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Direct and converse theorems for King operators

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Abstract. For the sequence of King operators, we establish a direct approximation theorem via the first order Ditzian-Totik modulus of smoothness, and a converse approximation theorem of Berens-Lorentz-type.

1 Introduction

Studying the connection between regular summability matrices and convergent positive linear operators, King [3] introduced an interesting Bernstein-type operator defined as follows:

$$(V_n f)(x) \equiv V_n(f;x) = \sum_{k=0}^{n} p_{n,k}(r_n(x)) f\left(\frac{k}{n}\right), \tag{1}$$

where $x \in [0,1]$, $f \in C[0,1]$, $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ and

$$r_n(x) = \begin{cases} x^2, & \text{if} \quad n = 1 \\ -\frac{1}{2(n-1)} + \sqrt{\frac{n}{n-1}x^2 + \frac{1}{4(n-1)^2}}, & \text{if} \quad n = 2,3,\ldots \end{cases} \tag{2}$$

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For $r_n(x) = x$, $x \in [0,1]$, we recover from (1) the classical Bernstein operator:

$$(B_n f)(x) \equiv B_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right).$$

It is known that

$$(B_n e_0)(x) = 1, \quad (B_n e_1)(x) = x \quad \text{and} \quad (B_n e_2)(x) = x^2 + \frac{x(1-x)}{n}, \tag{3}$$

where $e_j(x) = x^j$, $x \in [0,1]$ and $j \in \{0,1,2,\ldots\}$. In contrast with (3), we have for V_n the relations (see [3, pp. 204-205]):

$$(V_n e_0)(x) = 1, \quad (V_n e_1)(x) = r_n(x) \quad \text{and} \quad (V_n e_2)(x) = x^2. \tag{4}$$

The goal of the paper is to obtain direct and converse approximation theorems for the operators given by (1)-(2). The direct result is established with the aid of the first order Ditzian-Totik modulus of smoothness defined by

$$\omega_\varphi^1(f;\delta) = \sup_{0<h\leq\delta} \sup_{x\pm\frac{1}{2}h\varphi(x)\in[0,1]} \left| f(x+\frac{1}{2}h\varphi(x)) - f(x-\frac{1}{2}h\varphi(x)) \right|, \tag{5}$$

where $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$. It is known [2, Theorem 2.1.1] that (5) is equivalent with the K-functional

$$K_{1,\varphi}(f;\delta) = \inf_{g\in W(\varphi)} \{\|f-g\| + \delta\|\varphi g'\|\}, \quad \delta > 0,$$

where $W(\varphi) = \{g \,|\, g \in A.C._{loc}[0,1], \|\varphi g'\| < \infty\}$, i.e. there exists $C_1 > 0$ such that

$$C_1^{-1}\omega_\varphi^1(f;\delta) \leq K_{1,\varphi}(f;\delta) \leq C_1\omega_\varphi^1(f;\delta). \tag{6}$$

Finally, a converse result of Berens-Lorentz-type is established for the operators V_n (see [1, p. 312, Lemma 5.2] and Lemma 3 below). Throughout this paper $C_1, C_2, \ldots, C_{13}$ denote absolute positive constants.

2 Direct theorem

We have the following result for the functions defined by (2).

Lemma 1 *The functions* r_n, $n = 1,2,\ldots,$ *satisfy the properties*

a) $0 \le r_n'(x) \le 2$ *for* $x \in [0,1]$ *and* $n = 1, 2, \dots;$

b) $r_n(0) = 0$, $r_n(1) = 1$ *and* r_n *is strictly increasing function on* $[0,1]$ *for* $n = 1, 2, \dots;$

c) $0 \le r_n(x) \le x \le 1$ *for* $x \in [0,1]$ *and* $n = 1, 2, \dots;$

d) $0 \le x - r_n(x) \le \frac{2}{n}(1-x)$ *for* $x \in [0,1]$ *and* $n = 1, 2, \dots;$

e) $x \le 2r_n(x)$ *for* $x \in \left[\frac{1}{n}, 1\right]$ *and* $n = 1, 2, \dots$

Proof. a) Obviously $r_1'(x) = 2x$, $x \in [0,1]$. For $n \ge 2$, by simple computations, we obtain

$$r_n'(x) = \begin{cases} \lim\limits_{x \searrow 0} \dfrac{r_n(x) - r_n(0)}{x - 0} = 0, & \text{if} \quad x = 0 \\ \dfrac{\frac{n}{n-1}x}{\sqrt{\frac{n}{n-1}x^2 + \frac{1}{4(n-1)^2}}}, & \text{if} \quad 0 < x \le 1. \end{cases}$$

Hence $0 \le r_n'(x) \le \dfrac{\frac{n}{n-1}x}{\sqrt{\frac{n}{n-1}}x} = \sqrt{\dfrac{n}{n-1}} \le \sqrt{2}$ for $x \in (0,1]$. Thus $0 \le r_n'(x) \le 2$ for $n = 1, 2, \dots$ and $x \in [0,1]$.

b) It follows from (2) and a).

c) It follows from (2) by direct computations.

d) Obviously $0 \le x - r_1(x) = x(1-x) \le 2(1-x)$, $x \in [0,1]$. Using b) and c), we have $0 \le x - r_n(x) \le \frac{2}{n}(1-x)$ for $x = 0$ and $n \ge 2$, and

$$\begin{aligned} 0 \le x - r_n(x) &= x + \frac{1}{2(n-1)} - \sqrt{\frac{n}{n-1}x^2 + \frac{1}{4(n-1)^2}} \\ &= \frac{\frac{x(1-x)}{n-1}}{x + \frac{1}{2(n-1)} + \sqrt{\frac{n}{n-1}x^2 + \frac{1}{4(n-1)^2}}} \\ &\le \frac{\frac{x(1-x)}{n-1}}{x} = \frac{1-x}{n-1} \le \frac{2}{n}(1-x) \end{aligned}$$

for $x \in (0,1]$ and $n \ge 2$.

e) For $n = 1$ the statement is obvious. For $n \ge 2$, we consider the function $h(x) = \frac{x}{r_n(x)}$, $x \in \left[\frac{1}{n}, 1\right]$. Then, by (2),

$$h'(x) = \frac{r_n(x) - x r_n'(x)}{r_n^2(x)} = r_n^{-2}(x)\left(r_n(x) + \frac{1}{2(n-1)}\right)^{-1/2}$$

$$\times \frac{1}{2(n-1)}\left[\frac{1}{2(n-1)}-\sqrt{\frac{n}{n-1}x^2+\frac{1}{4(n-1)^2}}\right]<0$$

for $x \in \left[\frac{1}{n}, 1\right]$. Hence

$$\begin{aligned} h(x) &\leq h\left(\frac{1}{n}\right)=\frac{\frac{1}{n}}{-\frac{1}{2(n-1)}+\sqrt{\frac{n}{n-1}\frac{1}{n^2}+\frac{1}{4(n-1)^2}}} \\ &= \frac{2(n-1)}{\sqrt{n}}\frac{1}{\sqrt{5n-4}-\sqrt{n}}=\frac{1}{2}\frac{\sqrt{5n-4}+\sqrt{n}}{\sqrt{n}}\leq 2 \end{aligned}$$

for $x \in \left[\frac{1}{n}, 1\right]$, which was to be proved. □

The operators V_n given by (1)-(2) are linear and positive. By Lemma 1, b), we have

$$(V_n f)(0)=f(0) \quad \text{and} \quad (V_n f)(1)=f(1) \tag{7}$$

for all $f \in C[0,1]$.

In the next theorem we establish the direct result.

Theorem 1 *There exists* $C_2 > 0$ *such that*

$$\|V_n f - f\| \leq C_2\, \omega_\varphi^1\left(f; \frac{1}{\sqrt{n}}\right) \tag{8}$$

for all $f \in C[0,1]$ *and* $n = 1, 2, \ldots$

Proof. Let $x \in (0,1)$ and $t \in [0,1]$. Taking into account [2, Lemma 9.6.1], we have

$$\left|\int_x^t \frac{du}{\varphi(u)}\right| \leq \varphi^{-1}(x)|t-x|^{1/2}\left|\int_x^t \frac{du}{|t-u|^{1/2}}\right| = 2\varphi^{-1}(x)|t-x|. \tag{9}$$

Further, for $g \in W(\varphi)$, we have $g(t) = g(x) + \int_x^t g'(u)\,du$, $t \in [0,1]$ and $x \in (0,1)$. Hence, by (9), Hölder's inequality, (4) and Lemma 1, d), we get

$$\begin{aligned} |V_n(g;x)-g(x)| &= \left| V_n\left(\int_x^t g'(u)\,du; x\right)\right| \\ &\leq V_n\left(\left|\int_x^t |g'(u)|\,du\right|; x\right) \leq \|\varphi g'\| V_n\left(\left|\int_x^t \frac{du}{\varphi(u)}\,du\right|; x\right) \end{aligned}$$

$$
\begin{aligned}
&\leq\ 2\varphi^{-1}(x)\|\varphi g'\|V_n(|t-x|;x) \leq 2\varphi^{-1}(x)\|\varphi g'\|\left(V_n((t-x)^2;x)\right)^{1/2}\\
&=\ 2\varphi^{-1}(x)\|\varphi g'\|\left(V_n(e_2;x)-2xV_n(e_1;x)+x^2V_n(e_0;x)\right)^{1/2}\\
&=\ 2\varphi^{-1}(x)\|\varphi g'\|(x^2-2xr_n(x)+x^2)^{1/2} = 2\varphi^{-1}(x)\|\varphi g'\|(2x(x-r_n(x)))^{1/2}\\
&\leq\ 2\varphi^{-1}(x)\|\varphi g'\|\left(2x\frac{2}{n}(1-x)\right)^{1/2} = \frac{4}{\sqrt{n}}\|\varphi g'\|. \qquad (10)
\end{aligned}
$$

Due to (7), the estimation (10) is also valid for $x \in \{0, 1\}$.

On the other hand, by (4), we obtain $|(V_n f)(x)| \leq \sum_{k=0}^{n} p_{n,k}(r_n(x)) \left| f\left(\frac{k}{n}\right) \right| \leq \|f\| \sum_{k=0}^{n} p_{n,k}(r_n(x)) \leq \|f\|$, therefore

$$\|V_n f\| \leq \|f\| \qquad (11)$$

for all $f \in C[0, 1]$.

Now, in view of (4), (10) and (11), we find that

$$
\begin{aligned}
|V_n(f;x)-f(x)| &\leq |V_n(f-g;x)| + |V_n(g;x)-g(x)| + |g(x)-f(x)|\\
&\leq 2\|f-g\| + \frac{4}{\sqrt{n}}\|\varphi g'\| \leq 4\left\{\|f-g\| + \frac{1}{\sqrt{n}}\|\varphi g'\|\right\}.
\end{aligned}
$$

Taking the infimum on the right hand side over all $g \in W(\varphi)$, we obtain

$$\|V_n f - f\| \leq 4\, K_{1,\varphi}\left(f; \frac{1}{\sqrt{n}}\right).$$

Hence, by (6), we arrive at (8), which completes the proof. □

3 Converse theorem

We begin with the following remark.

Remark 1 *Due to* (8), *the condition* $\omega_\varphi^1(f;\delta) \leq C_3\delta^\alpha$, $\delta > 0$, $0 < \alpha < 1$ *implies that* $\|V_n f - f\| \leq C_4 n^{-\alpha/2}$, $n \geq 1$.

In what follows, we establish the converse result of the statement given in Remark 1. To achieve this we need some lemmas.

Lemma 2 *We have*

a) $\|\varphi(V_n f)'\| \le 8\sqrt{n}\|f\|$ *for* $f \in C[0,1]$ *and* $n = 1, 2, \ldots$;

b) $\|\varphi(V_n g)'\| \le 32\|\varphi g'\|$ *for* $g \in W(\varphi)$ *and* $n = 1, 2, \ldots$

Proof. a) Let $x \in (0,1)$. By [1, p. 305, (2.1)], we have for the derivatives of $p_{n,k}$ that

$$p'_{n,k}(x) = n\,[p_{n-1,k-1}(x) - p_{n-1,k}(x)] = \varphi^{-2}(x)(k - nx)p_{n,k}(x), \tag{12}$$

where $k = 1, 2, \ldots, n$ and $p_{n-1,-1}(x) = p_{n-1,n}(x) = 0$. We distinguish two cases: $\underline{x \in (0, \frac{1}{n}]}$. By (1), (12), Lemma 1, a) and (4), we get

$$\begin{aligned}
|\varphi(x)(V_n f)'(x)| &= \varphi(x)r'_n(x)\left|\sum_{k=0}^{n} p'_{n,k}(r_n(x))\, f\left(\frac{k}{n}\right)\right| \\
&= n\varphi(x)r'_n(x)\left|\sum_{k=0}^{n} [p_{n-1,k-1}(r_n(x)) - p_{n-1,k}(r_n(x))]\; f\left(\frac{k}{n}\right)\right| \\
&= n\varphi(x)r'_n(x)\left|\sum_{k=0}^{n-1} p_{n-1,k}(r_n(x))\;\left[f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right)\right]\right| \\
&\le n\varphi(x)r'_n(x)\sum_{k=0}^{n-1} p_{n-1,k}(r_n(x))\left|f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right)\right| \\
&\le 2n\varphi(x)r'_n(x)\|f\|\sum_{k=0}^{n-1} p_{n-1,k}(r_n(x)) \le 4n\sqrt{x(1-x)}\|f\| \\
&\le 4\sqrt{n}\|f\|.
\end{aligned} \tag{13}$$

$\underline{x \in [\frac{1}{n}, 1)}$. Using (1), Lemma 1, a), (12), Hölder's inequality, (4) and Lemma 1, c), d), e), we get

$$\begin{aligned}
&|\varphi(x)(V_n f)'(x)| = \varphi(x)r'_n(x)\left|\sum_{k=0}^{n} p'_{n,k}(r_n(x))\, f\left(\frac{k}{n}\right)\right| \\
&\le\; 2\varphi(x)\|f\|\sum_{k=0}^{n} \varphi^{-2}(r_n(x))|k - nr_n(x)|p_{n,k}(r_n(x)) \\
&\le\; 2\varphi(x)\varphi^{-2}(r_n(x))\|f\|\left(\sum_{k=0}^{n} (k - nr_n(x))^2 p_{n,k}(r_n(x))\right)^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&= 2n\varphi(x)\varphi^{-2}(r_n(x))\|f\|\left(V_n(e_2;x)-2r_n(x)V_n(e_1;x)+r_n^2(x)V_n(e_0;x)\right)^{1/2}\\
&= 2n\varphi(x)\varphi^{-2}(r_n(x))\|f\|\left(x^2-2r_n^2(x)+r_n^2(x)\right)^{1/2}\\
&= 2n\varphi(x)\varphi^{-2}(r_n(x))\|f\|(x+r_n(x))^{1/2}(x-r_n(x))^{1/2}\\
&\le 2n\varphi(x)\varphi^{-2}(r_n(x))\|f\|\sqrt{2x}\sqrt{\frac{2}{n}(1-x)}=4\sqrt{n}\frac{\varphi^2(x)}{\varphi^2(r_n(x))}\|f\|\\
&= 4\sqrt{n}\frac{x}{r_n(x)}\frac{1-x}{1-r_n(x)}\|f\|\le 4\sqrt{n}\cdot 2\cdot 1\cdot\|f\|=8\sqrt{n}\|f\|. \qquad (14)
\end{aligned}
$$

Finally, by Lemma 1, a), we get $\varphi(0)(V_nf)'(0)=\varphi(0)\sum_{k=0}^{n}p_{n,k}(r_n(0))r_n'(0)f\left(\frac{k}{n}\right)$ $=0$ and $\varphi(1)(V_nf)'(1)=\varphi(1)\sum_{k=0}^{n}p_{n,k}(r_n(1))r_n'(1)f\left(\frac{k}{n}\right)=0$. Hence, due to (13) and (14), we obtain $\|\varphi(V_nf)'\|\le 8\sqrt{n}\|f\|$, which was to be proved.

b) The proof is similar to the above. Let $x\in\left(0,\frac{1}{n}\right]$. Taking into account (1), (12), Lemma 1, a), (9), Hölder's inequality and (4), we get for $g\in W(\varphi)$ that

$$
|(V_ng)'(x)|=nr_n'(x)\left|\sum_{k=0}^{n}\left[p_{n-1,k-1}(r_n(x))-p_{n-1,k}(r_n(x))\right]g\left(\frac{k}{n}\right)\right|
$$

$$
\begin{aligned}
&= nr_n'(x)\left|\sum_{k=0}^{n-1}p_{n-1,k}(r_n(x))\left[g\left(\frac{k+1}{n}\right)-g\left(\frac{k}{n}\right)\right]\right|\\
&\le 2n\sum_{k=0}^{n-1}p_{n-1,k}(r_n(x))\left|g\left(\frac{k+1}{n}\right)-g\left(\frac{k}{n}\right)\right|\\
&\le 2n\sum_{k=0}^{n-1}p_{n-1,k}(r_n(x))\left\{\left|g\left(\frac{k+1}{n}\right)-g(x)\right|+\left|g\left(\frac{k}{n}\right)-g(x)\right|\right\}\\
&\le 2n\sum_{k=0}^{n-1}p_{n-1,k}(r_n(x))\left\{\left|\int_x^{\frac{k+1}{n}}|g'(u)|\,du\right|+\left|\int_x^{\frac{k}{n}}|g'(u)|\,du\right|\right\}\\
&\le 2n\|\varphi g'\|\sum_{k=0}^{n-1}p_{n-1,k}(r_n(x))\left\{\left|\int_x^{\frac{k+1}{n}}\frac{du}{\varphi(u)}\right|+\left|\int_x^{\frac{k}{n}}\frac{du}{\varphi(u)}\right|\right\}\\
&\le 4n\varphi^{-1}(x)\|\varphi g'\|\sum_{k=0}^{n-1}p_{n-1,k}(r_n(x))\left\{\left|\frac{k+1}{n}-x\right|+\left|\frac{k}{n}-x\right|\right\}
\end{aligned}
$$

$$\leq \quad 4n\varphi^{-1}(x)\|\varphi g'\| \left\{ \left(\sum_{k=0}^{n-1} p_{n-1,k}(r_n(x)) \left(\frac{k+1}{n} - x \right)^2 \right)^{1/2} \right.$$

$$\left. + \left(\sum_{k=0}^{n-1} p_{n-1,k}(r_n(x)) \left(\frac{k}{n} - x \right)^2 \right)^{1/2} \right\}. \tag{15}$$

Further, by (3), Lemma 1, c) and $x \in \left(0, \frac{1}{n}\right]$, we obtain

$$\sum_{k=0}^{n-1} p_{n-1,k}(r_n(x)) \left(\frac{k}{n} - x \right)^2 = \left(\frac{n-1}{n} \right)^2 \sum_{k=0}^{n-1} p_{n-1,k}(r_n(x)) \left(\frac{k}{n-1} \right)^2$$

$$-2x\frac{n-1}{n} \sum_{k=0}^{n-1} p_{n-1,k}(r_n(x)) \frac{k}{n-1} + x^2 \sum_{k=0}^{n-1} p_{n-1,k}(r_n(x))$$

$$= \left(\frac{n-1}{n} \right)^2 \left[r_n^2(x) + \frac{1}{n-1} r_n(x)(1 - r_n(x)) \right] - 2x\frac{n-1}{n} r_n(x) + x^2$$

$$= \frac{(n-1)(n-2)}{n^2} r_n^2(x) + \frac{n-1}{n} \left(\frac{1}{n} - 2x \right) r_n(x) + x^2$$

$$\leq \frac{(n-1)(n-2)}{n^2} r_n^2(x) + \frac{n-1}{n} \left(\frac{1}{n} + 2x \right) r_n(x) + x^2$$

$$\leq \frac{(n-1)(n-2)}{n^2} \frac{1}{n^2} + \frac{n-1}{n} \left(\frac{1}{n} + \frac{2}{n} \right) \frac{1}{n} + \frac{1}{n^2}$$

$$\leq \frac{1}{n^2} + \frac{3}{n^2} + \frac{1}{n^2} = \frac{5}{n^2}. \tag{16}$$

Using the inequality $(a+b)^2 \leq 2(a^2 + b^2)$ with a and b real numbers, (16) and (3), we obtain

$$\sum_{k=0}^{n-1} p_{n-1,k}(r_n(x)) \left(\frac{k+1}{n} - x \right)^2$$

$$\leq \quad 2\sum_{k=0}^{n-1} p_{n-1,k}(r_n(x)) \left(\frac{k}{n} - x \right)^2 + 2\sum_{k=0}^{n-1} p_{n-1,k}(r_n(x)) \frac{1}{n^2}$$

$$\leq \quad \frac{10}{n^2} + \frac{2}{n^2} = \frac{12}{n^2}. \tag{17}$$

Combining (15), (16) and (17), we get

$$|\varphi(x)(V_n g)'(x)| \leq 4n\|\varphi g'\| \left(\frac{\sqrt{12}}{n} + \frac{\sqrt{5}}{n} \right) \leq 23\|\varphi g'\|. \tag{18}$$

Let $x \in [\frac{1}{n}, 1)$. For $g \in W(\varphi)$, by (1), (4), (12), Lemma 1, a), (9), Hölder's inequality, (4) and Lemma 1, c), d), e), we find that

$$\begin{aligned}
&|\varphi(x)(V_n g)'(x)| = \varphi(x)|(V_n g)'(x) - g(x)(V_n e_0)'(x)| \\
= &\quad \varphi(x)\left|\sum_{k=0}^{n} p'_{n,k}(r_n(x))r'_n(x)\left[g(\frac{k}{n}) - g(x)\right]\right| \\
= &\quad \varphi(x)r'_n(x)\left|\sum_{k=0}^{n} p'_{n,k}(r_n(x))\int_x^{\frac{k}{n}} g'(u)\,du\right| \\
\leq &\quad 2\varphi(x)\left|\sum_{k=0}^{n} \varphi^{-2}(r_n(x))(k - nr_n(x))p_{n,k}(r_n(x))\int_x^{\frac{k}{n}} g'(u)\,du\right| \\
\leq &\quad 2\varphi(x)\varphi^{-2}(r_n(x))\sum_{k=0}^{n} |k - nr_n(x)|p_{n,k}(r_n(x))\left|\int_x^{\frac{k}{n}} |g'(u)|\,du\right| \\
\leq &\quad 4\varphi^{-2}(r_n(x))\|\varphi g'\|\sum_{k=0}^{n} |k - nr_n(x)|\left|\frac{k}{n} - x\right|p_{n,k}(r_n(x)) \\
\leq &\quad 4n\varphi^{-2}(r_n(x))\|\varphi g'\|\left(\sum_{k=0}^{n}\left(\frac{k}{n} - r_n(x)\right)^2 p_{n,k}(r_n(x))\right)^{1/2} \\
&\quad \times\left(\sum_{k=0}^{n}\left(\frac{k}{n} - x\right)^2 p_{n,k}(r_n(x))\right)^{1/2} \\
= &\quad 4n\varphi^{-2}(r_n(x))\|\varphi g'\|\left(V_n(e_2;x) - 2r_n(x)V_n(e_1;x) + r_n^2(x)V_n(e_0;x)\right)^{1/2} \\
&\quad \times\left(V_n(e_2;x) - 2xV_n(e_1;x) + x^2V_n(e_0;x)\right)^{1/2} \\
= &\quad 4n\varphi^{-2}(r_n(x))\|\varphi g'\|\left(x^2 - r_n^2(x)\right)^{1/2}\left(2x^2 - 2xr_n(x)\right)^{1/2} \\
= &\quad 4\sqrt{2}n\varphi^{-2}(r_n(x))\|\varphi g'\|\,(x + r_n(x))^{1/2}\sqrt{x}\,(x - r_n(x)) \\
\leq &\quad 4\sqrt{2}n\varphi^{-2}(r_n(x))\|\varphi g'\|\sqrt{2x}\sqrt{x}\frac{2}{n}(1 - x) = 16\frac{\varphi^2(x)}{\varphi^2(r_n(x))}\|\varphi g'\| \\
= &\quad 16\frac{x}{r_n(x)}\frac{1 - x}{1 - r_n(x)}\|\varphi g'\| \leq 32\|\varphi g'\|. \qquad (19)
\end{aligned}$$

Finally, we have $\varphi(0)(V_n g)'(0) = 0 = \varphi(1)(V_n g)'(1)$. Hence, by (18) and (19), we obtain $\|\varphi(V_n g)'\| \leq 32\|\varphi g'\|$, which completes the proof. □

The next result is a weak-type version of the Berens-Lorentz lemma (see [1,

p. 312, Lemma 5.2]).

Lemma 3 *Let* $\phi : [0, a] \to [0, \infty)$ *be an increasing function with* $\phi(0) = 0$ *and* $0 < \alpha < 1$. *If* $0 < a \leq 1$, *then the inequalities*

$$\phi(a) \leq C_5 a^\alpha \tag{20}$$

and

$$\phi(x) \leq C_5 \left(y^\alpha + \frac{x}{y}\phi(y) \right), \quad 0 \leq x \leq y \leq a \tag{21}$$

imply for some $C_6 = C_6(\alpha) > 0$ *that*

$$\phi(x) \leq C_6 C_5 x^\alpha, \quad 0 \leq x \leq a. \tag{22}$$

Proof. Following the proof of Lemma 5.2 in [1, p. 312], it is easy to prove our result taking into account the slight modification on α. For completeness we give the proof.

For $0 < q < 1$, we define $x_k = q^k a$, $k = 0, 1, 2, \ldots$ If we take $C \geq 1$, then (20) implies (22) for $x = x_0$. We prove (22) for all $x = x_k$ by induction. Let $\phi(x_k) \leq CC_5 x_k^\alpha$, then, by (21),

$$\phi(x_{k+1}) \leq C_5(x_k^\alpha + q\phi(x_k)) \leq C_5(1 + qCC_5)x_k^\alpha \leq C_5 C x_{k+1}^\alpha,$$

provided $1 + qCC_5 \leq Cq^\alpha$. To achieve this, we first take q so small that $q^\alpha > C_5 q$, because $0 < \alpha < 1$, and then C sufficiently large. After this, for any $0 < x < a$, we select a k with $x_{k+1} \leq x \leq x_k$ and get with $C_6 := Cq^{-\alpha}$ the estimations $\phi(x) \leq \phi(x_k) \leq CC_5 x_k^\alpha \leq C_6 C_5 x^\alpha$. $\square$

In the next theorem we establish the converse result. We set $C_{01}[0, 1] = \{f \in C[0, 1] : f(0) = f(1)\}$.

Theorem 2 *For* $f \in C_{01}[0, 1]$, $0 < \alpha < 1$ *and* V_n *defined by (1)-(2), the estimation*

$$\|V_n f - f\| \leq C_7 n^{-\alpha/2}, \quad n = 1, 2, \ldots \tag{23}$$

implies $\omega_\varphi^1(f; \delta) \leq C_8 \delta^\alpha$, $0 < \delta \leq 1$.

Proof. The proof is based on Lemma 3 with $\phi(t) = \omega_\varphi^1(f; t)$, $t \in [0, 1]$. For $f \in C_{01}[0, 1]$, by Lemma 1, b), we have

$$\begin{aligned}(V_1 f)(x) &= p_{1,0}(r_1(x))f(0) + p_{1,1}(r_1(x))f(1) = (1 - x^2)f(0) + x^2 f(1) \\ &= f(0) + x^2(f(1) - f(0)) = f(0).\end{aligned}$$

Therefore, by (5), $\omega_\varphi^1(f-V_1f;t)=\omega_\varphi^1(f;t)$, $t>0$. Hence, due to (5) and (23),

$$\omega_\varphi^1(f;1)=\omega_\varphi^1(f-V_1f;1)\le 2\|f-V_1f\|\le 2C_7. \tag{24}$$

Let $x\in[0,1]$ and $h>0$ such that $x\pm\frac{h}{2}\in[0,1]$, and let $\Delta_h^1f(x)=f(x+\frac{h}{2})-f(x-\frac{h}{2})$. Then, by (23),

$$\begin{aligned}|\Delta_h^1f(x)|&\le|\Delta_h^1(f-V_nf)(x)|+|\Delta_h^1(V_nf)(x)|\\ &\le\ 2\|f-V_nf\|+|\Delta_h^1(V_nf)(x)|\le 2C_7n^{-\alpha/2}+|\Delta_h^1(V_nf)(x)|.\end{aligned} \tag{25}$$

Using (6), we can choose $g=g_\delta\in A.C._{loc}[0,1]$ such that $\|f-g\|\le C_9\,\omega_\varphi^1(f;\delta)$ and $\|\varphi g'\|\le C_{10}\delta^{-1}\omega_\varphi^1(f;\delta)$. Hence, in view of Lemma 2,

$$\begin{aligned}|(V_nf)'(x)|&\le|(V_n(f-g))'(x)|+|(V_ng)'(x)|\\ &\le\ 8\sqrt{n}\varphi^{-1}(x)\|f-g\|+32\varphi^{-1}(x)\|\varphi g'\|\\ &\le\ 8\sqrt{n}C_9\varphi^{-1}(x)\omega_\varphi^1(f;\delta)+32C_{10}\delta^{-1}\varphi^{-1}(x)\omega_\varphi^1(f;\delta)\\ &\le\ C_{11}\varphi^{-1}(x)\left(\sqrt{n}+\frac{1}{\delta}\right)\omega_\varphi^1(f;\delta),\end{aligned}$$

where $C_{11}=8C_9+32C_{10}$. This implies that

$$|\Delta_h^1(V_nf)(x)|=\left|\int_{x-\frac{h}{2}}^{x+\frac{h}{2}}(V_nf)'(u)\,du\right|\le C_{11}\left(\sqrt{n}+\frac{1}{\delta}\right)\omega_\varphi^1(f;\delta)\left|\int_{x-\frac{h}{2}}^{x+\frac{h}{2}}\frac{du}{\varphi(u)}\right|.$$

Because of $x\pm\frac{h}{2}\in[0,1]$, we have $x\in(0,1)$. Using (9), we obtain

$$\begin{aligned}\left|\int_{x-\frac{h}{2}}^{x+\frac{h}{2}}\frac{du}{\varphi(u)}\right|&\le\left|\int_{x}^{x+\frac{h}{2}}\frac{du}{\varphi(u)}\right|+\left|\int_{x}^{x-\frac{h}{2}}\frac{du}{\varphi(u)}\right|\\ &\le\ 2\varphi^{-1}(x)\frac{h}{2}+2\varphi^{-1}(x)\frac{h}{2}=2\varphi^{-1}(x)h.\end{aligned}$$

Hence, by (25), we get

$$\begin{aligned}|\Delta_h^1f(x)|&\le\ 2C_7n^{-\alpha/2}+C_{11}\left(\sqrt{n}+\frac{1}{\delta}\right)\omega_\varphi^1(f;\delta)2\varphi^{-1}(x)h\\ &\le\ C_{12}\left\{n^{-\alpha/2}+\left(h\sqrt{n}+\frac{h}{\delta}\right)\varphi^{-1}(x)\omega_\varphi^1(f;\delta)\right\}.\end{aligned}$$

Replacing h by $h\varphi(x)$ gives

$$|\Delta_{h\varphi(x)}^1f(x)|\le C_{12}\left\{n^{-\alpha/2}+\left(h\sqrt{n}+\frac{h}{\delta}\right)\omega_\varphi^1(f;\delta)\right\}.$$

Now we choose $n \geq 1$ such that $\frac{1}{\sqrt{n}} \leq \delta \leq \frac{2}{\sqrt{n}}$, where $0 < \delta \leq 1$. Then we find that

$$|\Delta^1_{h\varphi(x)} f(x)| \leq C_{13} \left\{ \delta^\alpha + \frac{h}{\delta} \omega^1_\varphi(f; \delta) \right\}$$

for all x with $x \pm \frac{h}{2}\varphi(x) \in [0, 1]$. Taking supremum over all h with $0 < h \leq t$, we obtain

$$\omega^1_\varphi(f; t) \leq C_{13} \left\{ \delta^\alpha + \frac{t}{\delta} \omega^1_\varphi(f; \delta) \right\}, \quad 0 < t \leq \delta. \tag{26}$$

Now (24) and (26) yield the assertion of our theorem by Lemma 3. □

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