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Modified Hadamard product properties of certain class of analytic functions with varying arguments defined by the convolution of Sălăgean and Ruscheweyh derivative

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Abstract. In this paper we study the Hadamard product properties of certain class of analytic functions with varying arguments defined by the convolution of Sălăgean and Ruscheweyh derivative.
The obtained results are sharp and they improve known results.

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic and univalent in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. Let $g \in \mathcal{A}$ where

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k. \tag{2}$$

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The Hadamard product of two functions f and g of the form (1) and (2) is defined by (see also [3, 7, 8, 9])

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z).$$

The modified Hadamard product is

$$(f \otimes g)(z) = z - \sum_{k=2}^{\infty} a_k b_k z^k = (g \otimes f)(z).$$

Definition 1 [8]

For $f \in \mathcal{A}, n \in \mathbb{N}$, the Sălăgean differential operator $\mathcal{S}^n$ is defined by $\mathcal{S}^n : \mathcal{A} \to \mathcal{A}$,

$$\mathcal{S}^0 f(z) = f(z),$$

$$\mathcal{S}^{n+1} f(z) = z \left(\mathcal{S}^n f(z)\right)', z \in U$$

Remark 1 *If $f \in \mathcal{A}$ and $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then*

$$\mathcal{S}^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, \ z \in U.$$

Definition 2 [6]

For $f \in \mathcal{A}, n \in \mathbb{N}$, the operator $\mathcal{R}^n$ is defined by $\mathcal{R}^n : \mathcal{A} \to \mathcal{A}$,

$$\mathcal{R}^0 f(z) = f(z),$$

$$(n+1)\, \mathcal{R}^{n+1} f(z) = z \left(\mathcal{R}^n f(z)\right)' + n \mathcal{R}^n f(z), z \in U.$$

Remark 2 *If $f \in \mathcal{A}$ and $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then*

$$\mathcal{R}^n f(z) = z + \sum_{k=2}^{\infty} \frac{(n+k-1)!}{n!\,(k-1)!} a_k z^k, \ z \in U.$$

Definition 3 *Let $n \in \mathbb{N}$. Denote by $\mathcal{S}\mathcal{R}^n$ the operator given by the Hadamard product (convolution) of the Sălăgean operator $\mathcal{S}^n$ and the Ruscheweyh operator $\mathcal{R}^n$, $\mathcal{S}\mathcal{R}^n : \mathcal{A} \to \mathcal{A}$,*

$$\mathcal{S}\mathcal{R}^n f(z) = \mathcal{S}^n \left(\frac{z}{1-z}\right) * \mathcal{R}^n f(z), z \in U.$$

Remark 3 *If* $f \in \mathcal{A}$ *and* $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, *then*

$$\mathcal{SR}^n f(z) = z + \sum_{k=2}^{\infty} \frac{k^n (n+k-1)!}{n!(k-1)!} a_k z^k, \ z \in U.$$

Definition 4 [4] *Let* f *and* g *be analytic functions in* U. *We say that the function* f *is subordinate to the function* g, *if there exists a function* w, *which is analytic in* U *and* $w(0) = 0; |w(z)| < 1; z \in U$, *such that* $f(z) = g(w(z))$; $\forall z \in U$. *We denote by* $\prec$ *the subordination relation.*

Definition 5 *For* $\lambda \geq 0; -1 \leq A < B \leq 1; 0 < B \leq 1; n \in \mathbb{N}_0$ *let* $P(n, \lambda, A, B)$ *denote the subclass of* $\mathcal{A}$ *which contain functions* $f(z)$ *of the form* (1) *such that*

$$(1-\lambda)(\mathcal{SR}^n f(z))' + \lambda(\mathcal{SR}^{n+1} f(z))' \prec \frac{1+Az}{1+Bz}. \tag{3}$$

Attiya and Aouf defined in [2] the class $\mathcal{R}(n, \lambda, A, B)$ with a condition like (3), but there instead of the operator $\mathcal{SR}^n$ they used the Ruscheweyh operator.

Definition 6 [10] *A function* $f(z)$ *of the form* (1) *is said to be in the class* $V(\theta_k)$ *if* $f \in \mathcal{A}$ *and* $\arg(a_k) = \theta_k$, $\forall k \geq 2$. *If* $\exists \delta \in \mathbb{R}$ *such that* $2\theta_k + (k-1)\delta \equiv \pi (\bmod 2\pi), \forall k \geq 2$ *then* $f(z)$ *is said to be in the class* $V(\theta_k, \delta)$. *The union of* $V(\theta_k, \delta)$ *taken over all possible sequences* $\{\theta_k\}$ *and all possible real numbers* δ *is denoted by* V.

Let $VP(n, \lambda, A, B)$ denote the subclass of V consisting of functions $f(z) \in P(n, \lambda, A, B)$.

Theorem 1 [5] *Let the function* $f(z)$ *defined by* (1) *be in V. Then* $f(z) \in VP(n, \lambda, A, B)$, *if and only if*

$$T(f) = \sum_{k=2}^{\infty} (1+B) k^{n+1} C_k |a_k| \leq B - A, \tag{4}$$

where

$$C_k = [n + 1 + \lambda (k-1)(n+k+1)] \frac{(n+k-1)!}{(n+1)!(k-1)!}.$$

The extremal functions are:

$$f(z) = z + \frac{B-A}{k^{n+1} C_k (1+B)} e^{i\theta_k} z^k, (k \geq 2).$$

Main results

Theorem 2 *If* $f \in VP(n, \lambda, A_1, B), g \in VP(n, \lambda, A_2, B)$ *then*

$$f \otimes g \in VP(n, \lambda, A^*, B), \ \textit{where}\ A^* = B - \frac{(B - A_1)(B - A_2)}{2^{n+1} C_2 \,(1 + B)}.$$

The result is sharp.

Proof. Let $f \in VP(n, \lambda, A_1, B), g \in VP(n, \lambda, A_2, B)$ and suppose they have the form (1). Since $f \in VP(n, \lambda, A_1, B)$ we have

$$\frac{\sum_{k=2}^{\infty} k^{n+1} C_k \,(1 + B)\, |a_k|}{B - A_1} \leq 1 \tag{5}$$

and for $g \in VP(n, \lambda, A_2, B)$ we have

$$\frac{\sum_{k=2}^{\infty} k^{n+1} C_k \,(1 + B)\, |b_k|}{B - A_2} \leq 1. \tag{6}$$

We know from Theorem 1 that $f \otimes g \in VP(n, \lambda, A^*, B)$ if and only if

$$\frac{\sum_{k=2}^{\infty} k^{n+1} C_k \,(1 + B)\, |a_k b_k|}{B - A^*} \leq 1. \tag{7}$$

By using the Cauchy-Schwarz inequality for (5) and (6) we have

$$\frac{\sum_{k=2}^{\infty} k^{n+1} C_k \,(1 + B)\, \sqrt{|a_k b_k|}}{\sqrt{(B - A_1)(B - A_2)}} \leq 1.$$

We note that

$$\frac{\sum_{k=2}^{\infty} k^{n+1} C_k \,(1 + B)\, |a_k b_k|}{B - A^*} \leq \frac{\sum_{k=2}^{\infty} k^{n+1} C_k \,(1 + B)\, \sqrt{|a_k b_k|}}{\sqrt{(B - A_1)(B - A_2)}}$$

implies (7). But the above inequality holds provided that

$$\frac{|a_k b_k|}{B - A^*} \leq \frac{\sqrt{|a_k b_k|}}{\sqrt{(B - A_1)(B - A_2)}}$$

or

$$\sqrt{|a_k b_k|} \leq \frac{B - A^*}{\sqrt{(B - A_1)(B - A_2)}}. \tag{8}$$

From Theorem 1 we have:

$$|a_k| \leq \frac{B - A_1}{k^{n+1} C_k (1 + B)} \text{ and } |b_k| \leq \frac{B - A_2}{k^{n+1} C_k (1 + B)}, (k \geq 2)$$

this implies that

$$\sqrt{|a_k b_k|} \leq \frac{\sqrt{(B - A_1)(B - A_2)}}{k^{n+1} C_k (1 + B)}, (k \geq 2). \tag{9}$$

From (9) we obtain that (8) holds if

$$\frac{\sqrt{(B - A_1)(B - A_2)}}{k^{n+1} C_k (1 + B)} \leq \frac{B - A^*}{\sqrt{(B - A_1)(B - A_2)}}$$

or equivalently

$$A^* \leq B - \frac{(B - A_1)(B - A_2)}{k^{n+1} C_k (1 + B)}.$$

But $k^{n+1} C_k < (k+1)^{n+1} C_{k+1}, (k \geq 2)$ so

$$B - \frac{(B - A_1)(B - A_2)}{k^{n+1} C_k (1 + B)} \geq B - \frac{(B - A_1)(B - A_2)}{2^{n+1} C_2 (1 + B)}, (k \geq 2),$$

consequently the above inequality holds provided that

$$A^* = B - \frac{(B - A_1)(B - A_2)}{2^{n+1} C_2 (1 + B)}.$$

The result is sharp, because if

$$f(z) = z + \frac{B - A_1}{2^{n+1} C_2 (1 + B)} e^{i\theta_1} z^2 \in VP(n, \lambda, A_1, B)$$

$$g(z) = z + \frac{B - A_2}{2^{n+1} C_2 (1 + B)} e^{i\theta_2} z^2 \in VP(n, \lambda, A_2, B)$$

$$f \otimes g \in VP(n, \lambda, A^*, B)$$

and satisfy (4) with equality. Indeed,

$$2^{n+1}C_2\,(1+B)\,\frac{(B-A_1)(B-A_2)}{2^{2n+2}C_2^2\,(1+B)^2}=B-A^*$$

because

$$B-A^*=\frac{(B-A_1)(B-A_2)}{2^{n+1}C_2\,(1+B)}.$$

□

Corollary 1 *If* $f,g\in VP(n,\lambda,A,B)$ *then* $f\otimes g\in VP(n,\lambda,A^*,B)$, *where*

$$A^*=B-\frac{(B-A)^2}{2^{n+1}C_2\,(1+B)}.$$

The result is sharp.

Theorem 3 *If* $f\in VP(n,\lambda,A,B_1), g\in VP(n,\lambda,A,B_2)$ *then* $f\otimes g\in VP(n,\lambda,A,B^*)$, *where*

$$B^*=A+\frac{(B_1-A)(B_2-A)\,(A+1)}{2^{n+1}C_2\,(1+B_1)\,(1+B_2)-(B_1-A)\,(B_2-A)}.$$

The result is sharp.

Proof. Let $f\in VP(n,\lambda,A,B_1), g\in VP(n,\lambda,A,B_2)$ and suppose they have the form (1). Since $f\in VP(n,\lambda,A,B_1)$ we have

$$\frac{\sum\limits_{k=2}^{\infty}k^{n+1}C_k\,(1+B_1)\,|a_k|}{B_1-A}\leq 1 \tag{10}$$

and for $g\in VP(n,\lambda,A,B_2)$ we have

$$\frac{\sum\limits_{k=2}^{\infty}k^{n+1}C_k\,(1+B_2)\,|b_k|}{B_2-A}\leq 1. \tag{11}$$

We know from Theorem 1 that $f\otimes g\in VP(n,\lambda,A,B^*)$ if and only if

$$\frac{\sum\limits_{k=2}^{\infty}k^{n+1}C_k\,(1+B^*)\,|a_kb_k|}{B^*-A}\leq 1. \tag{12}$$

By using the Cauchy-Schwarz inequality for (10) and (11) we have

$$\frac{\sum_{k=2}^{\infty} k^{n+1} C_k \sqrt{|a_k b_k|}\sqrt{(1+B_1)(1+B_2)}}{\sqrt{(B_1-A)(B_2-A)}} \leq 1.$$

We note that

$$\frac{\sum_{k=2}^{\infty} k^{n+1} C_k (1+B^*)|a_k b_k|}{B^*-A} \leq \frac{\sum_{k=2}^{\infty} k^{n+1} C_k \sqrt{|a_k b_k|}\sqrt{(1+B_1)(1+B_2)}}{\sqrt{(B_1-A)(B_2-A)}}$$

implies (12). But the above inequality holds provided that

$$\frac{|a_k b_k|(1+B^*)}{B^*-A} \leq \frac{\sqrt{|a_k b_k|}\sqrt{(1+B_1)(1+B_2)}}{\sqrt{(B_1-A)(B_2-A)}}$$

or

$$\sqrt{|a_k b_k|} \leq \frac{(B^*-A)\sqrt{(1+B_1)(1+B_2)}}{(1+B^*)\sqrt{(B_1-A)(B_2-A)}}. \tag{13}$$

From Theorem 1 we have:

$$|a_k| \leq \frac{B_1-A}{k^{n+1} C_k (1+B_1)} \text{ and } |b_k| \leq \frac{B_2-A}{k^{n+1} C_k (1+B_2)}, (k \geq 2)$$

this implies that

$$\sqrt{|a_k b_k|} \leq \frac{\sqrt{(B_1-A)(B_2-A)}}{k^{n+1} C_k \sqrt{(1+B_1)(1+B_2)}}, (k \geq 2). \tag{14}$$

from (14) we obtain that (13) holds if

$$\frac{\sqrt{(B_1-A)(B_2-A)}}{k^{n+1} C_k \sqrt{(1+B_1)(1+B_2)}} \leq \frac{(B^*-A)\sqrt{(1+B_1)(1+B_2)}}{(1+B^*)\sqrt{(B_1-A)(B_2-A)}}$$

or equivalently

$$B^* \geq A + \frac{(B_1-A)(B_2-A)(A+1)}{k^{n+1} C_k (1+B_1)(1+B_2) - (B_1-A)(B_2-A)}.$$

But $k^{n+1} C_k < (k+1)^{n+1} C_{k+1}, (k \geq 2)$ so :

$$A + \frac{(B_1-A)(B_2-A)(A+1)}{k^{n+1} C_k (1+B_1)(1+B_2) - (B_1-A)(B_2-A)} \leq$$

$$\leq A + \frac{(B_1 - A)(B_2 - A)\,(A+1)}{2^{n+1}C_2\,(1+B_1)\,(1+B_2) - (B_1 - A)\,(B_2 - A)}, (k \geq 2),$$

consequently the above inequality holds provided that

$$B^* = A + \frac{(B_1 - A)(B_2 - A)\,(A+1)}{2^{n+1}C_2\,(1+B_1)\,(1+B_2) - (B_1 - A)\,(B_2 - A)}.$$

The result is sharp, because if

$$f(z) = z + \frac{B_1 - A}{2^{n+1}C_2\,(1+B_1)} e^{i\theta_1} z^2 \in VP(n, \lambda, A, B_1)$$

$$g(z) = z + \frac{B_2 - A}{2^{n+1}C_2\,(1+B_2)} e^{i\theta_2} z^2 \in VP(n, \lambda, A, B_2)$$

$$f * g \in VP(n, \lambda, A, B^*)$$

and satisfy (4) with equality. Indeed,

$$(1 + B^*)\, 2^{n+1} C_2 \frac{(B_1 - A)(B_2 - A)}{2^{2n+2} C_2^2\,(1+B_1)\,(1+B_2)} = B^* - A$$

because

$$B^* - A = \frac{(B_1 - A)(B_2 - A)\,(A+1)}{2^{n+1}C_2\,(1+B_1)\,(1+B_2) - (B_1 - A)\,(B_2 - A)}.$$

□

Corollary 2 *If* $f, g \in VP(n, \lambda, A, B)$ *then* $f \otimes g \in VP(n, \lambda, A, B^*)$, *where*

$$B^* = A + \frac{(B - A)^2\,(A+1)}{2^{n+1}C_2\,(1+B)^2 - (B - A)^2}.$$

The result is sharp.

Theorem 4 *If* $f_j \in VP(n, \lambda, A_j, B), j = \overline{1, s}, s \in \{2, 3, 4, \ldots\}$ *then* $f_1 \otimes f_2 \otimes \ldots \otimes f_s \in VP(n, \lambda, A^{(s-1)*}, B)$, *where*

$$A^{(s-1)*} = B - \frac{\prod_{j=1}^{s}(B - A_j)}{2^{(n+1)(s-1)} C_2^{s-1}\,(1+B)^{s-1}}.$$

The result is sharp.

Proof. For the proof we use the mathematical induction method and suppose that $f_j, \forall j$ have the form (1).

Let $s = 2$. If $f_j \in VP(n, \lambda, A_j, B), j = \overline{1,2}$ then $f_1 \otimes f_2 \in VP(n, \lambda, A^*, B)$ where $A^* = B - \dfrac{(B-A_1)(B-A_2)}{2^{n+1} C_2 (1+B)}$, from Theorem 2 is true.

Assume, for $s = m$, that the formula displayed below holds.

If $f_j \in VP(n, \lambda, A_j, B), j = \overline{1,m}, m \in \{2,3,4,\ldots\}$ then $f_1 \otimes f_2 \otimes \ldots \otimes f_m \in VP(n, \lambda, A^{(m-1)*}, B)$, where

$$A^{(m-1)*} = B - \frac{\prod_{j=1}^{m} (B - A_j)}{k^{(n+1)(m-1)} C_k^{m-1} (1+B)^{m-1}}.$$

Let $s = m+1$: if $f_1 \otimes f_2 \otimes \ldots \otimes f_m \in VP(n, \lambda, A^{(m-1)*}, B), m \in \{2,3,4,\ldots\}$ and $f_{m+1} \in VP(n, \lambda, A_{m+1}, B)$ then we have to prove

$f_1 \otimes f_2 \otimes \ldots \otimes f_m \otimes f_{m+1} \in VP(n, \lambda, A^{m*}, B)$, where $A^{m*} = B - \dfrac{\prod_{j=1}^{m+1} (B-A_j)}{k^{(n+1)m} C_k^m (1+B)^m}$.

For the proof we use the result of Theorem 2:

$$A^{m*} \le B - \frac{(B - A^{(m-1)*})(B - A_{m+1})}{k^{(n+1)} C_k (1+B)}$$

$$A^{m*} \le B - \frac{\dfrac{\prod_{j=1}^{m} (B-A_j)}{k^{(m-1)(n+1)} C_k^{m-1} (1+B)^{m-1}} (B - A_{m+1})}{k^{(n+1)} C_k (1+B)}$$

$$A^{m*} \le B - \frac{\prod_{j=1}^{m+1} (B-A_j)}{k^{m(n+1)} C_k^m (1+B)^m}.$$

But $k^{(n+1)} C_k < (k+1)^{(n+1)} C_{k+1}, (k \ge 2)$ so:

$$B - \frac{\prod_{j=1}^{m+1} (B-A_j)}{k^{m(n+1)} C_k^m (1+B)^m} \ge B - \frac{\prod_{j=1}^{m+1} (B-A_j)}{2^{m(n+1)} C_2^m (1+B)^m}, (k \ge 2),$$

consequently the above inequality holds provided that

$$A^{m*} = B - \frac{\prod_{j=1}^{m+1} (B-A_j)}{2^{m(n+1)} C_2^m (1+B)^m}.$$

The result is sharp, because if

$$f(z) = z + \frac{B - A^{(s-1)*}}{2^{n+1}C_2\,(1+B)} e^{i\theta_1} z^2 \in VP(n, \lambda, A^{(s-1)*}, B)$$
$$g(z) = z + \frac{B - A_s}{2^{n+1}C_2\,(1+B)} e^{i\theta_2} z^2 \in VP(n, \lambda, A_s, B)$$
$$f \otimes g \in VP(n, \lambda, A^{s*}, B)$$

and satisfy (4) with equality. Indeed,

$$2^{n+1}C_2\,(1+B)\,\frac{(B - A^{(s-1)*})(B - A_s)}{2^{2(n+1)}C_2^2\,(1+B)^2} = B - A^{s*}$$

because

$$B - A^{s*} = \frac{(B - A^{(s-1)*})(B - A_s)}{2^{n+1}C_2\,(1+B)} \Leftrightarrow B - A^{s*} = \frac{\prod_{j=1}^{s+1}(B - A_j)}{2^{s(n+1)}C_2^s\,(1+B)^s}.$$

□

Theorem 5 *If* $f_j \in VP(n, \lambda, A, B_j), j = \overline{1, s}, s \in \{2, 3, 4, \ldots\}$ *then* $f_1 \otimes f_2 \otimes \ldots \otimes f_s \in VP(n, \lambda, A, B^{(s-1)*})$, *where*

$$B^{(s-1)*} = A + \frac{(A+1)\prod_{j=1}^{s}(B_j - A)}{2^{(s-1)(n+1)}C_2^{s-1}\prod_{j=1}^{s}(1+B_j) - \prod_{j=1}^{s}(B_j - A)}.$$

The result is sharp.

Proof. For the proof we use the mathematical induction method and suppose that $f_j, \forall j$ have the form (1).
Let $s = 2$. If $f_j \in VP(n, \lambda, A, B_j), j = \overline{1, 2}$ then $f_1 \otimes f_2 \in VP(n, \lambda, A, B^*)$ where

$$B^* = A + \frac{(A+1)(B_1 - A)(B_2 - A)}{2^{n+1}C_2(1+B_1)(1+B_2) - (B_1 - A)(B_2 - A)},$$

from Theorem 3 is true.
Assume, for $s = m$, that the formula displayed below holds.

If $f_j \in VP(n, \lambda, A, B_j), j = \overline{1, m}, m \in \{2, 3, 4, \ldots\}$ then $f_1 \otimes f_2 \otimes \ldots \otimes f_m \in VP(n, \lambda, A, B^{(m-1)*})$, where

$$B^{(m-1)*} = A + \frac{(A+1)\prod_{j=1}^{m}(B_j - A)}{2^{(m-1)(n+1)}C_2^{m-1}\prod_{j=1}^{m}(1+B_j) - \prod_{j=1}^{m}(B_j - A)}.$$

Let $s = m+1$: if $f_1 \otimes f_2 \otimes \ldots \otimes f_m \in VP(n, \lambda, A, B^{(m-1)*}), m \in \{2, 3, 4, \ldots\}$ and $f_{m+1} \in VP(n, \lambda, A, B_{m+1})$ then we have to prove $f_1 \otimes f_2 \otimes \ldots \otimes f_m \otimes f_{m+1} \in VP(n, \lambda, A, B^{m*})$, where

$$B^{m*} = A + \frac{(A+1)\prod_{j=1}^{m+1}(B_j - A)}{2^{m(n+1)}C_2^{m}\prod_{j=1}^{m}(1+B_j) - \prod_{j=1}^{m}(B_j - A)}.$$

For the proof we use the result of Theorem 3:

$$B^{m*} \geq A + \frac{(A+1)(B^{(m-1)*} - A)(B_{m+1} - A)}{k^{n+1}C_k(1+B_1)(1+B_{m+1}) - (B^{(m-1)*} - A)(B_{m+1} - A)}$$

or equivalently

$$B^{m*} \geq A + \frac{(A+1)\prod_{j=1}^{m+1}(B_j - A)}{k^{m(n+1)}C_k^{m}\prod_{j=1}^{m}(1+B_j) - \prod_{j=1}^{m}(B_j - A)}.$$

But $k^{n+1}C_k < (k+1)^{n+1}C_{k+1}, (k \geq 2)$ so:

$$A + \frac{(A+1)\prod_{j=1}^{m+1}(B_j - A)}{k^{m(n+1)}C_k^{m}\prod_{j=1}^{m}(1+B_j) - \prod_{j=1}^{m}(B_j - A)} \leq$$

$$A + \frac{(A+1)\prod_{j=1}^{m+1}(B_j - A)}{2^{m(n+1)}C_2^{m}\prod_{j=1}^{m}(1+B_j) - \prod_{j=1}^{m}(B_j - A)}, \ (k \geq 2),$$

consequently the above inequality holds provided that

$$B^{m*} = A + \frac{(A+1)\prod_{j=1}^{m+1}(B_j - A)}{2^{m(n+1)}C_2^m \prod_{j=1}^{m+1}(1+B_j) - \prod_{j=1}^{m+1}(B_j - A)}.$$

The result is sharp, because if

$$f(z) = z + \frac{B^{(s-1)*} - A}{2^{n+1}C_2\left(1 + B^{(s-1)*}\right)} e^{i\theta_1} z^2 \in VP(n, \lambda, A, B^{(s-1)*})$$

$$g(z) = z + \frac{B_s - A}{2^{n+1}C_2\,(1 + B_{s+1})} e^{i\theta_2} z^2 \in VP(n, \lambda, A, B_{s+1})$$

$$f \otimes g \in VP(n, \lambda, A, B^{s*})$$

and satisfy (4) with equality. Indeed,

$$2^{n+1}C_2\,(1 + B^{s*})\,\frac{(B^{(s-1)*} - A)(B_{s+1} - A)}{2^{2(n+1)}C_2^2\,(1 + B_{s+1})^2} = B^{s*} - A$$

because

$$B^{s*} = A + \frac{(A+1)\prod_{j=1}^{s+1}(B_j - A)}{2^{s(n+1)}C_2^s \prod_{j=1}^{s+1}(1+B_j) - \prod_{j=1}^{s+1}(B_j - A)}.$$

□

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