Acta Univ. Sapientiae, Mathematica, **12**, 1 (2020) 5–13

DOI: 10.2478/ausm-2020-0001

On weak (σ, δ)-rigid rings over Noetherian rings

Vijay Kumar Bhat
School of Mathematics,
SMVD University, India
email: vijaykumarbhat2000@yahoo.com

Pradeep Singh
School of Mathematics,
SMVD University, India
email: pradeep333singh@gmail.com

Sunny Sharma
School of Mathematics,
SMVD University, India
email: sunnysrrm94@gmail.com

Abstract. Let R be a Noetherian integral domain which is also an algebra over $\mathbb{Q}$ ($\mathbb{Q}$ is the field of rational numbers). Let σ be an endomorphism of R and δ a σ-derivation of R. We recall that a ring R is a weak (σ, δ)-rigid ring if $a(\sigma(a) + \delta(a)) \in N(R)$ if and only if $a \in N(R)$ for $a \in R$ ($N(R)$ is the set of nilpotent elements of R). With this we prove that if R is a Noetherian integral domain which is also an algebra over $\mathbb{Q}$, σ an automorphism of R and δ a σ-derivation of R such that R is a weak (σ, δ)-rigid ring, then $N(R)$ is completely semiprime.

1 Introduction and preliminaries

Throughout this paper R will denote an associative ring with identity $1 \neq 0$, unless otherwise stated. The prime radical of a ring R denoted by $P(R)$ is the intersection of all prime ideals of R. The set of nilpotent elements of R is denoted by $N(R)$. The ring of integers, the field of rational numbers, the field

2010 Mathematics Subject Classification: 16N40, 16P40, 16W20
Key words and phrases: automorphism, derivation, nilpotent element, completely semiprime ideal

of real numbers and the field of complex numbers are denoted by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ respectively, unless otherwise stated.

Krempa [4] introduced σ-rigid rings and proved that if σ is a rigid endomorphism of R, then it is a monomorphism preserving every minimal prime ideal and annihilator in R. Several properties of σ-rigid rings have been studied in [2, 3]. Weak σ-rigid rings were studied by Ouyang [7]. Bhat [1] gave a necessary and sufficient condition for a commutative Noetherian ring to be weak (σ, δ)-rigid ring.

In this article we investigate weak (σ, δ)-rigid rings over Noetherian rings.

Now let R be a ring, σ an endomorphism of R and δ a σ-derivation of R. Recall that $\delta : R \to R$ an additive map such that

$$\delta(ab) = \delta(a)\sigma(b) + a\delta(b), \text{ for all } a, b \in R$$

is called a σ-derivation of R.

Example 1 *Let F be a field, $R = F[x]$ be the polynomial ring over F. Then $\sigma : R \to R$ defined as*

$$\sigma(f(x)) = f(-x) \textit{ is an automorphism.}$$

Define $\delta : R \to R$ by

$$\delta(f(x)) = f(x) - \sigma(f(x)).$$

Then δ is a σ-derivation of R.

1.1 σ-rigid ring

Recall that in Krempa [4], an endomorphism σ of a ring R is said to be rigid if $a\sigma(a) = 0$ implies that $a = 0$, for all $a \in R$. A ring R is said to be σ-rigid if there exists a rigid endomorphism σ of R.

Example 2 *Let $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$, where F is a field. Let $\sigma : R \to R$ be defined by*

$$\sigma\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \textit{ for } a, b, c \in F.$$

Then it can be seen that σ is an endomorphism of R.

Let $0 \neq a \in F$. Then $\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \sigma\left(\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

But $\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Hence R *is not a* σ*-rigid ring.*

We recall that σ-rigid rings are reduced rings by Hong et. al. [3]. Recall that a ring R is reduced if it has no non-zero nilpotent elements. Observe that reduced rings are abelian.

1.2 Weak σ-rigid ring

Note that as in Ouyang [7], a ring R with an endomorphism σ is called a weak σ-rigid ring if $a\sigma(a) \in N(R)$ if and only if $a \in N(R)$ for $a \in R$.

Example 3 (Example 2.1 of [7]) *Let* σ *be an endomorphism of a ring* R*. Let*

$$A = \left\{ \begin{bmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{bmatrix} \mid a, b, c, d \in R \right\}$$

be a subring of $T_3(R)$*, the ring of upper triangular matrices over* R*. Now* σ *can be extended to an endomorphism say* $\overline{\sigma}$ *of* A *by* $\overline{\sigma}((a_{ij})) = (\sigma(a_{ij}))$*. Then it can be seen that* A *is a weak* $\overline{\sigma}$*-rigid ring.*

Example 4 *Let* F *be a field and* $R = F(x)$*, the field of rational polynomials in one variable* x *over* F*. Then* $N(R) = \{0\}$*. Let* $\sigma : R \to R$ *be an endomorphism defined by*

$$\sigma(f(x)) = f(0).$$

Then R *is not a weak* σ*-rigid ring. For let* $f(x) = xa$*,* $f(0) = 0$ *and* $f(x)\sigma(f(x)) = xa.0 = 0 \in N(R)$*. But* $0 \neq f(x) \notin N(R)$*.*

Clearly the notion of a weak σ-rigid ring generalizes that of a σ-rigid ring. Also in [5], it has been shown that if R is a weak σ-rigid ring, then $N(R)$ is completely semi-prime where R is a Noetherian ring and σ an automorphism of R.The converse is not true.

Example 5 [6] *Let* F *be a field,* $R = F \times F$ *and* σ *an automorphism of* R *defined by*

$$\sigma((a, b)) = (b, a), \text{ for } a, b \in F.$$

Then R *is a reduced ring and so* $N(R) = \{0\}$ *is completely semi-prime. But* R *is not a weak* σ*-rigid ring. Since* $(1, 0)\sigma((1, 0)) = (0, 0) \in N(R)$*, but* $(1, 0) \notin N(R)$*.*

1.3 Weak (σ, δ)-rigid rings

We generalize the above mentioned notions by involving σ and δ together as follows:

1.3.1 (σ, δ)-ring

Definition 1 (Definition 7 of [1]) *Let* R *be a ring,* σ *an endomorphism of* R *and* δ *a* σ*-derivation of* R*. Then* R *is said to be a* (σ, δ)*-rigid ring if*

$$a(\sigma(a) + \delta(a)) \in P(R) \textit{ implies that } a \in P(R) \textit{ for } a \in R.$$

Definition 2 *Let* R *be a ring,* σ *an endomorphism of* R *and* δ *a* σ*-derivation of* R*. Then* R *is said to be a* (σ, δ)*-rigid ring if*

$$a(\sigma(a) + \delta(a)) = 0 \textit{ implies that } a = 0 \textit{ for } a \in R.$$

Example 6 *Let* $R = \mathbb{C}$ *and* $\sigma : R \to R$ *be defined by*

$$\sigma(a + ib) = a - ib, \textit{ for all } a, b \in \mathbb{R}.$$

Then σ *is an automorphism of* R*.*
Define δ *a* σ*-derivation of* R *as*

$$\delta(z) = z - \sigma(z) \textit{ for } z \in R.$$

i.e., $\delta(a + ib) = a + ib - \sigma(a + ib) = a + ib - (a - ib) = 2ib.$

Let $A = a + ib$*. Then* $A[\sigma(A) + \delta(A)] = 0$ *implies that*

$$(a + ib)[\sigma(a + ib) + \delta(a + ib)] = 0$$

i.e. $(a + ib)[(a - ib) + 2ib] = 0$ *or* $(a + ib)(a + ib) = 0$ *which implies that* $a = 0, b = 0$*. Therefore,* $A = a + ib = 0$*. Hence* R *is a* (σ, δ)*-rigid ring.*

1.3.2 Weak (σ, δ)-rigid rings

Definition 3 *Let* R *be a ring. Let* σ *be an endomorphism of* R *and* δ *a* σ*-derivation of* R*. Then* R *is said to be a weak* (σ, δ)*-rigid ring if* $a(\sigma(a)+\delta(a)) \in N(R)$ *implies and is implied by* $a \in N(R)$ *for* $a \in R$*.*

Example 7 *Let* $R = \mathbb{Z}[\sqrt{2}]$*. Then* $\sigma : R \to R$ *defined as*

$$\sigma(a + b\sqrt{2}) = (a - b\sqrt{2}) \textit{ for } a + b\sqrt{2} \in R$$

is an endomorphism of R.
For any $s \in R$. *Define* $\delta_s : R \to R$ *by*

$$\delta_s(a+b\sqrt{2}) = (a+b\sqrt{2})s - s\sigma(a+b\sqrt{2}) \text{ for } a+b\sqrt{2} \in R.$$

Then δ_s *is a* σ*-derivation of* R. *Here* $N(R) = \{0\}$.
Further,

$$(a+b\sqrt{2})\{\sigma(a+b\sqrt{2}) + \delta_s(a+b\sqrt{2})\} \in N(R)$$

implies that

$$(a+b\sqrt{2})\{(a-b\sqrt{2}) + (a+b\sqrt{2})s - s\sigma(a+b\sqrt{2})\} \in N(R)$$

or

$$(a+b\sqrt{2})\{a-b\sqrt{2}+as+bs\sqrt{2}-sa+sb\sqrt{2}\} \in N(R)$$

i.e.

$$(a+b\sqrt{2})\{a+(2s-1)b\sqrt{2}\} \in N(R) = \{0\}$$

which gives $a=0, b=0$. *Hence* $a+b\sqrt{2} = 0+0\sqrt{2} \in N(R)$. *Thus* R *is a weak* (σ,δ)*-rigid ring.*

With this we prove the following:

Theorem A: Let R be Noetherian, integral domain which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R and δ a σ-derivation of R. Then R a weak (σ,δ)-rigid ring implies that $N(R)$ is completely semi-prime.

The statement is proven in Theorem 1, to be found below.

2 Proof of the main result

We have the following before we prove the main result of this paper:

Recall that an ideal I of a ring R is called completely semi-prime if $a^2 \in I$ implies that $a \in I$ for $a \in R$.

Example 8 *Let* $R = \begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{bmatrix}$. *Then* $P_1 = \begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & 0 \end{bmatrix}$, $P_2 = \begin{bmatrix} 0 & \mathbb{Z} \\ 0 & \mathbb{Z} \end{bmatrix}$, $P_3 = \begin{bmatrix} 0 & \mathbb{Z} \\ 0 & 0 \end{bmatrix}$ *are prime ideals of* R.

It can be easily seen that P_1, P_2, P_3 *are completely semi-prime ideals.*

Proposition 1 *Let* R *be a ring,* σ *an automorphism of* R *and* δ *a* σ*-derivation of* R*. Then for* $u \neq 0$*,* $\sigma(u) + \delta(u) \neq 0$*.*

Proof. Let $0 \neq u \in R$, we show that $\sigma(u) + \delta(u) \neq 0$. Let for $0 \neq u$, $\sigma(u) + \delta(u) = 0$. This implies that

$$\delta(u) = -\sigma(u), \ \forall 0 \neq u \in R. \tag{1}$$

We know that for

$$0 \neq a, 0 \neq b \in R, \ \delta(ab) = \delta(a)\sigma(b) + a\delta(b).$$

By using equation 1, we have

$$\begin{aligned} -\sigma(ab) &= -\sigma(a)\sigma(b) + a(-\sigma(b)) \\ \Rightarrow \quad -\sigma(ab) &= -[\sigma(a) + a]\sigma(b) \\ \Rightarrow \quad \sigma(a)\sigma(b) &= -[\sigma(a) + a]\sigma(b) \\ \Rightarrow \quad \sigma(a) &= \sigma(a) + a \end{aligned}$$

Therefore, $a = 0$ which is not possible. Hence $\sigma(u) + \delta(u) \neq 0$. □

We now prove the main result of this paper in the form of the following Theorem:

Theorem 1 *Let* R *be a Noetherian integral domain which is also an algebra over* $\mathbb{Q}$*. Let* σ *be an automorphism of* R *and* δ *a* σ*-derivation of* R*. Then* R *a weak* (σ, δ)*-rigid ring implies that* $N(R)$ *is completely semi-prime.*

Proof. Let R be a weak (σ, δ)-rigid ring. Then we will show that $N(R)$ is completely semi-prime. Let $a \in R$. Since R is a weak (σ, δ)-rigid ring. Let

$$\{a(\sigma(a) + \delta(a))\}^2 \in N(R).$$

Then there exists a positive integer n such that $[a^2(\sigma(a) + \delta(a))^2]^n = 0$ which implies that $a^{2n}(\sigma(a) + \delta(a))^{2n} = 0$. But by Proposition (1), $\sigma(a) + \delta(a) \neq 0$. Hence $a^{2n} = 0$ which implies that $a^n = 0$, because R is an integral domain. Therefore, $a^n(\sigma(a) + \delta(a))^n = 0$ or $\{a(\sigma(a) + \delta(a))\}^n = 0$. Thus $a(\sigma(a) + \delta(a)) \in N(R)$. Hence $N(R)$ is completely semi-prime. □

The converse is not true.

Example 9 *Let* F *be a field. Let* $R = F \times F$ *and* σ *an automorphism of* R *defined by*

$$\sigma((a, b)) = (b, a) \text{ for } a, b \in F.$$

Then R *is a reduced ring and so* $N(R) = \{0\}$ *and therefore, it is completely semi-prime. Let* $r \in F$. *Define* $\delta_r : R \to R$ *by*

$$\delta_r((a, b)) = (a, b)r - r\sigma((a, b)) \text{ for } a, b \in F.$$

Then δ_r *is a* σ*-derivation of* R. *Also* R *is not a weak* (σ, δ)*-rigid ring. For take* $(1, -1) \in R$, $r = \frac{1}{2}$. *Then*

$$\begin{aligned} &\delta_r((1, -1)) = (1, -1)\tfrac{1}{2} - \tfrac{1}{2}\sigma((1, -1)) \\ &= (1, -1) \text{ and } (1, -1)[\sigma(1, -1) + \delta_r(1, -1)] \\ &= (1, -1)[(-1, 1) + (1, -1)] \\ &= (1, -1)(0, 0) = (0, 0) \in N(R). \end{aligned}$$

But $(1, -1) \notin N(R)$.

Corollary 1 *Let* R *be a commutative Noetherian, integral domain which is also an algebra over* $\mathbb{Q}$. *Let* σ *be an automorphism of* R *and* δ *a* σ*-derivation of* R. *Then* R *a weak* (σ, δ)*-rigid ring implies that* $N(R)$ *is completely semi-prime.*

Also we note that if R is a (σ, δ)-rigid ring then it is a weak (σ, δ)-rigid ring, but the converse need not be true as in the following example:

Example 10 *Let* σ *be an endomorphism of a ring* R *and* δ *a* σ*-derivation of* R. *Let* R *be a* (σ, δ)*-rigid ring. Then*

$$R_3 = \left\{ \begin{bmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{bmatrix} : a, b, c, d \in R \right\}$$

is a subring of $T_3(R)$. *The endomorphism* σ *of* R *can be extended to the endomorphism* $\overline{\sigma} : R_3 \to R_3$ *defined by* $\overline{\sigma}((a_{ij})) = (\sigma(a_{ij}))$ *and* δ *can be extended to* $\overline{\delta} : R_3 \to R_3$ *by* $\overline{\delta}((a_{ij})) = (\delta(a_{ij}))$. *Let*

$$\begin{bmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{bmatrix} \left\{ \overline{\sigma}\left(\begin{bmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{bmatrix} \right) + \overline{\delta}\left(\begin{bmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{bmatrix} \right) \right\} \in N(R).$$

Then there is some positive integer n *such that*

$$\left(\begin{bmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{bmatrix} \left\{ \begin{bmatrix} \sigma(a) & \sigma(b) & \sigma(c) \\ 0 & \sigma(a) & \sigma(d) \\ 0 & 0 & \sigma(a) \end{bmatrix} + \begin{bmatrix} \delta(a) & \delta(b) & \delta(c) \\ 0 & \delta(a) & \delta(d) \\ 0 & 0 & \delta(a) \end{bmatrix} \right\} \right)^n = 0$$

which implies that

$$\left(\begin{bmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{bmatrix} \left\{ \begin{bmatrix} \sigma(a)+\delta(a) & \sigma(b)+\delta(b) & \sigma(c)+\delta(c) \\ 0 & \sigma(a)+\delta(a) & \sigma(d)+\delta(d) \\ 0 & 0 & \sigma(a)+\delta(a) \end{bmatrix} \right\}\right)^n = 0$$

or

$$\begin{bmatrix} a(\sigma(a)+\delta(a)) & a(\sigma(b)+\delta(b))+b(\sigma(a)+\delta(a)) & a(\sigma(c)+\delta(c))+b(\sigma(d)+\delta(d))+c(\sigma(a)+\delta(a)) \\ 0 & a(\sigma(a)+\delta(a)) & a(\sigma(d)+\delta(d))+d(\sigma(a)+\delta(a)) \\ 0 & 0 & a(\sigma(a)+\delta(a)) \end{bmatrix}^n$$

$= 0$, *which gives*

$$a(\sigma(a)+\delta(a)) \in N(R).$$

Since R *is reduced, we have*

$$a(\sigma(a)+\delta(a)) = 0$$

which implies that $a = 0$, *since* R *is a* (σ, δ)*-rigid ring. Hence*

$$\begin{bmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{bmatrix} = \begin{bmatrix} 0 & b & c \\ 0 & 0 & d \\ 0 & 0 & 0 \end{bmatrix} \in N(R).$$

Conversely, assume that

$$\begin{bmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{bmatrix} \in N(R).$$

Then there is some positive integer n *such that*

$$\begin{bmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{bmatrix}^n = \begin{bmatrix} a^n & * & * \\ 0 & a^n & * \\ 0 & 0 & a^n \end{bmatrix} = 0$$

which implies that $a = 0$, *because* R *is reduced (Here* $*$ *are non-zero terms involving summation of powers of some or all of a, b, c, d). So*

$$\begin{bmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{bmatrix} \left\{ \overline{\sigma}\left(\begin{bmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{bmatrix}\right) + \overline{\delta}\left(\begin{bmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{bmatrix}\right) \right\}$$

$$= \begin{bmatrix} 0 & b & c \\ 0 & 0 & d \\ 0 & 0 & 0 \end{bmatrix} \left\{ \begin{bmatrix} 0 & \sigma(b) & \sigma(c) \\ 0 & 0 & \sigma(d) \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \delta(b) & \delta(c) \\ 0 & 0 & \delta(d) \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} 0 & b & c \\ 0 & 0 & d \\ 0 & 0 & 0 \end{bmatrix} \left\{ \begin{bmatrix} 0 & \sigma(b)+\delta(b) & \sigma(c)+\delta(c) \\ 0 & 0 & \sigma(d)+\delta(d) \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} 0 & 0 & b(\sigma(d)+\delta(d)) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in N(R).$$

Therefore, R_3 *is a weak* $(\overline{\sigma}, \overline{\delta})$*-rigid ring. Also since* R_3 *is not reduced,* R_3 *is not a* $(\overline{\sigma}, \overline{\delta})$*-rigid ring.*

References

[1] M. Abrol, V. K. Bhat, Ore Extensions Over (σ, δ)-Rings, *Eur. J. Pure Appl. Math.*, **8** (4) (2015), 462—468.

[2] Y. Hirano, On the uniqueness of rings of coefficients in skew polynomial rings, *Publ. Math. Debrecen*, **54** (3, 4) (1999), 489–495.

[3] C. Y. Hong, N. K. Kim and T. K. Kwak, Ore extensions of Baer and p.p - rings, *J. Pure Appl. Algebra*, **151** (3) (2000), 215–226.

[4] J. Krempa, Some examples of reduced rings, *Algebra Colloq.*, **3** (4) (1996), 289–300.

[5] N. Kumari, S. Gosani, V. K. Bhat, Skew polynomial rings over weak σ-rigid rings and $\sigma(*)$-rings, *Eur. J. Pure Appl. Math.*, **6** (1) (2003), 59–65.

[6] T. K. Kwak, Prime radicals of Skew polynomial rings, *Int. J. Math. Sci.*, **2** (2) (2003), 219–227.

[7] L. Ouyang, Extensions of generalized α-rigid rings, Wiley, *Int. Electron. J. Algebra*, **3** (2008), 103–116.

Received: December 16, 2019