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Coefficient estimates and Fekete-Szegö inequality for a class of analytic functions satisfying subordinate condition associated with Chebyshev polynomials

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Abstract. In this paper, we define a class of analytic functions, $\mathcal{F}(\mathcal{H}, \alpha, \delta, \mu)$, satisfying the following condition

$$\left(\alpha\left[\frac{zf'(z)}{f(z)}\right]^{\delta} + (1-\alpha)\left[\frac{zf'(z)}{f(z)}\right]^{\mu}\left[1+\frac{zf''(z)}{f'(z)}\right]^{1-\mu}\right) \prec \mathcal{H}(z,t),$$

where $\alpha \in [0,1], \delta \in [1,2]$ and $\mu \in [0,1]$.

We give coefficient estimates and Fekete-Szegö inequality for this class.

1 Introduction

Let $\mathcal{A}$ denote the class of analytic functions in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$, of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, z \in \mathbb{U}. \tag{1}$$

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Definition 1 [7, p.4] *Let* f, g *analytic functions in the open unit disk. The function* f *is said to be subordinate to* g*, written* $f \prec g$ *, or* $f(z) \prec g(z)$*, if there exists a function* w*, analytic in* $\mathbb{U}$*, with* $w(0) = 0$ *and* $|w(z)| < 1$ *(i.e.* w *is a Schwarz function), such that* $f(z) = g[w(z)], z \in \mathbb{U}$*.*

Remark 1 [7, p.4] *If* g *is univalent, then* $f \prec g$ *if and only if* $f(0) = g(0)$ *and* $f(\mathbb{U}) \subset g(\mathbb{U})$*.*

Chebyshev polynomials are of four kinds, but the most common are the Chebyshev polynomials of the first kind,

$$T_n(x) = \cos n\theta, x \in [-1, 1],$$

and the second kind,

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, x \in [-1, 1],$$

where n denotes the polynomial degree and $x = \cos\theta$.
Applications of Chebyshev polynomials for analytic functions can be found in [1, 2, 3, 4].
Let

$$\mathcal{H}(z, t) = \frac{1}{1 - 2tz + z^2},$$

where $t = \cos\theta, \theta \in \left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$.
We have

$$\begin{aligned} \mathcal{H}(z, t) &= 1 + \sum_{n=1}^{\infty} \frac{\sin(n+1)\theta}{\sin\theta} z^n \\ &= 1 + 2\cos\theta z + (3\cos^2\theta - \sin^2\theta)z^2 + \cdots \\ &= 1 + U_1(t)z + U_2(t)z^2 + \cdots, z \in \mathbb{U}, t \in \left(\frac{1}{2}, 1\right], \end{aligned} \tag{2}$$

where

$$U_{n-1} = \frac{\sin(n\cos^{-1} t)}{\sqrt{1 - t^2}}, n \in \mathbb{N},$$

are the Chebyshev polynomials of second kind.
Furthermore, we know that

$$U_n(t) = 2tU_{n-1}(t) - U_{n-2}(t)$$

and

$$U_1(t) = 2t, U_2(t) = 4t^2 - 1, \cdots .$$

In this paper, we define a new class of analytic functions, being motivated by the following result.

Corollary 1 [5] *Let* $f \in \mathcal{A}$ *and also let* $\alpha \in [0,1], a \in [0,1], \delta \in [1,2]$ *and* $\mu \in [0,1]$. *If*

$$\Re\left(\alpha\left[\frac{zf'(z)}{f(z)}\right]^{\delta} + (1-\alpha)\left[\frac{zf'(z)}{f(z)}\right]^{\mu}\left[1+\frac{zf''(z)}{f'(z)}\right]^{1-\mu}\right) > a, z \in \mathbb{U},$$

then

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > a, z \in \mathbb{U},$$

so f is starlike of order a *in* $\mathbb{U}$.

Definition 2 *We say that* $f \in \mathcal{A}$ *of the form* (1) *belongs to* $\mathcal{F}(\mathcal{H}, \alpha, \delta, \mu)$ *if*

$$\left(\alpha\left[\frac{zf'(z)}{f(z)}\right]^{\delta} + (1-\alpha)\left[\frac{zf'(z)}{f(z)}\right]^{\mu}\left[1+\frac{zf''(z)}{f'(z)}\right]^{1-\mu}\right) \prec \mathcal{H}(z,t), \tag{3}$$

the power is considered to have principal value, $\alpha \in [0,1], \delta \in [1,2]$ *and* $\mu \in [0,1]$.

Taking $\alpha = \delta = t = 1$ and $w(z) = z$, we obtain the following example.

Example 1 *The function* $f(z) = \frac{z}{1-z}e^{\frac{z}{1-z}}$ *with the series expansion* $f(z) = z + 2z^2 + \frac{7}{2}z^3 + \cdots$ *belongs to* $\mathcal{F}(\mathcal{H}, \alpha, \delta, \mu)$.

For the purpose of our results, we need the following lemma.

Lemma 1 [6] *Let the Schwarz function* w *be given by*

$$w(z) = w_1 z + w_2 z^2 + w_3 z^3 + \cdots, z \in \mathbb{U}. \tag{4}$$

Then

$$|w_1| \leq 1, \ |w_2 - t w_1^2| \leq 1 + (|t|-1)|w_1|^2 \leq \max\{1, |t|\},$$

where $t \in \mathbb{C}$.

2 Main result

Our main result in this paper is stated as the following theorem.

Theorem 1 *Let* $f \in \mathcal{A}$ *of the form* (1) *belong to the class* $\mathcal{F}(\mathcal{H}, \alpha, \delta, \mu)$*. Then*

$$|a_2| \leq \frac{2t}{\alpha\delta + (1-\alpha)(2-\mu)}, \tag{5}$$

and, for $\lambda \in \mathbb{C}$*,*

$$\left|a_3 - \lambda a_2^2\right| \leq \frac{t}{\alpha\delta + (1-\alpha)(3-2\mu)} \max\left\{1, \left|2t\left(\frac{2\lambda\big(\alpha\delta + (1-\alpha)(3-2\mu)\big)}{\big(\alpha\delta + (1-\alpha)(2-\mu)\big)^2} - \frac{3 + \frac{2(1-\alpha)(1-\mu)-\alpha(\delta^2-\mu^2)-\mu^2}{\alpha\delta+(1-\alpha)(2-\mu)}}{2\big(\alpha\delta + (1-\alpha)(2-\mu)\big)}\right) - \frac{4t^2-1}{2t}\right|\right\}. \tag{6}$$

Proof. Let $f \in \mathcal{F}(\mathcal{H}, \alpha, \delta, \mu)$, then from (3) we have

$$\alpha\left[\frac{zf'(z)}{f(z)}\right]^{\delta} + (1-\alpha)\left[\frac{zf'(z)}{f(z)}\right]^{\mu}\left[1 + \frac{zf''(z)}{f'(z)}\right]^{1-\mu} = \mathcal{H}\big(w(z), t\big), z \in \mathbb{U}. \tag{7}$$

Using (2) and (4), we obtain

$$\mathcal{H}\big(w(z), t\big) = 1 + U_1(t)w_1 z + \big(U_2(t)w_1^2 + U_1(t)w_2\big)z^2 + \cdots. \tag{8}$$

Making use of (1), (7) and (8), we get

$$\big(\alpha\delta + (1-\alpha)(2-\mu)\big)a_2 = U_1(t)w_1, \tag{9}$$

and

$$\begin{aligned} &2\big(\alpha\delta + (1-\alpha)(3-2\mu)\big)a_3 + \frac{\alpha\delta(\delta-3) + (1-\alpha)(\mu^2+5\mu-8)}{2}a_2^2 \\ &\quad = U_2(t)w_1^2 + U_1(t)w_2. \end{aligned} \tag{10}$$

From (9), we obtain

$$a_2 = \frac{U_1(t)w_1}{\alpha\delta + (1-\alpha)(2-\mu)}. \tag{11}$$

Using Lemma 1 and (11), we obtain (5).
Putting (11) in (10), we have

$$2\big(\alpha\delta + (1-\alpha)(3-2\mu)\big)a_3 = U_2(t)w_1^2 + U_1(t)w_2$$

$$+\frac{\big(\alpha\delta(3-\delta)-(1-\alpha)(\mu^2+5\mu-8)\big)U_1^2(t)w_1^2}{2\big(\alpha\delta+(1-\alpha)(2-\mu)\big)^2}.$$

Therefore,

$$a_3=\frac{U_1(t)}{2\big(\alpha\delta+(1-\alpha)(3-2\mu)\big)}\left(w_2+\left(\frac{\left(3+\frac{2(1-\alpha)(1-\mu)-\alpha(\delta^2-\mu^2)-\mu^2}{\alpha\delta+(1-\alpha)(2-\mu)}\right)U_1(t)}{2\big(\alpha\delta+(1-\alpha)(2-\mu)\big)}+\frac{U_2(t)}{U_1(t)}\right)w_1^2\right). \tag{12}$$

For $\lambda\in\mathbb{C}$, from (11) and (12) we obtain that

$$a_3-\lambda a_2^2=\frac{U_1(t)}{2\big(\alpha\delta+(1-\alpha)(3-2\mu)\big)}\Bigg(w_2+\frac{U_2(t)}{U_1(t)}w_1^2$$

$$-U_1(t)w_1^2\left(\frac{2\lambda\big(\alpha\delta+(1-\alpha)(3-2\mu)\big)}{\big(\alpha\delta+(1-\alpha)(2-\mu)\big)^2}-\frac{3+\frac{2(1-\alpha)(1-\mu)-\alpha(\delta^2-\mu^2)-\mu^2}{\alpha\delta+(1-\alpha)(2-\mu)}}{2\big(\alpha\delta+(1-\alpha)(2-\mu)\big)}\right)\Bigg). \tag{13}$$

Hence,

$$\left|a_3-\lambda a_2^2\right|=\frac{t}{\alpha\delta+(1-\alpha)(3-2\mu)}\Bigg|w_2-\Bigg(2t\bigg(\frac{2\lambda\big(\alpha\delta+(1-\alpha)(3-2\mu)\big)}{\big(\alpha\delta+(1-\alpha)(2-\mu)\big)^2}$$

$$-\frac{3+\frac{2(1-\alpha)(1-\mu)-\alpha(\delta^2-\mu^2)-\mu^2}{\alpha\delta+(1-\alpha)(2-\mu)}}{2\big(\alpha\delta+(1-\alpha)(2-\mu)\big)}\bigg)-\frac{4t^2-1}{2t}\Bigg)w_1^2\Bigg|. \tag{14}$$

Using Lemma 1 for inequality (14), we obtain inequality (6). □

Taking $\alpha=1-\beta,\delta=1$ and $\mu=0$ in Theorem 1, we obtain the following result:

Corollary 2 [2] *Let* $f\in\mathcal{A}$ *of the form* (1) *satisfying the condition*

$$\left((1-\beta)\frac{zf'(z)}{f(z)}+\beta\left(1+\frac{zf''(z)}{f'(z)}\right)\right)\prec\mathcal{H}(z,t),$$

where $\beta\in[0,1]$. *Then*

$$|a_2|\le\frac{2t}{1+\beta},$$

and, for $\lambda \in \mathbb{C}$,

$$\left|a_3 - \lambda a_2^2\right| \leq \frac{t}{1+2\beta} \max\left\{1, \left|2t\left(\frac{2\lambda(1+2\beta)}{(1+\beta)^2} - \frac{1+3\beta}{(1+\beta)^2}\right) - \frac{4t^2-1}{2t}\right|\right\}.$$

Taking $\alpha = 0$ in Theorem 1, we obtain the following result:

Corollary 3 [1] *Let* $f \in \mathcal{A}$ *of the form* (1) *satisfying the condition*

$$\left(\frac{zf'(z)}{f(z)}\right)^{\mu}\left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\mu} \prec \mathcal{H}(z,t),$$

where $\mu \in [0,1]$. *Then*

$$|a_2| \leq \frac{2t}{2-\mu},$$

and, for $\lambda \in \mathbb{C}$,

$$\left|a_3 - \lambda a_2^2\right| \leq \frac{t}{3-2\mu} \max\left\{1, \left|2t\left(\frac{2\lambda(3-2\mu)}{(2-\mu)^2} + \frac{\mu^2+5\mu-8}{2(2-\mu)^2}\right) - \frac{4t^2-1}{2t}\right|\right\}.$$

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