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Collatz conjecture revisited: an elementary generalization

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Abstract. Collatz conjecture states that iterating the map that takes even natural number n to $\frac{n}{2}$ and odd natural number n to $3n+1$, will eventually obtain 1. In this paper a new generalization of the Collatz conjecture is analyzed and some interesting results are obtained. Since Collatz conjecture can be seen as a particular case of the generalization introduced in this articule, several more general conjectures are also presented.

1 Introduction

The Collatz conjecture remains today unsolved; as it has been for almost over 80 years. Although its statement is very simple and easy to understand, the nature of the problem makes extremelly to demonstrate or refuse. Articles such as [3] and [4] contain a huge amount of publications dealing with this problem and somehow trying to solve it.

Although the Collatz conjecture can be stated in several ways, in this paper we will use the following notation, i.e. $\mathsf{modified\ Collatz\ function}$, that represents a slightly modification of the traditional formulation

$$C(n) = \begin{cases} \frac{3\cdot n+1}{2} & \text{if } n \equiv 1 (\text{mod } 2) \\ \\ \frac{n}{2} & \text{if } n \equiv 0 ((\text{mod } 2) \end{cases}$$

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By using the modified Collatz function, the Collatz conjecture can be stated in the following way:

Conjecture 1 *(Collatz) For every integer number* $n \in \mathbb{N}$*, there exists k such that* $C^{(k)}(n) = 1$*.*

In this sense, Terras [5] defined the total stopping time of an integer $n \in \mathbb{N}$, here we denote it by $\sigma_2(n)$, as the smallest integer k such that $C^{(k)}(n) = 1$ or $\sigma_2(n) = \infty$ if no such k exists. For example, if $n = 7$, then by successively applying C, we obtain the following sequence $7 \to 11 \to 17 \to 26 \to 13 \to 20 \to 10 \to 5 \to 8 \to 4 \to 2 \to 1$, and then $C^{11}(7) = 1$, so $\sigma_2(7) = 11$.

In this paper, we consider an elementary generalization of the Collatz problem. We also generalize the concept of total stopping time and, related with this new approach, we obtain several results for some classes of integer numbers that generalize some well-known results on this topic. In the last section of the paper, some calculations and new conjectures are introduced. These conjectures try to illustrate the idea that traditional Collatz problem is just a special case and that it can be seen as part of a more general point of view, what we have called of Collatz numbers.

1.1 A more general $b-$Collatz function, C_b

For any natural number $b \geq 2$, we define the following $b-$Collatz function $C_b : \mathbb{N} \to \mathbb{N}$

$$C_b(n) = \begin{cases} \frac{(b+1)\cdot n+(b-x)}{b} & \text{if } n \equiv x (\text{mod } b),\ 1 \leq x \leq b-1 \\ \\ \frac{n}{b} & \text{if } n \equiv 0((\text{mod } b) \end{cases}$$

Clearly in the previous formula, if $b = 2$, we obtain the modified Collatz function stated in section [1]. In this sense, we defined the b-total stopping time of an integer $n \in \mathbb{N}$, denoted $\sigma_b(n)$, as the smallest integer k such that $C_b^{(k)}(n) = 1$ or $\sigma_b(n) = \infty$ if no such k exists. For example, if $b = 5$ and $n = 7$, then by successively applying C_5, we obtain the following sequence $7 \to 9 \to 11 \to 14 \to 17 \to 21 \to 26 \to 32 \to 39 \to 47 \to 57 \to 69 \to 83 \to 100 \to 20 \to 4 \to 5 \to 1$, so $C_5^{17}(7) = 1$ and finally $\sigma_5(7) = 17$

Theorem 1 *Let* $b, k, r, s \in \mathbb{N}$*, such that* $b \geq 2$*,* $k, r \geq 1$*,* $n = b^k \cdot r - s > 1$ *and* $1 \leq s \leq b-1$*. Then*

$$C_b^{(k)}(n) = (b+1)^k \cdot r - s.$$

Proof. We proceed by induction on t, $0 \le t \le k$. Case $t = 0$.

$$C_b^{(0)}(b^k \cdot r - s) = b^k \cdot r - s = ((b+1)^0 \cdot b^{k-0}) \cdot r - s.$$

So, the initial case holds. Now let us assume that

$$C_b^{(t)}(b^k \cdot r - s) = ((b+1)^t \cdot b^{k-t}) \cdot r - s,$$

for some $0 \le t < k$. Since $t < k$, then $((b+1)^t \cdot b^{k-t}) \cdot r - s \equiv -s \equiv b - s (\mathrm{mod}\ b)$, so

$$\begin{aligned} C_b^{(t+1)}(b^k \cdot r - s) &= C_b(C_b^{(t)}(b^k \cdot r - s)) = C_b(((b+1)^t \cdot b^{k-t}) \cdot r - s) \\ &= \frac{(b+1) \cdot ((b+1)^t \cdot b^{k-t}) \cdot r - s) + s}{b} = \\ &= \frac{(b+1)^{(t+1)} \cdot b^{k-t} \cdot r - b \cdot s}{b} = \\ &= (b+1)^{(t+1)} \cdot b^{k-(t+1)} \cdot r - s. \end{aligned}$$

Thus,

$$C_b^{(t+1)}(b^k \cdot r - s) = (b+1)^{(t+1)} \cdot b^{k-(t+1)} \cdot r - s,$$

and the result follows. □

Corollary 1 *If* $b, k, r, s \in \mathbb{N}$*, such that* $b \ge 2$*,* $k, r \ge 1$*,* $n = b^k \cdot r - s > 1$ *and* $1 \le s \le b - 1$*, then*

$$\sigma_b(b^k \cdot r - s) = \sigma_b((b+1)^k \cdot r - s) + k.$$

Corollary 2 *If* $b \ge 2$*,* $k, r \ge 1$*,* $n = b^k - 1 > 1$*, then*

$$\sigma_b(b^k - 1) = \sigma_b((b+1)^k - 1) + k.$$

Corollary 3 *If* $b = 2$*,* $k, r \ge 1$*,* $n = 2^k \cdot r - 1$*, then*

$$\sigma_2(2^k \cdot r - 1) = \sigma_2(3^k \cdot r - 1) + k.$$

Theorem 2 *If* $b, k, r, t, s \in \mathbb{N}$*, such that* $b \ge 2$*,* $k > t \ge 1$*,* $r \ge 1$*, and* $1 \le s \le b - 1$*, then*

$$C_b^{(k)}(b^k \cdot r - b^t \cdot s) = (b+1)^{k-t} \cdot r - s.$$

Proof. Since $b^k \cdot r - b^t \cdot s$ is divisible by b^t, then $C_b^{(k)}(b^k \cdot r - b^t \cdot s) = C_b^{(k-t)}$ $(C_b^t(b^k \cdot r - b^t \cdot s)) = C_b^{(k-t)}(b^{k-t} \cdot r - s) = (b+1)^{k-t} \cdot r - s$, by Theorem 1. □

Corollary 4 *If* $b, k, r, t, s \in \mathbb{N}$, *such that* $b \geq 2$, $k > t \geq 1$, $r \geq 1$, *and* $1 \leq s \leq b-1$, *then*

$$\sigma_b(b^k \cdot r - b^t \cdot s) = \sigma_b((b+1)^{k-t} \cdot r - s) + k.$$

Theorem 3 *Let* $b \geq 2$, $k > 2$ *and* $r \geq 1$, *then*
i) *If* $b = 2$ *then* $\sigma_b(b^k \cdot r - (2b-1)) = \sigma_b((b+1)^{k-2} \cdot r - 1) + k$.
ii) *If* $b > 2$ *then* $\sigma_b(b^k \cdot r - (2b-1)) = \sigma_b((b+1)^{k-1} \cdot r - 2) + k$.

Proof. Let $n = b^k \cdot r - (2b-1)$. Since $n \equiv 1 (\mathrm{mod}\ b)$, then

$$\begin{aligned} C_b^{(k)}(n) &= C_b^{(k-1)}(C_b(n)) = C_b^{(k-1)}\left(\frac{(b+1) \cdot (b^k \cdot r - 2b + 1)) + b - 1}{b}\right) \\ &= C_b^{(k-1)}((b+1) \cdot b^{k-1} \cdot r - 2b) = C_b^{(k-2)}(C_b((b+1) \cdot b^{k-1} \cdot r - 2b)) \\ &= C_b^{(k-2)}((b+1) \cdot b^{k-2} \cdot r - 2). \end{aligned}$$

If $b = 2$, then $C_b^{(k)}(n) = C_b^{(k-2)}(3 \cdot 2^{k-2} \cdot r - 2) = C_b^{(k-3)}(2^{k-3} \cdot 3 \cdot r - 1) = 3^{k-2} \cdot r - 1$. If $b > 2$, then by Theorem 1,

$$C_b^{(k)}(n) = C_b^{(k-2)}((b+1) \cdot b^{k-2} \cdot r - 2) = (b+1)^{k-1} \cdot r - 2.$$

□

Corollary 5 [6] *If* $k > 2$ *and* $r \geq 1$, *then* $\sigma_2(2^k \cdot r - 3) = \sigma_2(3^{k-2} \cdot r - 1) + k$.

Theorem 4 *Let* $b \geq 2$, $k > 2$ *and* $r, t \geq 1$, *then*
i) *If* $b = 2$, *then* $\sigma_b(b^k \cdot r - b^t \cdot (2b-1)) = \sigma_b((b+1)^{(k-t-2)} \cdot r - 1) + k$.
i) *If* $b > 2$, *then* $\sigma_b(b^k \cdot r - b^t \cdot (2b-1)) = \sigma_b((b+1)^{(k-t-1)} \cdot r - 2) + k$.

Proof. Let $n = b^k \cdot r - b^t \cdot (2b-1)$. Since $n \equiv 0 (\mathrm{mod}\ b^t)$, then

$$C_b^{(k)}(n) = C_b^{(k-t)}\left(\frac{n}{b^t}\right) = C_b^{k-t}(b^{k-t} \cdot r - (2b-1)).$$

By Theorem 3, if $b = 2$, then

$$C_b^{(k)}(n) = C_b^{k-t}(b^{k-t} \cdot r - (2b-1)) = (b+1)^{(k-t-2)} \cdot r - 1$$

and then

$$\sigma_b(b^k \cdot r - b^t \cdot (2b-1)) = \sigma_b((b+1)^{(k-t-2)} \cdot r - 1) + k.$$

In othen hand, if $b > 2$, then

$$C_b^{(k)}(n) = C_b^{k-t}(b^{k-t} \cdot r - (2b-1)) = (b+1)^{(k-t-1)} \cdot r - 2$$

and then $\sigma_b(b^k \cdot r - b^t \cdot (2b-1)) = \sigma_b((b+1)^{(k-t-1)} \cdot r - 2) + k$. □

Corollary 6 [6] *If* $k > 2$ *and* $r \geq 1$, *then* $\sigma_2(2^k \cdot r - 6) = \sigma_2(3^{k-3} \cdot r - 1) + k$.

Theorem 5 *Let* $b \geq 2$ *and* $k \equiv 0 (\mathrm{mod}\ b)$, *then*
i) $\sigma_b(b^k - 1) = \sigma_b(b^{k-1} - 1) + 1$.
ii) $\sigma_b((b+1)^k - 1) = \sigma_b((b+1)^{k-1} - 1)$.

Proof. i) It enough to proof that

$$C_b^{(k+1)}(b^{k-1} - 1) = C_b^{(k+2)}(b^k - 1).$$

So, by Theorem 1, $C_b^{(k+1)}(b^{k-1}-1) = C_b^{(2)}((b+1)^{k-1}-1)$. Since $(b+1)^{k-1}-1 \equiv 0 (\mathrm{mod}\ b)$, then

$$C_b^{(2)}((b+1)^{k-1} - 1) = C_b(\frac{(b+1)^{k-1} - 1}{b}).$$

Finally, since $k \equiv 0 (\mathrm{mod}\ b)$ and $\frac{(b+1)^{k-1}-1}{b} \equiv b - 1 (\mathrm{mod}\ b)$, then

$$C_b^{(2)}((b+1)^{k-1} - 1) = C_b(\frac{(b+1)^{k-1} - 1}{b}) = \frac{(b+1)^k - 1}{b^2}.$$

In other hand, $C_b^{(k+2)}(b^k - 1) = C_b^{(2)}((b+1)^k - 1)$. So, since $k \equiv 0 (\mathrm{mod}\ b)$, then $(b+1)^k - 1 \equiv 0 (\mathrm{mod}\ b^2)$. Therefore,

$$C_b^{(k+2)}(b^k - 1) = C_b^{(2)}((b+1)^k - 1) = \frac{(b+1)^k - 1}{b^2},$$

and equality holds.
ii) From previous result i) and Corollary 4, we have

$$\begin{aligned} \sigma_b((b+1)^k - 1) &= \sigma_b(b^k - 1) - k = \sigma_b(b^{k-1} - 1) + 1 - k \\ &= \sigma_b((b+1)^{k-1} - 1) + (k-1) + 1 - k \\ &= \sigma_b((b+1)^{k-1} - 1). \end{aligned}$$

□

Corollary 7 [6] *If* k *is even and* $k > 2$*, then*

$$\sigma_2(2^k \cdot r - 1) = \sigma_2(2^{k-1} \cdot r - 1) + 1, \text{ and}$$
$$\sigma_2(3^k \cdot r - 1) = \sigma_2(3^{k-1} \cdot r - 1).$$

3. Some empirical results on C_b and open problems

The behaviour of C_b has been intensively studied during last years when $b = 2$. But, what can we state for C_b for other values of $b \geq 3$?. Let us see some examples for several values of b.

Case $b = 3$. Calculating $C_3(n)$ for some values of n

Iteration/$n =$	4	5	6	7	8	9	10	11	12
1	6	7	2	10	11	3	14	15	4
2	2	10	3	14	15	1	19	5	6
3	3	14	1	19	5	2	26	7	2
4	1	19	2	26	7	3	35	10	3
5	2	26	3	35	10	1	47	14	1
6	3	35	1	47	14	2	63	19	2
7	1	47	2	63	19	3	21	26	3
8	2	63	3	21	26	1	7	35	1
9	3	21	1	7	35	2	10	47	2
10	1	7	2	10	47	3	14	63	3
11	2	10	3	14	63	1	19	21	1
12	3	14	1	19	21	2	26	7	2
13	1	19	2	26	7	3	35	10	3
14	2	26	3	35	10	1	47	14	1
15	3	35	1	47	14	2	63	19	2
16	1	47	2	63	19	3	21	26	3
17	2	63	3	21	26	1	7	35	1
18	3	21	1	7	35	2	10	47	2
19	1	7	2	10	47	3	14	63	3

At first sight, there is no homogeneous behaviour for different values of n. In this case, for $n \in \{4, 6, 9, 12\}$ we found the cycle $2 \to 3 \to 1$ that repeats and contains the number 1.

In other hand, for $n \in \{5, 7, 8, 10, 11\}$, the behaviour is completely different than the previous one. In this case, the cycle $7 \to 10 \to 14 \to 19 \to 26 \to 35 \to 47 \to 63 \to 21$ repeats and does not contain the number 1. In fact, it seems that these are the only two cycles one can find when iterating C_3.

Case $b = 5$. Calculating $C_5(n)$ for some values of n

Iteration/$n =$	6	7	8	9	10	11	12	13	14
1	8	9	10	11	2	14	15	16	17
2	10	11	2	14	3	17	3	20	21
3	2	14	3	17	4	21	4	4	26
4	3	17	4	21	5	26	5	5	32
5	4	21	5	26	1	32	1	1	39
6	5	26	1	32	2	39	2	2	47
7	1	32	2	39	3	47	3	3	57
8	2	39	3	47	4	57	4	4	69
9	3	47	4	57	5	69	5	5	83
10	4	57	5	69	1	83	1	1	100
11	5	69	1	83	2	100	2	2	20
12	1	83	2	100	3	20	3	3	4
13	2	100	3	20	4	4	4	4	5
14	3	20	4	4	5	5	5	5	1
15	4	4	5	5	1	1	1	1	2
16	5	5	1	1	2	2	2	2	3
17	1	1	2	2	3	3	3	3	4
18	2	2	3	3	4	4	4	4	5
19	3	3	4	4	5	5	5	5	1
20	4	4	5	5	1	1	1	1	2
21	5	5	1	1	2	2	2	2	3

At first sight, there is an homogeneous behaviour for different values of n. Independently the value of n we select, we found the cycle $2 \to 3 \to 4 \to 5 \to 1$ that repeats and contains the number 1.

Case $b = 6$. Calculating $C_6(n)$ for some values of n

Iteration/ $n =$	7	8	9	10	11	12	13	14	15
1	9	10	11	12	13	2	16	17	18
2	11	12	13	2	16	3	19	20	3
3	13	2	16	3	19	4	23	24	4
4	16	3	19	4	23	5	27	4	5
5	19	4	23	5	27	6	32	5	6
6	23	5	27	6	32	1	38	6	1
7	27	6	32	1	38	2	45	1	2
8	32	1	38	2	45	3	53	2	3
9	38	2	45	3	53	4	62	3	4
10	45	3	53	4	62	5	73	4	5
11	53	4	62	5	73	6	86	5	6
12	62	5	73	6	86	1	101	6	1
13	73	6	86	1	101	2	118	1	2
14	86	1	101	2	118	3	138	2	3
15	101	2	118	3	138	4	23	3	4
16	118	3	138	4	23	5	27	4	5
17	138	4	23	5	27	6	32	5	6
18	23	5	27	6	32	1	38	6	1

By inspectioning this examples, it can be seen that there are two groups of numbers. First, numbers such as $n \in \{7, 9, 11, 13\}$ whose iterations contain the cycle $23 \to 27 \to 32 \to 38 \to 45 \to 53 \to 62 \to 73 \to 86 \to 101 \to 118 \to 138$. Secondly, numbers such as $n \in \{8, 10, 12, 14, 15\}$ whose iterations contain the cycle $2 \to 3 \to 4 \to 5 \to 6 \to 1$ that repeats and contains the number 1.

Case $b = 7$. Calculating $C_7(n)$ for some values of n

Iteration/ $n =$	8	9	10	11	12	13	14	15	16
1	10	11	12	13	14	15	2	18	19
2	12	13	14	15	2	18	3	21	22
3	14	15	2	18	3	21	4	3	26
4	2	18	3	21	4	3	5	4	30
5	3	21	4	3	5	4	6	5	35
6	4	3	5	4	6	5	7	6	5
7	5	4	6	5	7	6	1	7	6
8	6	5	7	6	1	7	2	1	7
9	7	6	1	7	2	1	3	2	1
10	1	7	2	1	3	2	4	3	2
11	2	1	3	2	4	3	5	4	3
12	3	2	4	3	5	4	6	5	4
13	4	3	5	4	6	5	7	6	5
14	5	4	6	5	7	6	1	7	6
15	6	5	7	6	1	7	2	1	7
16	7	6	1	7	2	1	3	2	1
17	1	7	2	1	3	2	4	3	2
18	2	1	3	2	4	3	5	4	3

By inspectioning this examples, it can be seen that in all cases, iterations contain the cycle $2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 7 \rightarrow 1$ that repeats and contains the number 1.

Case $b = 9$. Calculating $C_9(n)$ for some values of n

Iteration/ $n =$	10	11	12	13	14	15	16	17	31	35
1	12	13	14	15	16	17	18	19	35	39
2	14	15	16	17	18	19	2	22	39	44
3	16	17	18	19	2	22	3	25	44	49
4	18	19	2	22	3	25	4	28	49	55
5	2	22	3	25	4	28	5	32	55	62
6	3	25	4	28	5	32	6	36	62	69
7	4	28	5	32	6	36	7	4	69	77
8	5	32	6	36	7	4	8	5	77	86
9	6	36	7	4	8	5	9	6	86	96
10	7	4	8	5	9	6	1	7	96	107
11	8	5	9	6	1	7	2	8	107	119
12	9	6	1	7	2	8	3	9	119	133
13	1	7	2	8	3	9	4	1	133	148
14	2	8	3	9	4	1	5	2	148	165
15	3	9	4	1	5	2	6	3	165	184
16	4	1	5	2	6	3	7	4	184	205
17	5	2	6	3	7	4	8	5	205	228
18	6	3	7	4	8	5	9	6	228	254
19	7	4	8	5	9	6	1	7	254	283
20	8	5	9	6	1	7	2	8	283	315
21	9	6	1	7	2	8	3	9	315	35
22	1	7	2	8	3	9	4	1	35	39
23	2	8	3	9	4	1	5	2	39	44
24	3	9	4	1	5	2	6	3	44	49
25	4	1	5	2	6	3	7	4	49	55

By inspectioning this examples, it can be seen that for $n \in \{10, 11, 12, 13, 14, 15, 16, 17\}$, iterations contain the cycle $2 \to 3 \to 4 \to 5 \to 6 \to 7 \to 8 \to 9 \to 1$ that repeats and contains the number 1. In other hand, for $n \in \{31, 35\}$, for example, interations contain the cycle $35 \to 39 \to 44 \to 49 \to 55 \to 62 \to 69 \to 77 \to 86 \to 96 \to 107 \to 119 \to 133 \to 148 \to 165 \to 184 \to 205 \to 228 \to 254 \to 283 \to 315$.

Case $b = 10$. Calculating $C_{10}(n)$ for some values of n. We compute the fisrt 30 interations for esch value of n,

Iteration/$n =$	11	12	13	14	15	16	17	18	34	38
1	13	14	15	16	17	18	19	20	38	42
2	15	16	17	18	19	20	21	2	42	47
3	17	18	19	20	21	2	24	3	47	52
4	19	20	21	2	24	3	27	4	52	58
5	21	2	24	3	27	4	30	5	58	64
6	24	3	27	4	30	5	3	6	64	71
7	27	4	30	5	3	6	4	7	71	79
8	30	5	3	6	4	7	5	8	79	87
9	3	6	4	7	5	8	6	9	87	96
10	4	7	5	8	6	9	7	10	96	106
11	5	8	6	9	7	10	8	1	106	117
12	6	9	7	10	8	1	9	2	117	129
13	7	10	8	1	9	2	10	3	129	142
14	8	1	9	2	10	3	1	4	142	157
15	9	2	10	3	1	4	2	5	157	173
16	10	3	1	4	2	5	3	6	173	191
17	1	4	2	5	3	6	4	7	191	211
18	2	5	3	6	4	7	5	8	211	233
19	3	6	4	7	5	8	6	9	233	257
20	4	7	5	8	6	9	7	10	257	283
21	5	8	6	9	7	10	8	1	283	312
22	6	9	7	10	8	1	9	2	312	344
23	7	10	8	1	9	2	10	3	344	379
24	8	1	9	2	10	3	1	4	379	417
25	9	2	10	3	1	4	2	5	417	459
26	10	3	1	4	2	5	3	6	459	505
27	1	4	2	5	3	6	4	7	505	556
28	2	5	3	6	4	7	5	8	556	612
29	3	6	4	7	5	8	6	9	612	674
30	4	7	5	8	6	9	7	10	674	742

Then, we compute values for the rest of interations

Iteration/n =	11	12	13	14	15	16	17	18	34	38
31	5	8	6	9	7	10	8	1	742	817
32	6	9	7	10	8	1	9	2	817	899
33	7	10	8	1	9	2	10	3	899	989
34	8	1	9	2	10	3	1	4	989	1088
35	9	2	10	3	1	4	2	5	1088	1197
36	10	3	1	4	2	5	3	6	1197	1317
37	1	4	2	5	3	6	4	7	1317	1449
38	2	5	3	6	4	7	5	8	1449	1594
39	3	6	4	7	5	8	6	9	1594	1754
40	4	7	5	8	6	9	7	10	1754	1930
41	5	8	6	9	7	10	8	1	1930	193
42	6	9	7	10	8	1	9	2	193	213
43	7	10	8	1	9	2	10	3	213	235
44	8	1	9	2	10	3	1	4	235	259
45	9	2	10	3	1	4	2	5	259	285
46	10	3	1	4	2	5	3	6	285	314
47	1	4	2	5	3	6	4	7	314	346
48	2	5	3	6	4	7	5	8	346	381
49	3	6	4	7	5	8	6	9	381	420
50	4	7	5	8	6	9	7	10	420	42
51	5	8	6	9	7	10	8	1	42	47

In this case, for $n \in \{11, 12, 13, 14, 15, 16, 17, 18\}$ there is a common cycle $2 \to 3 \to 4 \to 5 \to 6 \to 7 \to 8 \to 9 \to 10 \to 1$ that repeats and contains the number 1. In other hand, for numbers such as $n \in \{34, 38\}$, one can find the large cycle $42 \to 47 \to 52 \to 58 \to 64 \to 71 \to 79 \to 87 \to 96 \to 106 \to 117 \to 129 \to 142 \to 157 \to 173 \to 191 \to 211 \to 233 \to 257 \to 283 \to 312 \to 344 \to 379 \to 417 \to 459 \to 505 \to 556 \to 612 \to 674 \to 742 \to 817 \to 899 \to 1088 \to 1197 \to 1317 \to 1449 \to 1594 \to 1754 \to 1930 \to 193 \to 213 \to 235 \to 259 \to 285 \to 314 \to 346 \to 381 \to 420$ that repeats and does not contain the number 1.

Case $b = 11$. Calculating $C_{11}(n)$ for some values of n

Iteration/n =	12	13	14	15	16	17	18	19	20	21
1	14	15	16	17	18	19	20	21	22	23
2	16	17	18	19	20	21	22	23	2	26
3	18	19	20	21	22	23	2	26	3	29
4	20	21	22	23	2	26	3	29	4	32
5	22	23	2	26	3	29	4	32	5	35
6	2	26	3	29	4	32	5	35	6	39
7	3	29	4	32	5	35	6	39	7	43
8	4	32	5	35	6	39	7	43	8	47
9	5	35	6	39	7	43	8	47	9	52
10	6	39	7	43	8	47	9	52	10	57
11	7	43	8	47	9	52	10	57	11	63
12	8	47	9	52	10	57	11	63	1	69
13	9	52	10	57	11	63	1	69	2	76
14	10	57	11	63	1	69	2	76	3	83
15	11	63	1	69	2	76	3	83	4	91
16	1	69	2	76	3	83	4	91	5	100
17	2	76	3	83	4	91	5	100	6	110
18	3	83	4	91	5	100	6	110	7	10
19	4	91	5	100	6	110	7	10	8	11
20	5	100	6	110	7	10	8	11	9	1
21	6	110	7	10	8	11	9	1	10	2
22	7	10	8	11	9	1	10	2	11	3
23	8	11	9	1	10	2	11	3	1	4
24	9	1	10	2	11	3	1	4	2	5
25	10	2	11	3	1	4	2	5	3	6
26	11	3	1	4	2	5	3	6	4	7
27	1	4	2	5	3	6	4	7	5	8
28	2	5	3	6	4	7	5	8	6	9
29	3	6	4	7	5	8	6	9	7	10
30	4	7	5	8	6	9	7	10	8	11
31	5	8	6	9	7	10	8	11	9	1
32	6	9	7	10	8	11	9	1	10	2
33	7	10	8	11	9	1	10	2	11	3
34	8	11	9	1	10	2	11	3	1	4
35	9	1	10	2	11	3	1	4	2	5

By inspectioning this examples, it can be seen that in all cases, iterations contain the cycle $2 \to 3 \to 4 \to 5 \to 6 \to 7 \to 8 \to 9 \to 10 \to 11 \to 1$ that repeats and contains the number 1.

From previous empirical results, following definitions and conjectures can be stated

Definition 1 *For any* $b \geq 2$*, set* $\{2, 3, ..., b, 1\}$ *is called a* b*-trivial cycle.*

Definition 2 *Let* $b \geq 2$ *an integer number. Then, for any* $n \in \mathbb{N}$*, we define*

$$\mathtt{Iter}_b(n) = \{C_b^{(k)}(n) \mid k \geq 1\}.$$

Definition 3 *An integer number* $b \geq 2$ *is called Collatz number if for every* $n \in \mathbb{N}$*,* $\mathtt{Iter}_b(n)$ *contains the* b*-trivial cycle.*

Lemma 1 *Let* $b \geq 2$ *an integer number. Then, there exist numbers* $n \in \mathbb{N}$*, such that*
i) $1 \in \mathtt{Iter}_b(n)$*, and*
ii) $\{2, 3, ..., b, 1\} \subset \mathtt{Iter}_b(n)$.

Proof. Let $n = 2 \cdot b$, then it is easy to verify that $C_b(n) = 2$, $C_b^{(2)}(n) = 3$, ..., $C_b^{b-1}(n) = b$, $C_b^{(b)}(n) = 1$ and $C_b^{(b+1)}(n) = 2$. Thus, both previous statements can be easily checked. □

If we analyze the list of positive integers numbers lower than 20, then we can see two completely different behaviour when iterating C_b function. In one hand, we can find values for b, such as $2, 5, 7, 8, ...$, in which interations of $C_b(n)$ on any integer number n, $\mathtt{it\ seems}$, always end in 1, more concretely, interations contain the b-trivial cycle. And in other hand, values of b, such as $3, 4, 6, 9, 10, ...$, in which interations of $C_b(n)$ on any integer number n either end in 1 or in another non trivial cycle. Below you can find a table for different values of b, where one can find the lowest value of n, for which $\mathtt{Iter}_b(n)$ does not contain 1, but however, it contains a non trivial cycle.

b	n
3	5
4	11
6	7
9	31
10	34
11	588
12	767
15	49
16	35
17	19

Conjecture 2 *Let* $b \geq 2$ *an integer number. Then, one can find only following two possibilities*

Case i) *For all* $n \in N$, $\mathrm{Iter}_b(n)$ *contains the* b*-trivial cycle, and then* b *is a Collatz number, or*

Case ii) *There are* $n \in N$, *such that* $\mathrm{Iter}_b(n)$ *contains a common non-trivial cycle. There are also other values of* $n \in N$, *for which* $\mathrm{Iter}_b(n)$ *contains the* b*-trivial cycle.*

Conjecture 3 *Numbers 2, 5, 7, 8, 13, 14, 18 and 19 are Collatz numbers.*

Conjecture 4 *There are infinite Collatz numbers.*

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