{1}{R_n} \left| \left\{k \leq R_n : x_k - N_{p,q}^k E_k^1 \geq \epsilon\right\} \right| = 0. \tag{19}$$

Now we consider case where $0 < t < 1$. From above we get that:

$$\begin{aligned} & x_n - N_{p,q}^n E_n^1 \\ & = \frac{R_{t_n}}{R_n - R_{t_n}} \left[\frac{1}{R_n} \sum_{j=0}^{n} p_j q_{n-j} \frac{1}{2^j} \sum_{\nu=0}^{j} \binom{j}{\nu} x_\nu - \frac{1}{R_{t_n}} \sum_{j=0}^{t_n} p_j q_{t_n - j} \frac{1}{2^j} \sum_{\nu=0}^{j} \binom{j}{\nu} x_\nu \right] \\ & \quad + \frac{1}{R_n - R_{t_n}} \sum_{j=t_n+1}^{n} p_j q_{n-j} \frac{1}{2^j} \sum_{\nu=0}^{j} \binom{j}{\nu} (x_n - x_\nu). \end{aligned}$$

For any $\epsilon > 0$,

$$\{k \leq R_n : x_k - N_{p,q}^k E_k^1 \geq \epsilon\} \subset \left\{k \leq R_n : \frac{R_{t_k}}{R_k - R_{t_k}}(N_{p,q}^k E_k^1 - N_{p,q}^{t_k} E_k^1) \geq \frac{\epsilon}{2}\right\} \cup$$

$$\left\{k \le R_n : \frac{1}{R_k - R_{t_k}} \sum_{j=t_k+1}^{k} p_j q_{k-j} \frac{1}{2^j} \sum_{\nu=0}^{j} \binom{j}{\nu} (x_k - x_\nu) \le -\frac{\epsilon}{2}\right\}.$$

For same reasons as in the case where $t > 1$, by Lemma 2, we have that for every $\epsilon > 0$,

$$\limsup_n \frac{1}{R_n} \left|\left\{k \le R_n : x_k - N_{p,q}^k E_k^1 \le -\epsilon\right\}\right| = 0. \tag{20}$$

Finally from relations (19) and (20) we get:

$$\limsup_n \frac{1}{R_n} \left|\left\{k \le R_n : |x_k - N_{p,q}^k E_k^1| \ge \epsilon\right\}\right| = 0. \quad \square$$

Remark 2 *Let us suppose that* $st - \lim_k x_k = L$, $st - \lim_k N_{p,q}^k E_k^1 = L$ *and relation* (6) *satisfies. Then for every* $t > 1$, *relation* (9) *holds, and in case where* $0 < t < 1$, *relation* (10) *is valid.*

Proof of Theorem 4. We omit it because it is similar to the Theorem 3. $\square$

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