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Some consequences of the rank normal form of a matrix

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Abstract. If A is a rectangular matrix of rank r, then A may be written as PSQ where P and Q are invertible matrices and $S = \begin{pmatrix} I_r & O \\ O & O \end{pmatrix}$. This is the rank normal form of the matrix A.

The purpose of this paper is to exhibit some consequences of this representation form.

1 Introduction and notation

In the following we shall denote by K a commutative field and by $M(m \times n, K)$ the set of matrices with m lines and n columns with elements in K. By $O_{m,n}$ we shall denote the null matrix with m lines and n columns and by $I_{r,r}$ the unit matrix with r lines and r columns.

The purpose of this paper is to exhibit some consequences of the following representation theorem from [1], [2], [3], [4], [5], [6], [7].

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Theorem 1 (*rank normal form of a matrix*). *Let* $A \in M(m \times n, K)$ *with* $\mathrm{rank} A = r$. *Then there are the invertible matrices* $P \in M(m \times m, K)$ *and* $Q \in M(n \times n, K)$ *and*

$$S = \begin{bmatrix} I_{r,r} & O_{r,n-r} \\ O_{m-r,r} & O_{m-r,n-r} \end{bmatrix} \in M(m \times n, K)$$

such that $A = PSQ$.

The exposition of the results is made in an unitary manner using the block matrices and contains some known inequalities as inequality of Sylvester and inequality of Frobenius.

The Theorem 9 is new.

2 Main results

In the following we give three representation theorems 2, 3 and 4 which are direct consequences of Theorem 1.

Theorem 2 *Let* $A \in M(m \times n, K)$ *with* rank $A = r$. *Then there are* $A_1, A_2, \ldots,$ $A_r \in M(m \times n, K)$ *matrices of* rank 1 *such that* $A = A_1 + A_2 + \ldots + A_r$.

Proof. The proof follows from Theorem 1 if we note that the matrix

$$S = \begin{bmatrix} I_{r,r} & O_{r,n-r} \\ O_{m-r,r} & O_{m-r,n-r} \end{bmatrix},$$

may be written as the sum of r matrices of rank 1. □

Theorem 3 *Let* $A \in M(m \times n, K)$ *with* rank $A = r$. *Then there are two matrices with rank equal with* r, $B \in M(m \times r, K)$ *and* $C \in M(r \times n, K)$ *such that* $A = BC$.

Proof. From Theorem 1 it follows that there are the invertible matrices $P \in M(m \times m, K)$ and $Q \in M(n \times n, K)$ and

$$S = \begin{bmatrix} I_{r,r} & O_{r,n-r} \\ O_{m-r,r} & O_{m-r,n-r} \end{bmatrix} \in M(m \times n, K)$$

such that $A = PSQ$.

Note that $S = S_1S_2$ where $S_1 = \begin{bmatrix} I_{r,r} \\ O_{m-r,r} \end{bmatrix} \in M(m \times r, K)$ and $S_2 = [I_{r,r}, O_{r,n-r}] \in M(r \times n, K)$.

If we put $B = PS_1$ and $C = S_2Q$ we have $A = BC$ and $\operatorname{rank} B = \operatorname{rank}(PS_1) = \operatorname{rank} S_1 = r$, $\operatorname{rank} C = \operatorname{rank}(S_2Q) = \operatorname{rank} S_2 = r$. □

Theorem 4 *Let* $A \in M(n \times n, K)$ *with* $\operatorname{rank} A = r$. *Then there exists two matrices with rank* r, $B \in M(n \times n, K)$ *and* $C \in M(n \times n, K)$ *such that* $A = BC$.

Proof. From Theorem 1 it follows that there are two invertible matrices $P, Q \in M(n \times n, K)$ and $S = \operatorname{diag}[I_{r,r}, O_{n-r,n-r}]$ such that $A = PSQ$. Let $B = PS$ and $C = SQ$.

From the equality $S^2 = S$ we have that $A = B \cdot C$. We have

$$\operatorname{rank} B = \operatorname{rank}(PS) = \operatorname{rank} S = r \text{ and } \operatorname{rank} C = \operatorname{rank}(SQ) = \operatorname{rank} S = r$$

□

Lemma 1 *Let* $A \in M(m \times n, K)$, $B \in M(p \times q, K)$, $S = \begin{bmatrix} I_{r,r} & O_{r,p-r} \\ O_{n-r,r} & O_{n-r,p-r} \end{bmatrix} \in M(n \times p, K)$, $S_1 = \begin{bmatrix} I_{r,r} \\ O_{n-r,r} \end{bmatrix} \in M(n \times r, K)$, $S_2 = [I_{r,p}, O_{r,p}] \in M(r \times p, K)$.

Then the following statements are true:

i). $\operatorname{rank}(AS) \geq \operatorname{rank} A + r - n$

ii). $\operatorname{rank}(AS_1) = \operatorname{rank}(AS_1S_2)$

iii). $\operatorname{rank}(S_1S_2B) = \operatorname{rank}(S_2B)$.

Proof.
i). Let $A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$ where $A_1 \in M(r \times r, K)$, $A_2 \in M(r \times (n-r), K)$, $A_3 \in M((m-r) \times r, K)$, $A_4 \in M((m-r) \times (n-r), K)$.

We have

$$\begin{aligned} \operatorname{rank}(AS) &= \operatorname{rank}\left(\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \cdot \begin{bmatrix} I_{r,r} & O_{r,p-r} \\ O_{n-r,r} & O_{n-r,p-r} \end{bmatrix} \right) \\ &= \operatorname{rank} \begin{bmatrix} A_1 & O_{r,p-r} \\ A_3 & O_{n-r,p-r} \end{bmatrix} = \operatorname{rank}\left(\begin{bmatrix} A_1A_2 \\ A_3A_4 \end{bmatrix} - \operatorname{rank} \begin{bmatrix} O_{r,r} & A_2 \\ O_{m-r,r} & A_4 \end{bmatrix} \right) \end{aligned}$$

$$\geq \operatorname{rank}\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} - \operatorname{rank}\begin{bmatrix} A_2 \\ A_4 \end{bmatrix} \geq \operatorname{rank} A + r - n$$

ii). We have

$$\begin{aligned} \operatorname{rank}(AS_1) &= \operatorname{rank}\left(\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}\begin{bmatrix} I_{r,r} \\ O_{n-r,r} \end{bmatrix}\right) = \operatorname{rank}\begin{bmatrix} A_1 \\ A_3 \end{bmatrix} \\ &= \operatorname{rank}\left(\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}\begin{bmatrix} I_{r,r} & O_{r,p-r} \\ O_{n-r,r} & O_{n-r,p-r} \end{bmatrix}\right) = \operatorname{rank}(AS_1S_2) \end{aligned}$$

iii). We can write

$$B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$$

where $B_1 \in M(r \times r, K), B_2 \in M(r \times q - r, K), B_3 \in M((p-r) \times r, K)$, $B_4 \in M((p-r) \times q - r, K)$.

We have

$$\begin{aligned} \operatorname{rank}(S_1S_2B) = \operatorname{rank}(S, B) &= \operatorname{rank}\left(\begin{bmatrix} I_{r,r} & O_{r,p-r} \\ O_{n-r,r} & O_{n-r,p-r} \end{bmatrix}\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}\right) \\ &= \operatorname{rank}\begin{bmatrix} B_1 & B_2 \\ O_{p-r,r} & O_{p-r,q-r} \end{bmatrix} = \operatorname{rank}[B_1B_2] \\ &= \operatorname{rank}\left([I_{r,r}, O_{r,p-r}]\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}\right) = \operatorname{rank}(S_2B). \end{aligned}$$

□

Theorems 5 and 6 may be found in the papers [1], [2], [3], [4], [5], [6], [7].

Theorem 5 (*Sylvester*). *Let* $A \in M(m \times n, K), B \in M(n \times p, K)$. *Then is true that*

$$\operatorname{rank}(AB) \geq \operatorname{rank} A + \operatorname{rank} B - n$$

Proof. The rank normal form of matrix B is $B = PSQ$, $\operatorname{rank} B = r$ where $P \in M(n \times n, K)$ and $Q \in M(p \times p, K)$ are invertible and

$$S = \begin{bmatrix} I_{r,r} & O_{r,p-r} \\ O_{n-r,r} & O_{n-r,p-r} \end{bmatrix} \in M(n \times p, K).$$

According with Lemma 1 i). we have

$$\begin{aligned} \operatorname{rank}(AB) &= \operatorname{rank}(APSQ) = \operatorname{rank}(APS) \geq \operatorname{rank}(AP) + r - n \\ &= \operatorname{rank} A + \operatorname{rank} B - n \end{aligned}$$

□

Theorem 6 (*Frobenius*). *Let* $A \in M(m \times n, K), B \in M(n \times p, K), C \in M(p \times q, K)$ *then the following inequality is true*

$$\operatorname{rank}(AB) + \operatorname{rank}(BC) \leq \operatorname{rank} B + \operatorname{rank}(ABC)$$

Proof. Let $r = \operatorname{rank} B$ and the rank normal form of B is $B = PSQ$ where $P \in M(n \times n, K)$ and $Q \in M(p \times p, K)$ are invertible. We shall apply Lemma 1, ii). and iii). and Sylvester inequality and we will obtain:

$$\begin{aligned}
\operatorname{rank}(ABC) &= \operatorname{rank}(APSQC) = \operatorname{rank}(APS, S_2QC) \\
&\geq \operatorname{rank}(APS_1) + \operatorname{rank}(S_2QC) - r \\
&= \operatorname{rank}(APS_1S_2) + \operatorname{rank}(S_1S_2QC) - r \\
&= \operatorname{rank}(APSQ) + \operatorname{rank}(PSQC) - r \\
&= \operatorname{rank}(AB) + \operatorname{rank}(BC) - \operatorname{rank} B
\end{aligned}$$

□

Theorem 7 *Let* $A \in M(n \times n, K)$ *and the sequence* $(a_p)_{p \geq 1}$ *defined by* $a_p = \operatorname{rank}(A^p)$, $p \geq 1$. *Then the following statements hold:*

i). $(a_p)_{p \geq 1}$ *is decreasing*

ii). $2a_{p+1} \leq a_p + a_{p+2}$ *for each* $p \geq 1$

iii). $a_p = a_{p+1}$ *implies that* $a_p = a_{p+t}$ *for each* $t \geq 1$.

Proof. i). We have

$$a_{p+1} = \operatorname{rank}\left(A^{p+1}\right) \leq \min\left(\operatorname{rank}(A)^p ; \operatorname{rank} A\right) \leq \operatorname{rank}(A^p) = a_p$$

for each $p \geq 1$.
ii). From Theorem 5 we have for $p \geq 1$

$$\operatorname{rank}(AA^p) + \operatorname{rank}(A^pA) \leq \operatorname{rank}(A^p) + \operatorname{rank}(AA^pA)$$

this inequality is equivalent with

$$2a_{p+1} \leq a_p + a_{p+2}$$

for each $p \geq 1$.
iii). It results from i) and ii). □

Theorem 8 *Let* $A \in M(n \times n, K)$ *and* q_A *the minimal polynomial of the matrix* A. *Then the following inequality hold*

$$\deg(q_A) \leq 1 + \operatorname{rank} A$$

Proof. We de note

$$r = \operatorname{rank} A$$

From Theorem 2 it follows that it exists $B \in M(n \times r, K)$ and $C \in M(r \times n, K)$ such that $A = BC$.

Let $D = CB \in M(r \times r, K)$ and $f_D(t) = \det(tI_r - D)$, $t \in K$.

We have from Hamilton-Cayley-Frobenius theorem that

$$f_D(D) = O_r$$

It follows that

$$Bf_D(CB)C = O_n$$

Because $B(CB)^p C = (BC)^p BC$ for $p \geq 1$.

We obtain that

$$f_D(BC)BC = O_n$$

Let $g(t) = tf_D(t), t \in K$ and note that $\deg g = r + 1$. We have $g(A) = O_n$ and that $q_A | g$, so

$$\deg q_A \leq \deg g = r + 1 = 1 + \operatorname{rank} A$$

$\square$

Theorem 9 *Let* $A \in M(n \times n, K)$. *Then the following statement are equivalent*

i). $A^2 = O_n$

ii). *There are* $B, C \in M(n \times n, K)$ *with the following properties*

$$A = BC \text{ and } CB = O_n$$

Proof. ii). $\Rightarrow$ i). Let $A = BC$ with $CB = O_n$. We have

$$A^2 = BCBC = BO_nC = O_n$$

i). $\Rightarrow$ ii). Let $A = PSQ$ the rank normal form of A. Note that $S^2 = S$. We put $B = PS$ and $C = SQ$. We have

$$O_n = A^2 = BCBC = PSSQPSSQ = PSQPSQ$$

Because P and Q are invertible, we obtain that $SQPS = O_n$. It follows that $CB = O_n$ and $A = BC$. $\square$

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