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Orbital shadowing property on chain transitive sets for generic diffeomorphisms

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Abstract. Let $f : M \to M$ be a diffeomorphism on a closed smooth $n(\geq 2)$ dimensional manifold M. We show that C^1 generically, if a diffeomorphism f has the orbital shadowing property on locally maximal chain transitive sets which admits a dominated splitting then it is hyperbolic.

1 Introduction

Let M be a closed smooth $n(n \geq 2)$-dimensional Riemannian manifold, and let $\mathrm{Diff}(M)$ be the space of diffeomorphisms of M endowed with the C^1-topology. Denote by d the distance on M induced from a Riemannian metric $\|\cdot\|$ on the tangent bundle TM. For any $\delta > 0$, a sequence $\{x_i\}_{i\in\mathbb{Z}}$ is called a δ-*pseudo orbit* of f if $d(f(x_i), x_{i+1}) < \delta$ for all $i \in \mathbb{Z}$. Let Λ be a closed f-invariant set. We say that f has the *shadowing property* on Λ if for any $\epsilon > 0$ there is $\delta > 0$ such that for any δ-pseudo orbit $\{x_i\}_{i\in\mathbb{Z}} \subset \Lambda$ there is $y \in M$ such that $d(f^i(y), x_i) < \epsilon$ for all $i \in \mathbb{Z}$. If $\Lambda = M$ then we say that f has the shadowing property. The shadowing property is very useful notion to investigate for hyperbolic structure. In fact, Robinson[22] and Sakai[24] proved that a diffeomorphism f has the C^1 robustly shadowing property if and only if it is structurally stable

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diffeomorphisms. Here, we say that f has the C^1 robustly shadowing property if there is a C^1 neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, g has the shadowing property. From the property, a general shadowing property was introduced by [20] which is called the orbital shadowing property. For the orbital shadowing property, many results published by the various view points (see [10, 13, 14, 15, 16, 17, 19]). We say that f has the *orbital shadowing property* on Λ if for any $\epsilon > 0$ there is $\delta > 0$ such that for any δ pseudo orbit $\xi = \{x_i\}_{\in\mathbb{Z}} \subset \Lambda$ such that there is a point $y \in M$ such that

$$\mathrm{Orb}(y) \subset B_\epsilon(\xi) \text{ and } \xi \subset B_\epsilon(\mathrm{Orb}(y)).$$

If $\Lambda = M$ then we say that f has the orbital shadowing property. Let Λ be a closed f-invariant set. We say that Λ is *hyperbolic* if the tangent bundle $T_\Lambda M$ has a Df-invariant splitting $E^s \oplus E^u$ and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|D_x f^n|_{E^s_x}\| \leq C\lambda^n \text{ and } \|D_x f^{-n}|_{E^u_x}\| \leq C\lambda^n$$

for all $x \in \Lambda$ and $n \geq 0$. If $\Lambda = M$ then we say that f is Anosov.

Pilyugin *et al* [20] proved that if a diffeomorphism f has the C^1 robustly orbital shadowing property then it is structurally stable diffeomorphisms. Lee and Lee [10] proved that a volume preserving diffeomorphism f has the C^1 robustly orbital shadowing property then it is Anosov. Moreover, we can find similar results [12, 13, 14, 15]. We say that the set Λ is *transitive* if there is a point $x \in \Lambda$ such that $\omega(x) = \Lambda$, where $\omega(x)$ is the omega limit set of x. An invariant closed set $\mathcal{C}$ is called a *chain transitive* if for any $\delta > 0$ and $x, y \in \mathcal{C}$, there is δ-pseudo orbit $\{x_i\}_{i=0}^n (n \geq 1) \subset \mathcal{C}$ such that $x_0 = x$ and $x_n = y$. It is clear that the transitive set Λ is the chain transitive set $\mathcal{C}$, but the converse is not true. We say that Λ is *locally maximal* if there is a neighborhood U of Λ such that $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(U)$. For the relation between chain transitive sets and C^1 robustly shadowing theories, In [16], Lee proved that if a robustly chain transitive set with orbital shadowing then it is hyperbolic. We say that f has the C^1 stably shadowing property on Λ if there are a C^1 neighborhood $\mathcal{U}(f)$ of f and a neighborhood U of Λ such that for any $g \in \mathcal{U}(f)$, g has the shadowing property on $\Lambda_g(U)$, where $\Lambda_g(U)$ is the continuation of Λ. For $f \in \mathrm{Diff}(M)$, we say that a compact f-invariant set Λ admits a *dominated splitting* if the tangent bundle $T_\Lambda M$ has a continuous Df-invariant splitting $E \oplus F$ and there exist $C > 0$, $0 < \lambda < 1$ such that for all $x \in \Lambda$ and $n \geq 0$, we have

$$\|Df^n|_{E(x)}\| \cdot \|Df^{-n}|_{F(f^n(x))}\| \leq C\lambda^n.$$

In [18], Lee proved that if a diffeomorphism f has the C^1 stably shadowing property on chain transitive set $\mathcal{C}$ then it admits a dominated splitting. Sakai [23] proved that a diffeomorphism f has the C^1 stably shadowing property on chain transitive set $\mathcal{C}$ then it is hyperbolic.

We say that a subset $\mathcal{G} \subset \text{Diff}(M)$ is *residual* if $\mathcal{G}$ contains the intersection of a countable family of open and dense subsets of Diff(M); in this case $\mathcal{G}$ is dense in Diff(M). A property "P" is said to be *(C^1)-generic* if "P" holds for all diffeomorphisms which belong to some residual subset of Diff(M). For C^1 generic differeomorphism f, Abdenur and Díaz [3] suggested the problem : *if a C^1 generic diffeomorphism f has the shadowing property then is it hyperbolic?*

Unfortunately, this question still is open. For the problem, there are partial results [4, 9, 11]. Ahn *et al* [4] proved that for C^1 generic diffeomorphism f, if f has the shadowing property on a locally maximal homoclinic class then it is hyperbolic. Lee and Wen [11] proved that for C^1 generic diffeomorphism f, if f has the shadowing property on a locally maximal chain transitive set then it is hyperbolic. Very recently, Lee and Lee [9] proved that for C^1 generic diffeomorphism f, if f has the shadowing property on chain recurrence classes then it is hyperbolic. From the results, we study the orbital shadowing property for C^1 generic diffeomorphisms. The following is the main theorem of the paper.

Theorem A *For C^1 generic f, if f has the orbital shadowing property on a locally maximal $\mathcal{C}$ which admits a dominated splitting $E \oplus F$ then it is hyperbolic.*

2 Proof of Theorem A

Let M be as before, and let $f \in \text{Diff}(M)$. A *periodic point* for f is a point $p \in M$ such that $f^{\pi(p)}(p) = p$, where $\pi(p)$ is the minimum period of p. Denote by $P(f)$ the set of all periodic points of f. Let p be a hyperbolic periodic point of f. A point $x \in M$ is called *chain recurrent* if for any $\delta > 0$, there is a finite δ-pseudo orbit $\{x_i\}_{i=0}^{n} (n \geq 1)$ such that $x_0 = x$ and $x_n = x$. Denote by $\mathcal{CR}(f)$ the set of all chain recurrent points of f. We define a relation $\leftrightsquigarrow$ on $\mathcal{CR}(f)$ by $x \leftrightsquigarrow y$ if for any $\delta > 0$, there is a finite δ pseudo orbit $\{x_i\}_{i=0}^{n}$ such that $x_0 = x$ and $x_n = y$ and a finite δ pseudo orbit $\{w_i\}_{i=0}^{n}$ such that $w_0 = y$ and $w_n = x$. Then we know that the relation $\leftrightsquigarrow$ is an equivalence relation on $\mathcal{CR}(f)$. the equivalence classes are called the *chain recurrence classes* of f, denote by C_f. Note that if the class C_f has a hyperbolic periodic point p then we denote as $C(p, f)$.

It is well known that if p is a hyperbolic periodic point of f with period k

then the sets

$$W^s(p) = \{x \in M : f^{kn}(x) \to p \text{ as } n \to \infty\} \quad \text{and}$$
$$W^u(p) = \{x \in M : f^{-kn}(x) \to p \text{ as } n \to \infty\}$$

are C^1-injectively immersed submanifolds of M. The homoclinic class of a hyperbolic periodic point p is the closure of the transverse intersection of the $W^s(p)$ and $W^u(p)$, and it is denoted by $H(p, f)$. It is clear that $H(p, f)$ is compact, transitive and invariant sets. Let q be a hyperbolic periodic point of f. We say that p and q are *homoclinically related*, and write $p \sim q$ if

$$W^s(p) \pitchfork W^u(q) \neq \emptyset \text{ and } W^u(p) \pitchfork W^s(q) \neq \emptyset.$$

It is clear that if $p \sim q$ then index(p) = index(q), that is, dim$W^s(p)$ = dim$W^s(q)$. By the Smale's transverse homoclinic point theorem, $H_f(p)$ coincides with the closure of the set of hyperbolic periodic points q of f such that $p \sim q$. Note that if p is a hyperbolic periodic point of f then there is a neighborhood U of p and a C^1-neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$ there exists a unique hyperbolic periodic point p_g of g in U with the same period as p and index(p_g) = index(p). Such a point p_g is called the *continuation* of $p = p_f$. The following are results for C^1 generic diffeomorphisms (see [2]).

Lemma 1 *There is a residual set $\mathcal{G} \subset$ Diff(M) such that if $f \in \mathcal{G}$,*

(a) $H(p, f) = C(p, f)$*, for some hyperbolic periodic point* p (see [5]).

(b) *A compact f-invariant set $\mathcal{C}$ is chain transitive if and only if $\mathcal{C}$ is the Hausdorff limit of a sequence of periodic orbits of* f (see [8]).

(c) *A locally maximal transitive set Λ is a locally maximal $H(p, f)$ for some periodic point* $p \in \Lambda$ (see [1]).

(d) $H(p, f) = \overline{W^s(p)} \cap \overline{W^u(p)}$ (see [7]).

Remark 1 *Applying Pugh's closing lemma, we know that any transitive set Λ of a C^1-generic diffeomorphism f is the Hausdorff limit of a sequence of periodic orbits $\mathrm{Orb}_f(p_n)$ of f, that is, $\lim_{n\to\infty} \mathrm{Orb}_f(p_n) = \Lambda$. By Lemma 1* (b) *and* (c)*, a chain transitive set $\mathcal{C}$ is a transitive set Λ and so, a locally maximal chain transitive set $\mathcal{C} = H(p, f)$ for some periodic point p.*

Let Λ be a closed f-invariant set. We say that Λ is *Lyapunov stable* for f if for any open neighborhood U of Λ there is a neighborhood $V \subset U$ such that

$f^j(V) \subset U$ for all $j \in \mathbb{N}$. We say that the closed set is *bi-Lyapunov stable* if it is Lyapunov stable for f and f^{-1}. Potrie [21, Theorem 1.1] proved that C^1 generically, if a homoclinic class $H(p, f)$ is a bi-Lyapunov stable then it admits a dominated splitting.

A diffeomorphism f has a *heterodimensiional cycle* associated with the hyperbolic periodic points p and q of f if (i) the indice of the points p and q are different, and (ii) the stable manifold of p meets the unstable manifold of q and the same holds for the stable manifold of p and the unstable manifold of q (see [6]). We say that f has C^1 *robustly heterodimensional cycle* if f has a heterodiemnsional cycle associated with the hyperbolic periodic points p and q of f and there is a C^1 neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, g has a heterodimensional cycle associated with the hyperbolic periodic points p_g and q_g, where p_g and q_g are the continuations of p and q for g.

Lemma 2 [6, Corollary 1.15] *There is a residual set $\mathcal{T} \subset \mathrm{Diff}(M)$ such that for any $f \in \mathcal{T}$ and every locally maximal chain recurrence class C_f of f there are two possibilities: either C_f is hyperbolic or it has a robustly heterodimensional cycle.*

Lemma 3 *Let $C(p, f)$ admits a dominated splitting $E \oplus F$ and let $C(p, f)$ be locally maximal. If a chain recurrence class $C(p, f)$ has heterodimensional cycle then f does not have the orbital shadowing property on $C(p, f)$.*

Proof. Suppose, by contradiction, that f has the orbital shadowing property on $C(p, f)$. Since $C(p, f)$ has heterodimensional cycle, there is q a hyperbolic periodic point in $C(p, f)$ such that $\mathrm{index}(p) \neq \mathrm{index}(q)$. Then we take $x, y \in C(p, f)$ such that $x \in W^s(p) \cap W^u(q)$ and $y \in W^u(p) \cap W^s(q)$. By assumption, x, y are not transverse intersection points. Since q is hyperbolic, $T_qM = E^s_q \oplus E^u_q$. Choose $\alpha > 0$ sufficiently small such that $W^s_{\alpha/4}(q) = \exp_q(E^s(\alpha/4))$ and $W^u_{\alpha/4}(q) = \exp_q(E^u(\alpha/4))$. Then we may assume that $y \in W^s_{\alpha/4}(q)$ and $x \in W^u_{\alpha/4}(q)$. Since $y \in W^u(p)$, there is $\eta > 0$ such that $y \in B_\eta(y) \cap W^u(p)$. Take a small arc $\mathcal{J}_y \subset B_\eta(y) \cap W^u(p)$ such that $T_y\mathcal{J}_y = T_yW^u(p)$. Since $C(p, f)$ admits a dominated splitting $E \oplus F$, we have $T_y\mathcal{J}_y = F_y = T_yW^u(p), F_q \subset E^u_q$ and $E^s_q \subset E_q$. Put $E^{u,1} = E_q \oplus E^u_q$ and $E^{u,2} = F_q$. Then $E^u_q = E^{u,1}_q \oplus E^{u,2}_q$, and $W^{u,1}_{\alpha/4}(q) = \exp_q(E^{u,1}_q(\alpha/4)), W^{u,2}_{\alpha/4}(q) = \exp_q(E^{u,2}_q(\alpha/4))$.

Let $P^u : B_{\alpha/4}(q) \to E^u_q$ and $P^s : B_{\alpha/4}(q) \to E^s_q$ be the projections parallel to E^s_q and E^u_q, respectively. Then $P^u(f^n(\mathcal{J}_y)) \cap B_{\alpha/4}(q) \to W^{u,1}_{\alpha/4}(q)$ and $P^s(f^n(\mathcal{J}_y)) \cap B_{\alpha/4}(q) \to q$ as $n \to \infty$.

Take $\epsilon = \min\{\alpha/4, \eta, d(x, W^{u,1}_{\alpha/4}(p))/2\}$, and let $0 < \delta < \epsilon$ be the number of the orbital shadowing property. Since $y \in W^s(q) \cap W^u(p)$ and $x \in W^u(q) \cap W^s(p)$, there are $i_1 > 0$ and $i_2 > 0$ such that (i) $d(f^{i_1}(y), f^{-i_1}(x)) < \delta$ and $d(f^{-i_2}(y), f^{i_2}(x)) < \delta$, (ii) $\max\{d_H(P^s(f^{i_1+j}(\mathcal{J}_y), q), d_H(P^u(f^{i_1+j}(\mathcal{J}_y)) \cap B_{\alpha/4}(q), W^{u,1}_{\alpha/4}(q))\} < \epsilon$ for all $j \geq 0$, where d_H is the Hausdorff metric. Then we have a δ-pseudo orbit as follows:

$$\begin{aligned}\xi = \{&y, f(y), \ldots, f^{i_1-1}(y), f^{-i_1}(x), \ldots, f^{-1}(x),\\ &x, f(x), \ldots, f^{i_2-1}(x), f^{-i_2}(y), \ldots, f^{-1}(y), y\} \subset C(p, f).\end{aligned}$$

Since f has the orbital shadowing property on $C(p, f)$ and $C(p, f)$ is locally maximal, there is a point $w \in C(p, f)$ such that

$$\mathrm{Orb}(w) \subset B_\epsilon(\xi) \text{ and } \xi \subset B_\epsilon(\mathrm{Orb}(w)).$$

First, we assume that there is $k > 0$ such that $f^k(w) \in \mathcal{J}_y \setminus \{y\}$. Then if $f^k(w) \in P^u(f^n(\mathcal{J}_y))$ then since $P^u(f^n(\mathcal{J}_y)) \to W^{u,1}(q)(n \to \infty)$, $f^{k+n}(w) \to W^{u,1}(q)$ as $n \to \infty$. Thus there is $j > 0$ such that $d(f^{k+j}(w), f^j(y)) > 8\epsilon$ and so, $d(f^{k+j}(w), q) > 2\epsilon$ which is a contradiction. If $f^k(w) \notin P^u(\mathcal{J}_y)$ then by λ-lemma, $f^n(\mathcal{J}_y) \to W^u(q)$ as $n \to \infty$. Then there is $l > 0$ such that $d(f^{k+l}(w), q) > 4\epsilon$. Since $x \in W^u(q)$, there is $m > 0$ such that $d(f^{-m}(x), q) < \epsilon$ for some $m < i_2$. Then we know that $f^{k+l}(w) \notin B_\epsilon(\xi)$ which is a contradiction by the orbital shadowing property on $C(p, f)$. Then for all $i \in \mathbb{Z}$, $f^i(w) \notin \mathcal{J}_y \setminus \{y\}$.

We assume that there is $k > 0$ such that $f^k(w) = y$. Since $y \in W^s(q) \cap W^u(p)$ and $x \in W^u(q) \cap W^s(p)$, we know $\mathrm{Orb}(x) \cap \mathrm{Orb}(y) = \emptyset$. Then we have $\xi \not\subset B_\epsilon(\mathrm{Orb}(w))$ which is a contradiction by the orbital shadowing property on $C(p, f)$. Thus we know $\mathrm{Orb}(w) \cap \mathcal{J}_y = \emptyset$.

Finally, we assume that there is $k > 0$ such that $f^k(w) \in B_\eta(y) \setminus \mathcal{J}_y$. Then for all $z \in B_\eta(y) \setminus \mathcal{J}_y$, there is $k > 0$ such that $d(f^{-k}(x), f^k(z)) > 2\epsilon$ since $x \in W^u(q)$ and q is hyperbolic saddle. Then we have $\xi \not\subset B_\epsilon(\mathrm{Orb}(w))$ which is a contradiction by the orbital shadowing property on $C(p, f)$. Consequently, if a locally maximal chain recurrence class $C(p, f)$ admits a dominated splitting and f has the orbital shadowing property on $C(p, f)$ then it does not the heterodimensional cycle. □

Proof of Theorem A. Let $f \in \mathcal{G} \cap \mathcal{T}$ and let f has the orbital shadowing property on a locally maximal chain transitive set $\mathcal{C}$. Since $f \in \mathcal{G}$, by Remark 1 $\mathcal{C} = C(p, f)$ for some hyperbolic periodic point p. Since chain transitive set $\mathcal{C}$ admits a dominated splitting $E \oplus F$, and f has the orbital shadowing property on a locally maximal chain transitive $\mathcal{C}$, by Lemma 3, $\mathcal{C} = C(p, f)$ does not have the heterodimensional cycles. It is clear that a locally maximal $C(p, f)$ does not have the robustly hetrodimensional cycle. Thus by Lemma 2, a locally maximal chain transitive set $\mathcal{C}$ is hyperbolic. □

In Abdenur *et al* [2] the authors proved that every locally maximal homoclinic class with a non-empty interior is the whole space.

Corollary 1 *Let* $f : M \to M$ *be a diffeomorphism with* $\dim M = 3$. *For* C^1 *generic* f, *if* f *has the orbital shadowing property on a locally maximal chain transitive set* $\mathcal{C}$ *which admits a dominated splitting* $E \oplus F$ *then it is Anosov.*

The following was proved in [21, Proposition 1.2] which means that a homoclinic class admits a codimension one dominated splitting then it has a non-empty interior.

Lemma 4 *There is a residual set* $\mathcal{R} \subset \mathrm{Diff}(M^3)$ *such that for any* $f \in \mathcal{R}$, *if a homoclinic class* H *admits a codimension one dominated splitting then it has non-empty interior.*

Proof of Corollary 1. Let $f \in \mathcal{G} \cap \mathcal{T} \cap \mathcal{R}$ and let f has the orbital shadowing property on a locally maximal chain transitive set $\mathcal{C}$ which admits a dominated splitting $E \oplus F$. By Remark 1, a locally maximal chain transitive $\mathcal{C} = H(p, f)$. Since $f \in \mathcal{R}$, by Lemma 4, a hoomoclinic class $H(p, f)$ has nonempty interior. Since $H(p, f)$ is locally maximal, by [2, Thereom 3], $H(p, f) = M$. Since f has the orbital shadowing property on a locally maximal chain transitive set $\mathcal{C}$ which admits a dominated splitting $E \oplus F$, by Theorem A, it is hyperbolic, and so it is Anosov. □

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