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Restrained domination in signed graphs

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Abstract. A signed graph Σ is a graph with positive or negative signs attatched to each of its edges. A signed graph Σ is balanced if each of its cycles has an even number of negative edges. Restrained dominating set D in Σ is a restrained dominating set of its underlying graph where the subgraph induced by the edges across $\Sigma[D : V \setminus D]$ and within $V \setminus D$ is balanced. The set D having least cardinality is called minimum restrained dominating set and its cardinality is the restrained domination number of Σ denoted by $\gamma_r(\Sigma)$. The ability to communicate rapidly within the network is an important application of domination in social networks. The main aim of this paper is to initiate a study on restrained domination in the realm of different classes of signed graphs.

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1 Introduction

Graphs used in this article, unless otherwise mentioned will be undirected, simple and finite. For a graph $G = (V, E)$, the degree of a vertex v, denoted by $\deg(v)$ is the number of edges incident to the vertex v(loops counted twice incase of multigraph). The maximum degree of G is denoted by $\Delta(G)$ and the minimum degree of G is denoted by $\delta(G)$. If $\deg(v) = 1$, then v is called a pendant vertex. For all terminology and notation in graph theory, we refer the reader to the text-book by Harary [1]. The graphs with positive or negative signs attatched to each of its arcs are called signed graphs. Zaslavasky [2], formally defines a signed graph as $\Sigma = (G, \sigma)$, where G is the underlying unsigned graph consisting of $G = (V, E)$ and $\sigma : E \rightarrow \{+, -\}$ is the function assigning signs to the edges of the graph. The edges which receive $+(-)$ signs, are called positive(negative) edges of Σ.

A signed graph Σ is all-positive(all-negative) if all its edges are positive (negative). If it is an all-positive or all-negative, then it is said to be homogenous else heterogenous. Switching Σ with respect to a marking μ where $\mu : V \rightarrow \{+1, -1\}$ is the operation of negating every edge whose end vertices are of opposite signs. Σ is said to be balanced if each of its cycle has an even number of negative edges. Equivalently, a signed graph is balanced if it can be switched to an all-positive signed graph. For further details on theory of signed graphs, the reader is referred to [3, 2].

Domination in graph theory for unsigned graphs is one of the continuing research of the well-researched region. Detailed survey of the same can be found in the book by Haynes et al. [4]. In 2013, Acharya [5] introduced the theory of dominance for signed graphs as well as signed digraphs. A subset $D \subseteq V$ of vertices of $\Sigma = (G, \sigma)$ is a dominating set of Σ, if there exists a marking $\mu : V \rightarrow \{+1, -1\}$ of Σ such that every vertex not in D is adjacent to at least one vertex in D and $\sigma(uv) = \mu(u)\mu(v)$, $\forall\ u \in V \setminus D$. The minimum cardinality of a dominating set in Σ is called its domination number, denoted by $\gamma(\Sigma)$. Germina and Ashraf [6, 7] gave characterization for open domination and double domination in signed graphs. In 2015, Walikar et al. [8] introduced the concept of signed domination for signed graphs.

In a social network, if all individuals are connected to at least one such person who can be reached directly, an emergency message can easily be sent to all participants in the network, thus reducing delay time. Nevertheless, it is also important to examine positive and negative relationships between individuals when examining social network interactions. This situation can be modeled on what is known as the dominating set problem in signed graphs.

In this paper, we introduce the concept of restrained domination for signed graphs. In addition, we determine the best possible bounds on $\gamma_r(G)$ for certain classes of signed graphs.

2 Definitions and results

The concept of restrained domination in graphs was introduced by Domke et al. [9] in 1999. A set $D \subseteq V$ is a restrained dominating set of graph $G = (V, E)$, if every vertex in $V \setminus D$ is adjacent to a vertex in D as well as another vertex in $V \setminus D$. The restrained domination number of graph G denoted by $\gamma_r(G)$ is the smallest cardinality of a restrained dominating set of G. We will now define the concept of restrained domination for signed graphs and then find the best possible general bounds for some classes of signed graph. In this paper, we will be using the notation $\Sigma[D : V \setminus D]$ when $D \subseteq V$, to denote the subgraph of Σ induced by the edges of Σ with one end point in D and the other end point in $V \setminus D$. Induced subgraph in $V \setminus D$ is denoted by $\Sigma[V \setminus D]$.

Definition 1 *A subset $D \subseteq V$ of vertices of $\Sigma = (V, E, \sigma)$ is a restrained dominating set if there exists a marking $\mu : V \to \{+1, -1\}$ of Σ such that every vertex in $V \setminus D$ is adjacent to a vertex in D as well a vertex in $V \setminus D$ and for every vertex u in $V \setminus D$, $\sigma(uv) = \mu(u)\mu(v)$ $\forall$ $v \in D$ and $v \in V \setminus D$.*

The minimum cardinality of a restrained dominating set D is called restrained domination number of Σ denoted by $\gamma_r(\Sigma)$. Every restrained dominating set of a signed graph Σ of order n follows the inequality $1 \leq \gamma_r(\Sigma) \leq n$. As each vertex in $V \setminus D$ is adjacent to at least one other vertex in $V \setminus D$, the cardinality of the set $V \setminus D$ is always greater than or equal to two. Hence, $\gamma_r(\Sigma)$ can never be equal to $n - 1$. Before proceeding further with results on bounds for few classes of signed graphs, we state some important results used for obtaining these bounds.

Proposition 1 *Let G be any graph of order n with $\delta(G) = 1$. Then, every restrained dominating set of graph G must necessarily have all its pendant vertices.*

Switching $\Sigma = (G, \sigma)$ with respect to μ means forming the switched graph $\Sigma^\mu = (\Sigma, \sigma^\mu)$, whose underlying graph is the same but whose sign function is defined on an edge $e : vw$ by $\Sigma^\mu(e) = \mu(v)\sigma(e)\mu(w)$. In case of balanced signed graphs, when Σ is switched with respect to μ, we obtain an all-positive signed graph. Hence, we can state the following lemma:

Lemma 1 [2] *A signed graph Σ is balanced if and only if it can be switched to an all-positive signed graph.*

Let, D^r_Σ be the set of all restrained dominating sets of signed graphs and $D^r_{|\Sigma|}$ be the set of all restrained dominating sets of its underlying graphs. Then, for balanced signed graphs Σ, $D^r_\Sigma = D^r_{|\Sigma|}$. But, note that this equality does not hold true for all unbalanced signed graphs. For example, consider a 6-cycle graph $\Sigma = C_6$ with three negative edges, denoted as $C_6^{(3)}$. The underlying graph $|\Sigma| = C_6$ will have all independent vertices in its minimum restrained dominating set. Whereas, in Σ there exists no such independent vertices in restrained dominating set. This leads to the conclusion of following proposition:

Proposition 2 *The set D^r_Σ of all restrained dominating sets of a signed graph Σ is contained in the set $D^r_{|\Sigma|}$ of all restrained dominating sets of its underlying graph $|\Sigma|$.*

Proposition 3 *For any finite balanced signed graph, $\gamma_r(\Sigma) = \gamma_r(|\ \Sigma\ |)$.*

Clearly, from Lemma 1, we can conclude for balanced signed graphs $D^r_\Sigma = D^r_{|\Sigma|}$ and hence the result holds true.

Further, since $D^r_\Sigma \subseteq D^r_{|\Sigma|}$, we can conclude $\gamma_r(|\Sigma|) \leq \gamma_r(\Sigma)$. In the following results, we derive bounds for some classes of signed graphs.

Theorem 1 *If $P_n^{(r)}$ is a signed path with n vertices and r negative edges, $\gamma_r(P_n^{(r)}) = \gamma_r(P_n) = n - 2\lfloor (n-1)/3 \rfloor$ for $n \geq 1$ and $0 < r \leq n-1$.*

Proof. In the case of restrained dominating set of P_n, it is proved in [9] that $\gamma_r(P_n) = n - 2\lfloor (n-1)/3 \rfloor$. Since signed paths are trivially balanced, then by Proposition 3, the theorem follows. □

Theorem 2 *If $C_n^{(r)}$ is a signed cycle with $n \equiv 1, 2$ (mod 3) and $0 < r \leq n$, then $\gamma_r(C_n^{(r)}) = \gamma_r(C_n) = n - 2\lfloor n/3 \rfloor$.*

Proof. Let Σ be a signed cycle $C_n^{(r)}$ with $n \equiv 1, 2 \pmod 3$ and $0 < r \leq n$. Restrained domination number as proved in [9] for C_n is $n - 2\lfloor n/3 \rfloor$. We consider following two different cases to derive bounds for $\gamma_r(C_n)$:

Case 1 *$r \equiv 0 (mod\ 2)$.*

Since, Σ has an even number of negative edges, therefore Σ is balanced. Thus by Proposition 3, $\gamma_r(C_n) = \gamma_r(\Sigma) = n - 2\lfloor n/3 \rfloor$.

Case 2 *$r \equiv 1 (mod\ 2)$.*

Let D be the minimum restrained dominating set of the underlying graph $|\Sigma|$. We need to check if D is a minimum restrained dominating set of Σ also. Let us suppose, D is restrained dominating set of Σ. Then, there exists at least one pair of adjacent vertices in D. Thus, $\Sigma[D : V \setminus D] \cup \Sigma[V \setminus D]$ will always be acyclic, which is trivially balanced. Since, set D satisfies the property given in the definition, therefore it is a restrained dominating set of Σ. We know, $D^r_\Sigma \subseteq D^r_{|\Sigma|}$ by Proposition 2. We can thus conclude, D is minimum restrained dominating set of Σ and hence follows the theorem. □

Theorem 3 *Let $C_n^{(r)}$ be a signed cycle with $n \equiv 0$ (mod 3) and $0 < r \leq n$, then*

$$\gamma_r(C_n^{(r)}) = \begin{cases} n - 2\lfloor n/3 \rfloor & \text{if } r \text{ is even} \\ n - 2\lfloor n/3 \rfloor + 2 & \text{if } r \text{ is odd} \end{cases}$$

Proof. Let Σ be a signed cycle $C_n^{(r)}$ with $n \equiv 0 \pmod 3$ and $0 < r \leq n$. Proceeding in a similar way as previous theorem, we form two cases for $\gamma_r(C_n)$:

Case 1 *$r \equiv 0 (mod\ 2)$*

Since, Σ has an even number of negative edges, therefore Σ is balanced. Thus by Proposition 3, $\gamma_r(C_n) = \gamma_r(\Sigma) = n - 2\lfloor n/3 \rfloor$.

Case 2 *$r \equiv 1 (mod\ 2)$*

Let D be the minimum restrained dominating set of the underlying graph $|\Sigma|$. We need to check, if D is restrained dominating set of Σ. In this case we observe that the set D has all vertices at a distance three from each other. Therefore, $\Sigma[D : V \setminus D] \cup \Sigma[V \setminus D]$ will be a cycle with odd number of negative edges and hence not balanced. Thus by definition, D will not be a restrained dominating set of Σ. We now need to add more vertices to D. Suppose, a vertex $v_1 \in V \setminus D$ is added to the set D. The neighboring vertex of v_1 in $V \setminus D$, say v_2 then has no neighboring vertex in $V \setminus D$ and is not a restrained dominating set. Thus, we will need to add more vertices to D. Let us add $N(v_1) \in V \setminus D$ to the set D. Then, there exists only signed paths in $\Sigma[D : V \setminus D] \cup \Sigma[V \setminus D]$, which is trivially balanced . Since, we added two more vertices to the set D, therefore $\gamma_r(\Sigma) = \gamma_r(C_n) + 2 = n - 2\lfloor n/3 \rfloor + 2$. □

Theorem 4 *If $K^{(r)}_{1,n-1}$ is a star signed graph with n vertices and r negative edges, then $\gamma_r(K^{(r)}_{1,n-1}) = n$.*

Proof. Let Σ be a star signed graph $K^{(r)}_{1,n-1}$ with r negative edges. Since, $\gamma_r(K_{1,n-1}) \leq \gamma_r(K^{(r)}_{1,n-1}) \leq n$ and $\gamma_r(K_{1,n-1}) = n$, the theorem holds true. □

For complete signed graph K_n, $n \geq 5$, we can derive a general bounds as shown in Theorem 5 to obtain γ_r. But, $K^{(r)}_4$ doesnot satisfy this theorem. Hence, we state the following proposition.

Note that, paw graph is the graph obtained by joining a vertex of cycle graph C_3 to a singleton graph K_1 . In the following proposition, $P_2 \cup P_2$ is the union of two disconnected paths P_2.

Proposition 4 *Let Σ be a $K^{(r)}_4$ graph with r negative edges and r is even and let $\langle I \rangle$ be all-negative edge induced subgraph of Σ. Then,*

$$\gamma_r(\Sigma) = \begin{cases} 1, & \text{if } \langle I \rangle \cong C_4 \\ 2, & \text{if } \langle I \rangle \cong P_3 \text{ or } \langle I \rangle \text{ is a paw graph} \\ 4, & \text{if } \langle I \rangle \cong K^{(6)}_4 \text{ or } \langle I \rangle \cong P_2 \cup P_2. \end{cases}$$

Theorem 5 *If p is the order of the subgraph induced by negative edges of a complete signed graph K_n with n vertices, n$\geq$ 5, then*

$$\gamma_r(K^{(r)}_n) = \begin{cases} p & \text{if } p < n-1 \\ n & \text{otherwise} \end{cases}$$

Proof. Let Σ be any complete signed graph having r negative edges and $n \geq 5$ and D be the minimum restrained dominating set of Σ. We need to show that all the vertices incident to any negative edge in Σ belongs to the set D. We prove this by contradiction. Suppose there exists at least one negative edge in Σ with end vertices say v_1 and v_2, such that either both or one end vertex is not in D. Then, the negative edge v_1v_2 will either be in $\Sigma[V \setminus D]$ or $\Sigma[D : V \setminus D]$. Now, there exists at least one C_3 in $\Sigma[V \setminus D]$ or $\Sigma[D : V \setminus D] \cup \Sigma[V \setminus D]$ having odd number of negative edges, and thus Σ is not balanced. This implies, by Definition 1 that D is not a minimum restrained dominating set. D satisfies Definition 1 only when there does not exists any negative edge in $\Sigma[V \setminus D]$ or $\Sigma[D : V \setminus D]$, which contradicts our assumption. Therefore, $\gamma_r(K^{(r)}_n) = p$, for $p < n-1$. By Definition 1, $\gamma_r(K^{(r)}_n)$ can never be equal to $n-1$.

Hence, $\gamma_r(K_n^{(r)}) > n-1$ for $p > n-1$. But, we know $\gamma_r(K_n^{(r)}) \leq n$. Therefore, $\gamma_r(K_n^{(r)}) = n$. □

Restrained domination for complete bipartite signed graph Σ varies based on the number of negative edges in Σ and hence, for large graphs it is difficult to find exact restrained domination number. In the following theorems, we have generalized some of those cases for complete bipartite signed graphs $K_{m,m}^{(r)}$ with $2m$ vertices and concluded by giving the bounds on γ_r for any complete bipartite signed graph.

Theorem 6 *Let $K_{m,m}^{(r)}$ be a complete bipartite signed graph with $2m$ vertices and r negative edges and $\langle I\rangle$ denote the subgraph induced by all negative edges, then $\gamma_r(K_{m,m}^{(r)}) = 2$ in any one of the following conditions:*

1. *$\langle I\rangle \cong K_{1,m-1}$ or $\langle I\rangle \cong K_{1,m}$.*
2. *All the edges are negative, i.e. $\langle I\rangle \cong K_{m,m}^{(r)}$, where $r = m$*

Proof. Let Σ be a complete bipartite graph $K_{m,m}^{(r)}$ with $2m$ vertices and r negative edges and D be the restrained dominating set of Σ. We denote $\langle I\rangle$ for the subgraph induced by all negative edges of Σ.

Case 1 $\langle I\rangle \cong K_{1,m-1}$ *or* $\langle I\rangle \cong K_{1,m}$

Any induced cycle in a complete bipartite graph is always even. Also, for a signed graph to be balanced, every cycle in the graph must have an even number of negative edges. Moreover, degree of every vertex in cycle is always 2. Let u be the vertex to which all the negative edges are incident. All the induced cycles of Σ not including vertex u are all positive and hence satisfy the marking $\sigma(vw) = \mu(v)\mu(w)\ \forall\ v, w \neq u$. Thus, we need to check for the induced cycles in Σ containing the vertex u. In case of $\langle I\rangle \cong K_{1,m}$, Σ can be switched to all positive signed graph, and hence, by Proposition 3, $\gamma_r(K_{m,m}) = \gamma_r(K_{m,m}^{(r)}) = 2$ for $\langle I\rangle \cong K_{1,m}$. In case of $\langle I\rangle \cong K_{1,m-1}$, every cycle incuding vertex u will either have two negative edges incident to vertex u, which is always an even cycle or it will have 1 negative and 1 positive edge incident to u, which is an odd cycle. Hence, in case of odd cycle if we take the end vertices of the positive edge incident to vertex u in the set D, we get the desired result.

Case 2 $\langle I\rangle \cong K_{m,m}$.

This implies Σ is all negative and switching graph Σ, we obtain all positive $K_{m,m}$ graph. Thus, by Lemma 1, $\gamma_r(\Sigma) = 2$. □

Theorem 7 *Let Σ be a complete bipartite signed graph $K^{(r)}_{m,m}$ with $2m$ vertices and r negative independent edges with $r \leq m$. Then*

$$\gamma_r(\Sigma) = \begin{cases} 2r, & \text{if } r < m \\ 2(r-1), & \text{if } r = m. \end{cases}$$

Proof. Let Σ be $K^{(r)}_{m,m}$ with $2m$ vertices and r negative independent edges with $r \leq m$. Let D be the restrained dominating set of Σ. Since, all the negative edges in Σ are independent, therefore no two negative edges have at least one end point in common. In this case, there always exists at least one cycle in Σ containing odd number of negative edges and hence Σ is not balanced. Thus, to satisfy Definition 1 we will need to choose all the end points of the negative edges in the set D such that $\Sigma[D : V \setminus D] \cup \Sigma[V \setminus D]$ is balanced.

Case 1 Suppose, there are r independent negative edges with $r < m$. Then, number of vertices in D will be twice the number of negative edges and hence $\gamma_r(\Sigma) = 2r$ for $r < m$.

Case 2 Now, suppose that the number of independent negative edges r is equal to m. Then, D must include all the vertices of Σ and hence, $\gamma_r(\Sigma)$ must be equal to $2r$. But, this is not the minimum restrained domination number and hence, we need to remove some vertices from the set D. Since, γ_r cannot be equal to $2m-1$, we will remove two vertices from set D. The set D is now minimum restrained dominating set. Thus, $\gamma_r(\Sigma) = 2(r-1)$.
□

Thus we can conclude with the following corollary:

Corollary 1 *Let Σ be any complete bipartite signed graph $K^{(r)}_{m,m}$ with $2m$ vertices and r negative edges, then $2 \leq \gamma_r(K^{(r)}_{m,m}) \leq 2(m-1)$.*

3 Conclusion

In this paper, we introduced the concept of restrained domination for signed graphs and determined the bounds on $\gamma_r(\Sigma)$ for certain classes of signed graphs. As for further work, it can be extended in finding bounds for other classes of derived signed graphs. Also, it would be interesting to study on the critical concepts of restrained domination in signed graphs. The above concepts is very much useful in fault tolerence analysis of communication networks, social networks and security systems.

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