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Commutative feebly nil-clean group rings

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Abstract. An arbitrary unital ring R is called *feebly nil-clean* if any its element is of the form $q + e - f$, where q is a nilpotent and e, f are idempotents with $ef = fe$. For any commutative ring R and any abelian group G, we find a necessary and sufficient condition when the group ring $R(G)$ is feebly nil-clean only in terms of R, G and their sections. Our result refines establishments due to McGovern et al. in J. Algebra Appl. (2015) on nil-clean rings and Danchev-McGovern in J. Algebra (2015) on weakly nil-clean rings, respectively.

1 Introduction and background

Throughout the text of this short paper, all rings R are assumed to be associative and commutative, containing identity element which differs from the zero element. Our terminology and notations are mainly in agreement with [10] and [11]. For instance, $J(R)$ denotes the Jacobson radical of R, and $N(R)$ denotes the nil-radical of R. Also, let everywhere in the text G be a multiplicative abelian group, and let $R(G)$ be the group ring of G over R. As usual, G_p stands for the p-torsion component of the group G with p-socle $G[p] = \{a \in G \mid a^p = 1\}$, and we shall say that the group G is a p-group, provided $G = G_p$. Likewise, we set $G^p = \{g^p \mid g \in G\}$ to be the p-th power subgroup of the group G.

A ring R is known to be *nil-clean* if, for every $r \in R$, there are a nilpotent $q \in R$ and an idempotent $e \in R$ such that $r = q + e$. The next necessary and

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sufficient condition for a commutative group ring to be nil-clean was recently obtained in [9]. Specifically, the following holds: *The commutative group ring* $R(G)$ *is nil-clean if, and only if, the ring* R *is nil-clean and the group* G *is a* 2*-group.*

Generalizing this, a ring R is known to be *weakly nil-clean* if, for every $r \in R$, there are a nilpotent $q \in R$ and an idempotent $e \in R$ such that $r = q + e$ or $r = q - e$. Alternatively, a necessary and sufficient condition for a commutative group ring to be weakly nil-clean was recently obtained in [4]. Precisely, the following holds: *The commutative group ring* $R(G)$ *is weakly nil-clean if, and only if, either* $G = \{1\}$ *and* R *is weakly nil-clean, or* R *is nil-clean and* G *is a non-trivial* 2*-group, or* $R/N(R) \cong \mathbb{Z}_3$ *and* G *is a non-trivial* 3*-group.*

As a common generalization of these two definitions for nil-clean and weakly nil-clean rings, a ring R is known to be *feebly nil-clean* (see, for example, [1] and [2]) if, for every $r \in R$, there are a nilpotent $q \in R$ and two idempotents $e, f \in R$ such that $r = q + e - f$. So, the leitmotif of writing up this brief article is to generalize somewhat the above two claims by obtaining a criterion for an arbitrary commutative group ring to be feebly nil-clean.

For completeness of the exposition, it is worthwhile noticing that an extension of the aforementioned nil-clean rings are the so-called *UU rings* that are rings whose units are unipotents (i.e., the sum of 1 and some nilpotent). In [3] were examined commutative UU group rings. Exactly, it was proved in Corollary 2.3 there that $R(G)$ *is a commutative UU ring if, and only if,* R *is a commutative UU ring and* G *is an abelian* 2*-group.*

2 Main results

Before proving our chief statement, we need the following two key formulas from [7] and [8], respectively. In fact, appealing to [7], one writes the formula

$$J(R(G)) = J(R)(G) + \langle r(g-1) \mid g \in G_p, pr \in J(R) \rangle.$$

In that aspect, consulting with [8], one writes the formula

$$N(R(G)) = N(R)(G) + \langle r(g-1) \mid g \in G_p, pr \in N(R) \rangle.$$

Standardly, $I(R(G); G)$ designates the fundamental (augmentation) ideal of $R(G)$ with respect to G with basis consisting of all elements of the type $1 - g$, where $g \in G$. It is well known that the isomorphisms

$$R(G)/[N(R)(G) + I(R(G); G)] \cong R/N(R)$$

and

$$R(G)/I(R(G);G) \cong R$$

are valid. We will now prove something similar and helpful for us for further use.

Proposition 1 *Let* R *be a ring and* G *a group. Then the following isomorphism is fulfilled:*

$$R(G)/N(R)(G) \cong (R/N(R))(G).$$

Proof. There is the natural ring surjection $R \to R/N(R)$ which induces by the usual element-wise manipulation the ring surjective homomorphism $R(G) \to (R/N(R))(G)$. This epimorphism obviously has kernel $N(R)G$ and henceforth the well-known Homomorphism Theorem applies to get the desired assertion. □

A ring is *boolean* if each its element is an idempotent. Let us recall that a ring is said to be *tripotent* if each its element x satisfies the equation $x^3 = x$. These rings are necessarily commutative being also a subdirect product of a family of single or isomorphic copies of the fields $\mathbb{Z}_2$ and $\mathbb{Z}_3$ (see, e.g., [6]). Likewise, as $6 = 0$ here, any tripotent ring R can be decomposed as the direct product of two rings $R_1 \times R_2$, where R_1 is boolean and R_2 is tripotent of characteristic 3. It is pretty evident that reduced feebly nil-clean rings are themselves tripotent.

We begin our work with a few useful technicalities.

Lemma 1 *The next two statements are true:*

(i) *A direct factor of a feebly nil-clean ring is a feebly nil-clean ring as well.*

(ii) *The direct product of two feebly nil-clean rings is also a feebly nil-clean ring.*

Proof. Straightforward by a direct check, so that we leave it to the interested reader. □

Lemma 2 *An epimorphic image of a feebly nil-clean ring is too a feebly nil-clean ring.*

Proof. Since nilpotents and idempotents map under any homomorphism again into nilpotents and idempotents, respectively, the claim follows elementarily. □

Proposition 2 *Suppose that* R *is a commutative ring. Then the following three points are equivalent:*

(i) R *is feebly nil-clean.*
(ii) $J(R)$ *is nil and* $R/J(R)$ *is tripotent.*
(iii) $R/N(R)$ *is tripotent.*

Proof. It suffices to prove only the equivalence (i) $\iff$ (iii), because whenever $J(R)$ is nil we will have that $J(R) = N(R)$ as well as, in accordance with [1] or [2], R being feebly nil-clean yields that $J(R)$ is nil. To this purpose, the implication (i) $\Rightarrow$ (iii) follows at once by the usage of Lemma 2.

As for the converse implication (i) $\Leftarrow$ (iii), we may write by consulting with [6, Theorem 1] accomplished with a simple trick that every element of $R/N(R)$ is the difference of two idempotents, say $r+N(R) = (e_1+N(R))-(e_2+N(R)) = e_1 - e_2 + N(R)$, where $r \in R$ is an arbitrary element and $e_1, e_2 \in R$ are some elements. But as it is well-known as a folklore fact, we may choose these e_1 and e_2 to be idempotents. Consequently, one follows that $r = t + e_1 - e_2$ for some nilpotent t in R, as expected. □

As an immediate consequence, one yields the following.

Corollary 1 *Let* I *be a nil ideal of a ring* R*. Then* R *is feebly nil-clean if, and only if,* R/I *is feebly nil-clean.*

Proof. The "necessity" follows by virtue of Lemma 2. As for the "sufficiency", because of the inclusion $I \subseteq N(R)$, there exists an epimorphism $R/I \to R/N(R)$ with kernel $N(R)/I = N(R/I)$. Hence $R/N(R)$ is feebly nil-clean, i.e., tripotent. Furthermore, we apply Proposition 2 to conclude the claim. □

We now have all the ingredients necessary to proceed by proving the following chief assertion, which gives a necessary and sufficient condition when a commutative group ring will be feebly nil-clean.

Theorem 1 *Suppose* R *is a commutative ring and* G *is an abelian group. Then the group ring* $R(G)$ *is feebly nil-clean if, and only if, exactly one of the following three items is valid:*

(1) $G = \{1\}$ *and* R *is feebly nil-clean.*

(2) $G \neq \{1\}$ *and* $R/N(R) \cong R_1 \times R_2$*, where* R_1 *is boolean and* R_2 *is tripotent of characteristic* 3 *such that*

(a) $R_1 = \{0\}$*, or* $R_1 \neq \{0\}$ *and* G *is a* 2*-group;*

(b) $R_2 = \{0\}$*, or* $R_2 \neq \{0\}$ *and either* $G = G_3$ *or* $G = G_3 \times G[2]$*.*

Proof. "**Left-to-right**". The assumption that G is the trivial group leads to $R(G) \cong R$, so that we may assume without loss of generality that G is non-trivial.

The epimorphism $R(G) \to R$ implies that R is feebly nil-clean and thus Proposition 2 (iii) enables us that $R/N(R)$ is tripotent. Therefore, the main result in [6] allows us to write that $R/N(R) \cong R_1 \times R_2$, where R_1 is a boolean ring and R_2 is a tripotent ring of characteristic 3.

On the other hand, as in the proof of Proposition 1, the surjection $R \to R/N(R)$ induces a surjection $R(G) \to (R/N(R))(G)$ and so in view of Lemma 2 the group ring $(R/N(R))(G) \cong R_1(G) \times R_2(G)$ has to be feebly nil-clean, too. Since $2 = 0$ in R_1, with the aid of Lemma 1 (i) it must be that $R_1(G)$ is feebly nil-clean of characteristic 2 whence it is necessarily nil-clean, because under these circumstances the sum of two idempotents is again an idempotent. Employing now the quoted above result from [9], we derive that either R_1 is zero, or R_1 is non-trivial and $G = G_2$ is a 2-primary group.

Further, concerning the second direct factor R_2, let us assume that it is non-zero and hence a subdirect product of the field $\mathbb{Z}_3$. Since there exist two epimorphisms, namely $R_2 \to \mathbb{Z}_3$ and $G \to G/G_3$, one infers that there is an induced epimorphism $R_2(G) \to \mathbb{Z}_3(G/G_3)$ which gives with the help of Lemma 2 that the epimorphic image $\mathbb{Z}_3(G/G_3)$ is feebly nil-clean as so is $R_2(G)$ being a direct factor of $(R/N(R))(G)$. According to the listed above formula of May from [8], we obtain that $\mathbb{Z}_3(G/G_3)$ is reduced and thus it is certainly tripotent by using once again Proposition 2 (iii). Consequently, the equation $z^3 = z$ holds in the factor-group G/G_3, that is, $z^2 = 1$. We may have $G/G_3 = \{\bar{1}\}$, that is, $G = G_3$. If now $G \neq G_3$, letting g be an arbitrary element in G, one deduces that $(gG_3)^2 = G_3$, i.e., $g^2G_3 = G_3$, i.e., $g^2 \in G_3$. But the 3-component G_3 is always 2-divisible, that is, $G_3 = G_3^2$ (see, e.g., [5]). This, in turn, forces that $g = g_3a \subseteq G_3G[2]$ for some $g_3 \in G_3$ and $a \in G[2]$ assuring the direct decomposition $G = G_3 \times G[2]$, as wanted.

"**Right-to-left**". Because item (1) implies at once that $R(G) \cong R$, the claim follows immediately.

We, therefore, will be concentrated on the non-trivial case for G, which is exactly point (2). With Proposition 1 at hand, we have that $R(G)/N(R)(G)$ is isomorphic to $(R/N(R))(G) \cong R_1(G) \times R_2(G)$ with nil $N(R)(G) \subseteq N(R(G))$. Therefore, applying Corollary 1, one needs to show the feebly nil-cleanness of $(R/N(R))(G)$ only. To that aim, condition (a) along with the major result from [9] rich us that the group ring $R_1(G)$ is nil-clean and so feebly nil-clean.

On the other side, concerning condition (b), the two possibilities $G = G_3$ and $G = G_3 \times G[2]$ will imply that either $R_2(G) \cong R_2(G_3)$ or $R_2(G) \cong R_2(G_3) \times R_2(G[2])$, where the validity of the latter isomorphism is formally assumed. Moreover, as the characteristic of R_2 is 3 and the equality $x^3 = x$ holds both in R_2 and in $G[2]$, it is readily verified by utilizing only technical arguments that it will hold in the group ring $R_2(G[2])$ as well. Thus $R_2(G[2])$ is feebly nil-clean. We claim, besides, that $R_2(G_3)$ is also feebly nil-clean. Indeed, referring to the noticed above formula of May from [8], one detects by using routine argumentation that $N(R_2(G_3)) = I(R_2(G_3); G_3)$. However, as noted before, $R_2(G_3)/N(R_2(G_3)) = R_2(G_3)/I(R_2(G_3); G_3) \cong R_2$ implies the tripotent property, which invoking Proposition 2 substantiates our claim, as expected. We, finally, just need to apply once again Lemma 1 (ii) to get the desired feebly nil-cleanness of the group ring $R_2(G)$, thus concluding the initial assertion for feebly nil-cleanness of the group ring $R(G)$, as promised.

As a new and somewhat more direct and comfortable confirmation that the group ring $R_2(G)$ is feebly nil-clean in the case when G is a decomposable group as stated above, we may process like this: Since $G = G_3 \times G[2]$, it follows at once that $R_2(G) \cong (R_2(G[2]))(G_3) = R'_2(G_3)$, where we putted $R'_2 := R_2(G[2])$. As we already showed above, R'_2 is a ring of characteristic 3 in which the equality $x^3 = x$ holds for all its elements. Thus, in particular, it should be reduced as well. Furthermore, as we have demonstrated, $N(R'_2(G_3)) = I(R'_2(G_3); G_3)$ and, consequently, $R'_2(G_3)/N(R'_2(G_3)) = R'_2(G_3)/I(R'_2(G_3); G_3) \cong R'_2$ is tripotent (i.e., reduced feebly nil-clean), as expected. This gives the desired feebly nil-cleanness of the group ring $R'_2(G_3)$ which, in turn, substantiates the promised feebly nil-cleanness of $R_2(G)$ after all. □

We close with some more comments.

Remark 1 *Utilizing the stated above formula of Karpilovsky from [7], we can deduce an equivalent necessary and sufficient condition for a commutative group ring to be feebly nil-clean in terms of* $J(R)$ *instead of* $N(R)$.

We end the work with a problem of interest.

Problem 1 *Find a criterion when an arbitrary (not necessarily commutative) group ring is feebly nil-clean.*

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