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On bounds of the sine and cosine along straight lines on the complex plane

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Abstract. In the paper, the author discusses and computes bounds of the sine and cosine along straight lines on the complex plane.

1 Motivations

In the theory of complex functions, the sine and cosine on the complex plane $\mathbb{C}$ are denoted and defined respectively by

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2},$$

where $z = x + iy$, $x, y \in \mathbb{R}$, and $i = \sqrt{-1}$ is the imaginary unit. When $z = x \in \mathbb{R}$, these two trigonometric functions become $\sin x$ and $\cos x$ which satisfy the periodicity

$$\sin(x + 2k\pi) = \sin x, \quad \cos(x + 2k\pi) = \cos x$$

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and the boundedness

$$0 \le |\sin x| \le 1, \quad 0 \le |\cos x| \le 1 \tag{1}$$

for $k \in \mathbb{Z}$. On the other hand, when $z = iy$ for $y \in \mathbb{R}$,

$$\sin(iy) = \frac{e^{-y} - e^{y}}{2i} \to \pm i\infty \quad \text{and} \quad \cos(iy) = \frac{e^{-y} + e^{y}}{2} \to +\infty$$

as $y \to \pm\infty$. These imply that the sine and cosine are bounded on the real x-axis, but unbounded on the imaginary y-axis.

Motivated by the above boundedness, we naturally guess that the complex functions $\sin z$ and $\cos z$ for $z \in \mathbb{C}$ are

1. bounded on all straight lines parallel to the real x-axis,
2. unbounded on all straight lines whose slopes are not horizontal.

In this paper, we will verify the above guesses and compute bounds for $\sin z$ and $\cos z$ on all horizontal straight lines.

2 Unboundedness of sine and cosine on sloped and vertical lines

On the sloped straight line $y = \alpha + \beta x$ for constants $\alpha \in \mathbb{R}$ and $\beta \ne 0$ on the complex plane $\mathbb{C}$, by the triangle inequality for complex numbers, we have

$$\begin{aligned}
|\sin z| &= |\sin(x + i(\alpha + \beta x))| \\
&= \left| \frac{e^{i[x+i(\alpha+\beta x)]} - e^{-i[x+i(\alpha+\beta x)]}}{2i} \right| \\
&= \left| \frac{e^{[ix-(\alpha+\beta x)]} - e^{-[ix-(\alpha+\beta x)]}}{2i} \right| \\
&\ge \frac{1}{2} \left| \left| e^{[ix-(\alpha+\beta x)]} \right| - \left| e^{-[ix-(\alpha+\beta x)]} \right| \right| \\
&= \frac{1}{2} \left| e^{-(\alpha+\beta x)} - e^{(\alpha+\beta x)} \right| \\
&\to +\infty, \quad x \to \pm\infty
\end{aligned}$$

and

$$|\cos z| = |\cos(x + i(\alpha + \beta x))|$$

$$
\begin{aligned}
&= \left|\frac{e^{i[x+i(\alpha+\beta x)]} + e^{-i[x+i(\alpha+\beta x)]}}{2}\right| \\
&= \left|\frac{e^{[ix-(\alpha+\beta x)]} + e^{-[ix-(\alpha+\beta x)]}}{2}\right| \\
&\ge \frac{1}{2}\left|\left|e^{[ix-(\alpha+\beta x)]}\right| - \left|e^{-[ix-(\alpha+\beta x)]}\right|\right| \\
&= \frac{1}{2}\left|e^{-(\alpha+\beta x)} - e^{(\alpha+\beta x)}\right| \\
&\to +\infty, \quad x \to \pm\infty.
\end{aligned}
$$

Consequently, the functions $\sin z$ and $\cos z$ are not bounded along any sloped straight line.

On the vertical straight line $x = \gamma$ for any constant $\gamma \in \mathbb{R}$ on the complex plane, by the triangle inequality for complex numbers, we have

$$
\begin{aligned}
|\sin z| = |\sin(\gamma + iy)| &= \left|\frac{e^{i(\gamma+iy)} - e^{-i(\gamma+iy)}}{2i}\right| \\
&\ge \frac{1}{2}\left||e^{i(\gamma+iy)}| - |e^{-i(\gamma+iy)}|\right| = \frac{1}{2}|e^{-y} - e^{y}| \to +\infty
\end{aligned}
$$

and

$$
\begin{aligned}
|\cos z| = |\cos(\gamma + iy)| &= \left|\frac{e^{i(\gamma+iy)} + e^{-i(\gamma+iy)}}{2}\right| \\
&\ge \frac{1}{2}\left||e^{i(\gamma+iy)}| - |e^{-i(\gamma+iy)}|\right| = \frac{1}{2}|e^{-y} - e^{y}| \to +\infty
\end{aligned}
$$

as $y \to \pm\infty$. Consequently, the functions $\sin z$ and $\cos z$ are not bounded along any vertical straight line.

3 Bounds of the sine on horizontal straight lines

On the horizontal straight line $y = \alpha$ for any constant $\alpha \in \mathbb{R}$ on the complex plane $\mathbb{C}$, by the triangle inequality for complex numbers, we have

$$
\begin{aligned}
|\sin z| = |\sin(x + i\alpha)| &= \left|\frac{e^{i(x+i\alpha)} - e^{-i(x+i\alpha)}}{2i}\right| \\
&= \left|\frac{e^{(ix-\alpha)} - e^{-(ix-\alpha)}}{2i}\right| = \frac{1}{2}\left|\frac{e^{ix}}{e^{\alpha}} - e^{-ix}e^{\alpha}\right|
\end{aligned}
$$

$$\leq \frac{1}{2}\left(\left|\frac{e^{ix}}{e^{\alpha}}\right| + \left|e^{-ix}e^{\alpha}\right|\right) = \frac{1}{2}\left(\frac{1}{e^{\alpha}} + e^{\alpha}\right)$$

and

$$\begin{aligned}
|\sin z| = |\sin(x+i\alpha)| &= \left|\frac{e^{i(x+i\alpha)} - e^{-i(x+i\alpha)}}{2i}\right| \\
&= \left|\frac{e^{(ix-\alpha)} - e^{-(ix-\alpha)}}{2i}\right| = \frac{1}{2}\left|\frac{e^{ix}}{e^{\alpha}} - e^{-ix}e^{\alpha}\right| \\
&\geq \frac{1}{2}\left|\left|\frac{e^{ix}}{e^{\alpha}}\right| - \left|e^{-ix}e^{\alpha}\right|\right| = \frac{1}{2}\left|\frac{1}{e^{\alpha}} - e^{\alpha}\right|.
\end{aligned}$$

Therefore, it follows that

$$\frac{1}{2}\left|\frac{1}{e^{\alpha}} - e^{\alpha}\right| \leq |\sin(x+i\alpha)| \leq \frac{1}{2}\left(\frac{1}{e^{\alpha}} + e^{\alpha}\right), \quad x, \alpha \in \mathbb{R}.$$

When $z = 2k\pi + i\alpha$ for $k \in \mathbb{Z}$, we have

$$\sin z = \sin(2k\pi + i\alpha) = \frac{e^{i(2k\pi+i\alpha)} - e^{-i(2k\pi+i\alpha)}}{2i} = -\frac{i}{2}\left(\frac{1}{e^{\alpha}} - e^{\alpha}\right).$$

When $z = 2k\pi + \frac{\pi}{2} + i\alpha$ for $k \in \mathbb{Z}$, we have

$$\begin{aligned}
\sin z = \sin\left(2k\pi + \frac{\pi}{2} + i\alpha\right) &= \frac{e^{i(2k\pi+\pi/2+i\alpha)} - e^{-i(2k\pi+\pi/2+i\alpha)}}{2i} \\
&= \frac{e^{i(\pi/2+i\alpha)} - e^{-i(\pi/2+i\alpha)}}{2i} = \frac{e^{-\alpha} + e^{\alpha}}{2} = \frac{1}{2}\left(\frac{1}{e^{\alpha}} + e^{\alpha}\right).
\end{aligned}$$

When $z = (2k+1)\pi + i\alpha$ for $k \in \mathbb{Z}$, we have

$$\begin{aligned}
\sin z = \sin((2k+1)\pi + i\alpha) &= \frac{e^{i((2k+1)\pi+i\alpha)} - e^{-i((2k+1)\pi+i\alpha)}}{2i} \\
&= \frac{e^{i(\pi+i\alpha)} - e^{-i(\pi+i\alpha)}}{2i} = \frac{1}{2i}\left(e^{\alpha} - \frac{1}{e^{\alpha}}\right) = -\frac{i}{2}\left(e^{\alpha} - \frac{1}{e^{\alpha}}\right).
\end{aligned}$$

When $z = (2k+1)\pi + \frac{\pi}{2} + i\alpha$ for $k \in \mathbb{Z}$, we have

$$\begin{aligned}
\sin z &= \sin\left((2k+1)\pi + \frac{\pi}{2} + i\alpha\right) \\
&= \frac{e^{i((2k+1)\pi+\pi/2+i\alpha)} - e^{-i((2k+1)\pi+\pi/2+i\alpha)}}{2i} \\
&= \frac{e^{i(3\pi/2+i\alpha)} - e^{-i(3\pi/2+i\alpha)}}{2i} = -\frac{e^{-\alpha} + e^{\alpha}}{2} = -\frac{1}{2}\left(\frac{1}{e^{\alpha}} + e^{\alpha}\right).
\end{aligned}$$

4 Bounds of the cosine on horizontal straight lines

On the horizontal straight line $y = \alpha$ for any constant $\alpha \in \mathbb{R}$ on the complex plane $\mathbb{C}$, by the triangle inequality for complex numbers, we have

$$\begin{aligned}
|\cos z| = |\cos(x + i\alpha)| &= \left|\frac{e^{i(x+i\alpha)} + e^{-i(x+i\alpha)}}{2}\right| \\
&= \left|\frac{e^{(ix-\alpha)} + e^{-(ix-\alpha)}}{2}\right| = \frac{1}{2}\left|\frac{e^{ix}}{e^{\alpha}} + e^{-ix}e^{\alpha}\right| \\
&\le \frac{1}{2}\left(\left|\frac{e^{ix}}{e^{\alpha}}\right| + \left|e^{-ix}e^{\alpha}\right|\right) = \frac{1}{2}\left(\frac{1}{e^{\alpha}} + e^{\alpha}\right)
\end{aligned}$$

and

$$\begin{aligned}
|\cos z| = |\cos(x + i\alpha)| &= \left|\frac{e^{i(x+i\alpha)} + e^{-i(x+i\alpha)}}{2}\right| \\
&= \left|\frac{e^{(ix-\alpha)} + e^{-(ix-\alpha)}}{2}\right| = \frac{1}{2}\left|\frac{e^{ix}}{e^{\alpha}} + e^{-ix}e^{\alpha}\right| \\
&\ge \frac{1}{2}\left|\left|\frac{e^{ix}}{e^{\alpha}}\right| - \left|e^{-ix}e^{\alpha}\right|\right| = \frac{1}{2}\left|\frac{1}{e^{\alpha}} - e^{\alpha}\right|.
\end{aligned}$$

Therefore, it follows that

$$\frac{1}{2}\left|\frac{1}{e^{\alpha}} - e^{\alpha}\right| \le |\cos(x + i\alpha)| \le \frac{1}{2}\left(\frac{1}{e^{\alpha}} + e^{\alpha}\right), \quad x, \alpha \in \mathbb{R}.$$

When $z = 2k\pi + i\alpha$ for $k \in \mathbb{Z}$, we have

$$\cos z = \cos(2k\pi + i\alpha) = \frac{e^{i(2k\pi+i\alpha)} + e^{-i(2k\pi+i\alpha)}}{2} = \frac{1}{2}\left(\frac{1}{e^{\alpha}} + e^{\alpha}\right).$$

When $z = 2k\pi + \frac{\pi}{2} + i\alpha$ for $k \in \mathbb{Z}$, we have

$$\begin{aligned}
\cos z = \cos\left(2k\pi + \frac{\pi}{2} + i\alpha\right) &= \frac{e^{i(2k\pi+\pi/2+i\alpha)} + e^{-i(2k\pi+\pi/2+i\alpha)}}{2} \\
= \frac{e^{i(\pi/2+i\alpha)} + e^{-i(\pi/2+i\alpha)}}{2} &= \frac{ie^{-\alpha} - ie^{\alpha}}{2} = \frac{i}{2}\left(\frac{1}{e^{\alpha}} - e^{\alpha}\right).
\end{aligned}$$

When $z = (2k + 1)\pi + i\alpha$ for $k \in \mathbb{Z}$, we have

$$\cos z = \cos((2k + 1)\pi + i\alpha) = \frac{e^{i((2k+1)\pi+i\alpha)} + e^{-i((2k+1)\pi+i\alpha)}}{2}$$

$$= \frac{e^{i(\pi+i\alpha)} + e^{-i(\pi+i\alpha)}}{2} = -\frac{1}{2}\left(\frac{1}{e^{\alpha}} + e^{\alpha}\right).$$

When $z = (2k+1)\pi + \frac{\pi}{2} + i\alpha$ for $k \in \mathbb{Z}$, we have

$$\begin{aligned}\cos z &= \cos\left((2k+1)\pi + \frac{\pi}{2} + i\alpha\right)\\ &= \frac{e^{i((2k+1)\pi+\pi/2+i\alpha)} + e^{-i((2k+1)\pi+\pi/2+i\alpha)}}{2}\\ &= \frac{e^{i(3\pi/2+i\alpha)} + e^{-i(3\pi/2+i\alpha)}}{2} = \frac{-ie^{-\alpha} + ie^{\alpha}}{2} = -\frac{i}{2}\left(\frac{1}{e^{\alpha}} - e^{\alpha}\right).\end{aligned}$$

5 Alternative proofs

Since $\sin\left(z + \frac{\pi}{2}\right) = \cos z$ for $z \in \mathbb{C}$, there is a similar behaviour of sine and cosine in the complex plane $\mathbb{C}$. Hence, in what follows, we just only consider sine.

It is easy to see that sine, cosine, hyperbolic sine, and hyperbolic cosine have relations

$$\sin(it) = i\sinh t, \quad \sinh(it) = i\sin t, \quad \cos(it) = \cosh t, \quad \cosh(it) = \cos t.$$

Accordingly, we have

$$\sin z = \sin(x+iy) = \sin x\cos(iy) + \cos x\sin(iy) = \sin x\cosh y + i\cos x\sinh y$$

and

$$|\sin z|^2 = \sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y.$$

On any horizontal line $y = c$, say, we have

$$\begin{aligned}|\sin z|^2 &= \sin^2 x\cosh^2 c + \cos^2 x\sinh^2 c\\ &= \sin^2 x\cosh^2 c + \cos^2 x\left(\cosh^2 c - 1\right) = \cosh^2 c - \cos^2 x\end{aligned}$$

or

$$|\sin z|^2 = 1 + \sinh^2 c - \cos^2 x = \sinh^2 c + \sin^2 x.$$

Consequently, sine is bounded on all horizontal lines.

Look at a non-horizontal line, where $z = \gamma + \alpha x + i\beta y$ for $\beta \neq 0$ (by non-horizontality). Here

$$|\sin z|^2 = \sin^2(\gamma+\alpha x)\cosh^2(\beta y) + \cos^2(\gamma+\alpha x)\sinh^2(\beta y).$$

If the line is sloped so that $\alpha \neq 0$ (by non-verticality), then both terms in the above equation are unbounded, so that sine is unbounded.

If the line is vertical, so that $\alpha = 0$, we have to be a tad careful! If γ is not a multiple of π, the term $\sin^2(\gamma + \alpha x)\cosh^2(\beta y)$ is unbounded; and if γ is a multiple of π, then the term $\cos^2(\gamma + \alpha x)\sinh^2(\beta y)$ is unbounded. In a word, sine is unbounded on all non-horizontal lines.

6 Conclusions

On the sloped straight line $y = \alpha + \beta x$ for $\alpha \in \mathbb{R}$ and $\beta \neq 0$ on the complex plane $\mathbb{C}$, the trigonometric functions $\sin z = \sin(x + i(\alpha + \beta x))$ and $\cos z = \cos(x + i(\alpha + \beta x))$ are unbounded.

On the vertical straight line $x = \gamma$ for any scalar $\gamma \in \mathbb{R}$ on the complex plane $\mathbb{C}$, the trigonometric functions $\sin z = \sin(\gamma + iy)$ and $\cos z = \cos(\gamma + iy)$ are unbounded.

On the horizontal straight line $y = \alpha$ for any constant $\alpha \in \mathbb{R}$ on the complex plane $\mathbb{C}$, the trigonometric functions $\sin z = \sin(x+i\alpha)$ and $\cos z = \cos(x+i\alpha)$ are bounded by the double inequalities

$$|\sinh \alpha| \le |\sin(x + i\alpha)| \le \cosh \alpha, \quad x, \alpha \in \mathbb{R} \tag{2}$$

and

$$|\sinh \alpha| \le |\cos(x + i\alpha)| \le \cosh \alpha, \quad x, \alpha \in \mathbb{R} \tag{3}$$

whose equalities are respectively attained at points

$$2k\pi + i\alpha, \quad 2k\pi + \frac{\pi}{2} + i\alpha, \quad (2k+1)\pi + i\alpha, \quad 2k\pi + \frac{3\pi}{2} + i\alpha$$

with concrete values

$$\begin{aligned}
&\sin(2k\pi + i\alpha) = \cos\left(2k\pi + \frac{3\pi}{2} + i\alpha\right) = i\sinh\alpha,\\
&\sin\left(2k\pi + \frac{\pi}{2} + i\alpha\right) = \cos(2k\pi + i\alpha) = \cosh\alpha,\\
&\sin((2k+1)\pi + i\alpha) = \cos\left(2k\pi + \frac{\pi}{2} + i\alpha\right) = -i\sinh\alpha,\\
&\sin\left(2k\pi + \frac{3\pi}{2} + i\alpha\right) = \cos((2k+1)\pi + i\alpha) = -\cosh\alpha
\end{aligned}$$

for $k \in \mathbb{Z}$.

Letting $\alpha \to 0$ in the double inequalities (2) and (3) recovers inequalities in (1) for $x \in \mathbb{R}$.

On the horizontal belt zones $0 \le A \le y \le B$ and $-B \le y \le -A \le 0$ on the complex plane $\mathbb{C}$, the trigonometric functions $\sin z = \sin(x + iy)$ and $\cos z = \cos(x + iy)$ are bounded by the double inequalities

$$\sinh A \le |\cos(x \pm iy)| \le \cosh B$$

and

$$\sinh A \le |\sin(x \pm iy)| \le \cosh B$$

for $x \in \mathbb{R}$.

7 An open problem

The inequalities in (1) can be refined as

$$\frac{2}{\pi}x \le \sin x \le x \quad \text{and} \quad 1 - \frac{2}{\pi}x \le \cos x \le 1 - \frac{x^2}{\pi} \tag{4}$$

for $0 \le x \le \frac{\pi}{2}$. See [1, p. 143], [3, p. 22], and [4, p. 33]. These two double inequalities in (4) are respectively called as Jordan's and Kober's inequality. These two double inequalities have been further refined, generalized, applied, and surveyed in the papers [2, 5, 6, 8, 9, 10] and closely related references therein. Motivated by these refinements, generalizations, and applications, we pose an open problem: can one refine, generalize, and apply the double inequalities (2) and (3) for $x \in \left[0, \frac{\pi}{2}\right]$ and $\alpha \ne 0$?

Finally we remark that this paper is a revised version of the preprint [7].

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