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Fixed points for a pair of weakly compatible mappings satisfying a new type of ϕ - implicit relation in $\mathcal{S}$ - metric spaces

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Abstract. The purpose of this paper is to introduce a new type of ϕ - implicit relation in $\mathcal{S}$ - metric spaces and to prove a general fixed point for a pair of weakly compatible mappings, which generalize Theorems 1, 2, 4 [23], Theorems 1-7 [13], Corollary 2.19 [13], Theorems 2.2, 2.4 [19], Theorems 3.2, 3.3, 3.4 [20] and other known results.

1 Introduction

Let X be a nonempty set and $f, g : X \to X$ two self mappings. A point $x \in X$ is said to be a coincidence point of f and g if $fx = gx = w$. The set of all coincidence points of f and g is denoted $\mathcal{C}(f, g)$ and w is said to be a point of coincidence of f and g.

In [8], Jungck defined f and g to be weakly compatible if $fgx = gfx$, for all $x \in \mathcal{C}(f, g)$.

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The notion of weakly compatible mappings is used to proof the existence of common fixed point for pairs of mappings.

A new class of generalized metric space, named D - metric space, is introduced in [5, 6]. In [11, 12], Mustafa and Sims proved that most of the claims concerning the fundamental topological structures on D - metric spaces are incorrect and introduced a new generalized metric spaces, named G - metric space. There exists a vast literature in the study of fixed points in G - metric spaces.

In [10], Mustafa initiated the study of fixed points for weakly compatible mappings in G - metric spaces.

Recently in [22], the authors introduced a new class of generalized metric space, named S - metric space. Quite recently in [7], the authors proved that the notions of G - metric spaces and S - metric space are independent.

Other results in the study of fixed points in S - metric space are obtained in [13, 19, 20, 21] and in other papers. Some results of fixed points for weakly compatible mappings in S - metric spaces are obtained in [23, 2].

In [14, 15], several classical fixed point theorems and common fixed point theorems have been unified considering a general condition by implicit function.

The study of fixed point for mappings satisfying an implicit relation in G - metric spaces is initiated in [16, 17] and in other papers.

The notion of ϕ - maps is introduced in [9]. In [3], Altun and Turkoglu introduced a new class of implicit relation satisfying a ϕ - map.

A general fixed point theorem for mappings satisfying ϕ - implicit relations in G - metric spaces is obtained in [18].

The purpose of this paper is to introduce a new type of ϕ - implicit relation in S - metric spaces and to prove a general fixed point theorem for a pair of weakly compatible mappings in S - metric spaces, generalizing Theorems 1, 2, 4 [23], Theorems 1-7 [13], Corollary 2.19 [13], Theorems 2.2, 2.4 [19], Theorems 3.2, 3.3, 3.4 [20] and other known results.

2 Preliminaries

Definition 1 ([21, 22]) *A S - metric on a nonempty set X is a function $S : X^3 \to \mathbb{R}_+$ such that for all $x, y, z, a \in X$:*

$(S_1) : S(x, y, z) = 0$ *if and only if* $x = y = z$;

$(S_2) : S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

The pair (X, S) is called a S - metric space.

Example 1 *Let* $X = \mathbb{R}$ *and* $S(x, y, z) = |x - z| + |y - z|$. *Then,* $S(x, y, z)$ *is a* S *- metric on* $\mathbb{R}$ *and is named the usual* S *- metric on* X.

Lemma 1 **([4, 5])** *If* S *is a* S *- metric on a nonempty set* X, *then*

$$S(x, x, y) = S(y, y, x) \text{ for all } x, y \in X.$$

Definition 2 **([22])** *Let* (X, S) *be a* S *- metric space. For* $r > 0$ *and* $x \in X$ *we define the open ball with center* x *and radius* r, *denoted* $B_S(x, r)$, *respectively closed ball, denoted* $\overline{B}_S(x, r)$, *the sets:*

$$B_S(x, r) = \{y \in X : S(x, x, y) < r\},$$

respectively,

$$\overline{B}_S(x, r) = \{y \in X : S(x, x, y) \leq r\}$$

The topology induced by S - metric on X is the topology determined by the base of all open balls in X.

Definition 3 **([22])** a) *A sequence* $\{x_n\}$ *in a* S *- metric space* (X, S) *is convergent to* x, *denoted* $x_n \to x$ *or* $\lim_{n\to\infty} x_n = x$, *if* $S(x_n, x_n, x) \to 0$ *as* $n \to \infty$, *that is, for* $\varepsilon > 0$, *there exists* $n_0 \in \mathbb{N}$ *such that for all* $n \geq n_0$ *we have* $S(x_n, x_n, x) < \varepsilon$.

b) *A sequence* $\{x_n\}$ *in* (X, S) *is a Cauchy sequence if* $S(x_n, x_n, x_m) \to 0$ *as* $n, m \to \infty$, *that is, for* $\varepsilon > 0$, *there exists* $n_0 \in \mathbb{N}$ *such that for all* $m, n \geq n_0$ *we have* $S(x_n, x_n, x_m) < \varepsilon$.

c) *A* S *- metric space* (X, S) *is complete if every Cauchy sequence is convergent.*

Example 2 (X, S) *by Example 1 is complete.*

Lemma 2 **([22])** *Let* (X, S) *be a* S *- metric space. If* $x_n \to x$ *and* $y_n \to y$, *then* $S(x_n, x_n, y_n) \to S(x, x, y)$.

Lemma 3 **([22])** *Let* (X, S) *be a* S *- metric space and* $x_n \to x$. *Then* $\lim_{n\to\infty} x_n$ *is unique.*

Lemma 4 **([4])** *Let* (X, S) *be a* S *- metric space and* $\{x_n\}$ *be a sequence in* X *such that*

$$\lim_{n\to\infty} S(x_n, x_n, x_{n+1}) = 0.$$

If $\{x_n\}$ *is not a Cauchy sequence, then there exists an* $\varepsilon > 0$ *and two sequences* $\{m_k\}$ *and* $\{n_k\}$ *of positive integers with* $n_k > m_k > k$ *such that*

$$S(x_{m_k}, x_{m_k}, x_{n_k}) \geq \varepsilon,\ S\left(x_{m_{k-1}}, x_{m_{k-1}}, x_{n_k}\right) < \varepsilon$$

and

(i) $\lim_{n\to\infty} S(x_{m_k}, x_{m_k}, x_{n_k}) = \varepsilon$,

(ii) $\lim_{n\to\infty} S\left(x_{m_k}, x_{m_k}, x_{n_{k-1}}\right) = \varepsilon$,

(iii) $\lim_{n\to\infty} S\left(x_{m_{k-1}}, x_{m_{k-1}}, x_{n_k}\right) = \varepsilon$,

(iv) $\lim_{n\to\infty} S\left(x_{m_{k-1}}, x_{m_{k-1}}, x_{n_{k-1}}\right) = \varepsilon$.

Definition 4 ([9]) *Let* Φ *be the set of all functions such that* $\phi : [0,\infty) \to [0,\infty)$ *is a nondecreasing function satisfying* $\lim_{n\to\infty} \phi^n(t) = 0$ *for all* $t \in [0,\infty)$*. If* $\phi \in \Phi$*, then* ϕ *is called* ϕ *- mapping. Furthermore, if* $\phi \in \Phi$*, then:*

(i) $\phi(t) < t$ *for all* $t \in (0,\infty)$,

(ii) $\phi(0) = 0$.

The following theorems are recently published in [23].

Theorem 1 (Theorem 1 [23]) *Let* (X, S) *be a* S *- metric space. Suppose that the mappings* $f, g : X \to X$ *satisfy*

$$S(fx, fy, gz) \leq \phi(\max\{S(gx, gx, fx), S(gy, gy, fy), S(gz, gz, fz)\}) \tag{1}$$

for all $x, y, z \in X$.

If $f(X) \subset g(X)$ *and one of* $f(X)$ *or* $g(X)$ *is a complete subspace of* X*, then* f *and* g *have a unique point of coincidence.*

Moreover, if f *and* g *are weakly compatible, then* f *and* g *have a unique common fixed point.*

Theorem 2 (Theorem 2 [23]) *Let* (X, S) *be a* S *- metric space. Suppose that the mappings* $f, g : X \to X$ *satisfy*

$$S(fx, fy, fz) \leq \max\{\phi(S(gx, gx, fx)), \phi(S(gy, gy, fy)), \phi(S(gz, gz, fz))\} \tag{2}$$

for all $x, y, z \in X$.

If $f(X) \subset g(X)$ *and one of* $f(X)$ *or* $g(X)$ *is a complete subspace of* X*, then* f *and* g *have a unique point of coincidence. Moreover, if* f *and* g *are weakly compatible, then* f *and* g *have a unique common fixed point.*

Theorem 3 (Theorem 4 [23]) *Let* (X, S) *be a* S *- metric space. Suppose that the mappings* $f, g : X \to X$ *satisfy*

$$S(fx, fy, fz) \le k_1\phi(S(gx, gx, fx)) + k_2\phi(S(gy, gy, fy)) + k_3\phi(S(gz, gz, fz)) \tag{3}$$

for all $x, y, z \in X$*,* $k_1 + k_2 + k_3 < 1$*.*

If $f(X) \subset g(X)$ *and one of* $f(X)$ *or* $g(X)$ *is a complete subspace of* X*, then* f *and* g *have a unique point of coincidence. Moreover, if* f *and* g *are weakly compatible, then* f *and* g *have a unique common fixed point.*

Remark 1 1) *Since* $\phi(t)$ *is nondecreasing, then*

$$\phi(\max\{t_2, t_3, t_4, t_5, t_6\}) = \max\{\phi(t_2), \phi(t_3), \phi(t_4), \phi(t_5), \phi(t_6)\}.$$

Hence, Theorem 2 is Theorem 1.

2) *By* (3) *we obtain*

$$\begin{aligned} S(fx, fy, fz) &\le (k_1 + k_2 + k_3)\max\{\phi(S(gx, gx, fx)), \\ &\qquad \phi(S(gy, gy, fy)), \phi(S(gz, gz, fz))\} \\ &= (k_1 + k_2 + k_3)\phi(\max\{S(gx, gx, fx), S(gy, gy, fy), \\ &\qquad S(gz, gz, fz)\}) \\ &\le \phi(\max\{\max\{S(gx, gx, fx), S(gy, gy, fy), S(gz, gz, fz)\}\}). \end{aligned}$$

Hence,

$$S(fx, fy, fz) \le \phi(\max\{S(gx, gx, fx), S(gy, gy, fy), S(gz, gz, fz)\}),$$

which is the inequality (1)*. Hence, Theorem 3 is a particular case of Theorem 1.*

3) *In the proof of Theorem 1 is used* $x = y$*. Hence in Theorem 1 we have a new form of inequality* (1)*:*

$$S(fx, fx, fy) \le \phi(\max\{S(gx, gx, fx), g(fy, gy, fy)\}).$$

3 ϕ - implicit relations

Let $\mathcal{F}_\phi$ be the set of all lower semi - continuous functions $F : \mathbb{R}_+^6 \to \mathbb{R}$ such that:

(F_1) : F is nonincreasing in variable t_6,

(F_2) : There exists $\phi \in \mathcal{F}_\phi$ such that for all $u, v \geq 0$, $F(u, v, v, u, 0, 2u + v) \leq 0$ implies $u \leq \phi(v)$;

(F_3) : $F(t, t, 0, 0, t, t) > 0, \ \forall t > 0$.

In all the following examples, (F_1) is obviously.

Example 3 $F(t_1, ..., t_6) = t_1 - k \max\{t_2, t_3, ..., t_6\}$, *where* $k \in \left[0, \frac{1}{3}\right)$.

(F_2) : Let $u, v \geq 0$ and $F(u, v, v, u, 0, 2u + v) = u - k(u + 2v) \leq 0$. If $u > v$, then $u(1 - 3k) \leq 0$, a contradiction. Hence, $u \leq v$, which implies $u \leq 3kv$ and F satisfies (F_2) for $\phi(t) = 3kt$.

(F_3) : $F(t, t, 0, 0, t, t) = t(1 - k) > 0, \ \forall t > 0$.

Example 4 $F(t_1, ..., t_6) = t_1 - k \max\left\{t_2, t_3, t_4, \frac{t_5 + t_6}{3}\right\}$, *where* $k \in [0, 1)$.

(F_2) : Let $u, v \geq 0$ and $F(u, v, v, u, 0, 2u + v) = u - k \max\left\{u, v, \frac{2u + v}{3}\right\} \leq 0$. If $u > v$, then $u(1 - k) \leq 0$, a contradiction. Hence, $u \leq v$, which implies $u \leq kv$ and F satisfies (F_2) for $\phi(t) = kt$.

(F_3) : $F(t, t, 0, 0, t, t) = t(1 - k) > 0, \ \forall t > 0$.

Example 5 $F(t_1, ..., t_6) = t_1 - at_2 - bt_3 - ct_4 - dt_5 - et_6$, *where* $a, b, c, d, e \geq 0$ *and* $a + b + c + 3e < 1$ *and* $a + d + e < 1$.

(F_2) : Let $u, v \geq 0$ and $F(u, v, v, u, 0, 2u + v) = u - av - bv - cu - e(2u + v) \leq 0$. If $u > v$, then $u[1 - (a + b + c + 3e)] \leq 0$, a contradiction. Hence, $u \leq v$, which implies $u \leq (a + b + c + 3e)v$ and F satisfies (F_2) for $\phi(t) = (a + b + c + 3e)t$.

(F_3) : $F(t, t, 0, 0, t, t) = t[1 - (a + d + e)] > 0, \ \forall t > 0$.

Example 6 $F(t_1, ..., t_6) = t_1^2 - t_1(at_2 + bt_3 + ct_4) - dt_5t_6$, *where* $a, b, c, d \geq 0$, $a + b + c < 1$ *and* $a + d < 1$.

(F_2) : Let $u, v \geq 0$ and $F(u, v, v, u, 0, 2u + v) = u^2 - u(av + bv + cu) \leq 0$. If $u > v$, then $u^2[1 - (a + b + c)] \leq 0$, a contradiction. Hence, $u \leq v$, which implies $u \leq (a + b + c)v$ and F satisfies (F_2) for $\phi(t) = (a + b + c)t$.

(F_3) : $F(t, t, 0, 0, t, t) = t^2[1 - (a + d)] > 0, \ \forall t > 0$.

Example 7 $F(t_1, ..., t_6) = t_1^2 - at_2^2 - \frac{bt_5t_6}{1 + t_3^2 + t_4^2}$, *where* $a, b \geq 0$ *and* $a + b < 1$.

(F_2): Let $u, v \geq 0$ and $F(u, v, v, u, 0, 2u + v) = u^2 - av^2 \leq 0$, which implies $u \leq \sqrt{a}v$. Hence, F satisfies (F_2) for $\phi(t) = \sqrt{a}t$.

(F_3): $F(t, t, 0, 0, t, t) = t^2[1 - (a + b)] > 0, \ \forall t > 0$.

In the following examples, if $\phi \in \Phi$, then F satisfy properties (F_1), (F_2), (F_3).

Example 8 $F(t_1, ..., t_6) = t_1 - \phi\left(\max\left\{t_2, t_3, t_4, \dfrac{t_5 + t_6}{3}\right\}\right)$.

(F_2): Let $u, v \geq 0$ and

$$F(u, v, v, u, 0, 2u + v) = u - \phi\left(\max\left\{u, v, \frac{2u + v}{3}\right\}\right) \leq 0.$$

If $u > v$, then $u \leq \phi(u) < u$, a contradiction. Hence, $u \leq v$, which implies $u \leq \phi(v)$.

(F_3): $F(t, t, 0, 0, t, t) = t - \phi(t) > 0, \ \forall t > 0$.

Example 9 $F(t_1, ..., t_6) = t_1 - \phi\left(\max\left\{t_2, \dfrac{t_3 + t_4}{2}, \dfrac{t_5 + t_6}{3}\right\}\right)$.

(F_2): Let $u, v \geq 0$ and

$$F(u, v, v, u, 0, 2u + v) = u - \phi\left(\max\left\{u, \frac{u + v}{2}, \frac{2u + v}{3}\right\}\right) \leq 0.$$

If $u > v$, then $u \leq \phi(u) < u$, a contradiction. Hence, $u \leq v$, which implies $u \leq \phi(v)$.

(F_3): $F(t, t, 0, 0, t, t) = t - \phi(t) > 0, \ \forall t > 0$.

Example 10 $F(t_1, ..., t_6) = t_1 - \phi(at_2 + b\max\{t_3, t_4\} + c\max\{t_5, t_6\})$, *where* $a, b, c \geq 0$ *and* $a + b + 3c < 1$.

(F_2): Let $u, v \geq 0$ and

$$F(u, v, v, u, 0, 2u + v) = u - \phi(av + b\max\{u, v\} + c(2u + v)) \leq 0.$$

If $u > v$, then $u - \phi((a + b + 3c)u) \leq 0$, which implies $u \leq \phi(u) < u$, a contradiction. Hence, $u \leq v$ and $u \leq \phi(v)$.

(F_3): $F(t, t, 0, 0, t, t) = t - \phi(at + ct) \geq t - \phi((a + b + 3c)t) \geq t - \phi(t) > 0, \ \forall t > 0$.

Example 11 $F(t_1, ..., t_6) = t_1 - \phi\left(a\sqrt{t_1t_2} + b\sqrt{t_3t_4} + c\sqrt{t_5t_6}\right)$, *where* $a, b, c \geq 0$ *and* $a + b + c < 1$.

(F_2) : Let $u, v \geq 0$ and $F(u, v, v, u, 0, 2u + v) = u - \phi\left(a\sqrt{uv} + b\sqrt{uv}\right) \leq 0$. If $u > v$, then $u \leq \phi((a + b) u) < u$, a contradiction. Hence, $u \leq v$, which implies $u \leq \phi(v)$.

(F_3) : $F(t, t, 0, 0, t, t) = t - \phi((a + c) t) \geq t - \phi((a + b + c) t) \geq t - \phi(t) > 0, \ \forall t > 0$.

Example 12 $F(t_1, ..., t_6) = t_1 - \phi\left(at_2, \dfrac{b\sqrt{t_5 t_6}}{1 + t_3 + t_4}\right)$, *where* $a, b \geq 0$ *and* $a + b < 1$.

(F_2) : Let $u, v \geq 0$ and $F(u, v, v, u, 0, 2u + v) = u - \phi(av) \leq 0$. If $u > v$, then $u - \phi(av) \leq 0$ implies $u \leq \phi(u) < u$, a contradiction. Hence, $u \leq v$, which implies $u \leq \phi(v)$.

(F_3) : $F(t, t, 0, 0, t, t) = t - \phi((a + b) t) \geq t - \phi(t) > 0, \ \forall t > 0$.

In the following examples, the proofs are similar to the proof of Example 12 and thus are omitted.

Example 13 $F(t_1, ..., t_6) = t_1 - at_2 - b\max\{t_3, t_4, t_5, t_6\}$, *where* $a, b \geq 0$ *and* $a + 3b < 1$.

If $F(u, v, v, u, 0, 2u + v) \leq 0$, then we have $u \leq \phi(v)$, where $\phi(t) = (a + 3b)t$.

Example 14 $F(t_1, ..., t_6) = t_1 - at_2 - bt_3 - ct_4 - d\max\{t_5, t_6\}$, *where* $a, b, c, d \geq 0$ *and* $a + b + c + 3d < 1$.

If $F(u, v, v, u, 0, 2u + v) \leq 0$ then we have $u \leq \phi(v)$, where $\phi(t) = (a + b + c + 3d)t$.

Example 15 $F(t_1, ..., t_6) = t_1 - at_2 - d\max\{t_3, t_4\} - bt_5 - ct_6$, *where* $a, b, c, d \geq 0$, $a + 3c + d \geq 0$, $a + 3c + d < 1$ *and* $a + b + c < 1$.

If $F(u, v, v, u, 0, 2u + v) \leq 0$ then $u \leq \phi(v)$, where $\phi(t) = (a + 3c + d)\, t$.

Example 16 $F(t_1, ..., t_6) = t_1 - at_2 - bt_3 - et_4 - ct_5 - dt_6 - f\max\{t_2, t_3, ..., t_6\}$, *where* $a, b, c, d, e, f \geq 0$, $a + b + e + 3d + 3f < 1$ *and* $a + c + e + f < 1$.

If $F(u, v, v, u, 0, 2u + v) \leq 0$ then $u \leq \phi(v)$, where $\phi(t) = (a + b + e + 3d + 3f)t$.

Example 17 $F(t_1, ..., t_6) = t_1 - a(t_5 + t_6) - bt_2 - c\max\{t_3, t_4\}$, *where* $a, b, c \geq 0$ *and* $3a + b + c < 1$.

If $F(u, v, v, u, 0, 2u+v) \le 0$ then $u \le \phi(v)$, where $\phi(t) = (3a+b+c)\,t$.

Example 18 $F(t_1, ..., t_6) = t_1 - a(t_3+t_4) - bt_2 - c\max\{t_5, t_6\}$, *where* $a, b, c \ge 0$ *and* $2a+b+3c < 1$.

If $F(u, v, v, u, 0, 2u+v) \le 0$ then $u \le \phi(v)$, where $\phi(t) = (2a+b+3c)\,t$.

Example 19 $F(t_1, ..., t_6) = t_1 - a\max\{t_4+t_5, t_3+t_6\} - bt_2$, *where* $a, b, c \ge 0$ *and* $4a+b < 1$.

If $F(u, v, v, u, 0, 2u+v) \le 0$ then $u \le \phi(v)$, where $\phi(t) = (4a+b)\,t$.

4 Main results

Lemma 5 ([1]) *Let* f *and* g *be weakly compatible self mappings of a nonempty set* X*. If* f *and* g *have a unique point of coincidence* $w = fx = gx$ *for some* $x \in X$*, then* w *is the unique common fixed point of* f *and* g*.*

Theorem 4 *Let* (X, S) *be a* S *- metric space and* $f, g : X \to X$ *such that*

$$F\left(\begin{array}{c} S(fx, fx, fy), S(gx, gx, gy), S(gx, gx, fx), \\ S(gy, gy, fy), S(gy, gy, fx), S(gx, gx, fy) \end{array}\right) \le 0 \tag{4}$$

for all $x, y \in X$ *and some* $F \in \mathcal{F}_\phi$*.*

If $f(X) \subset g(X)$ *(or* $g(X) \subset f(X)$*) and* $g(X)$ *(or* $f(X)$*) is a complete subspace of* (X, S)*, then* f *and* g *have a unique point of coincidence. Moreover, if* f *and* g *are weakly compatible, then* f *and* g *have a unique common fixed point.*

Proof. Let x_0 be an arbitrary point of X. Since $f(X) \subset g(X)$, there exists $x_1 \in X$ such that $fx_0 = gx_1$. Continuing this process we define the sequence $\{x_n\}$ satisfying

$$fx_n = gx_{n+1} \text{ for } n \in \mathbb{N}.$$

Then, by (4) for $x = x_{n-1}$ and $y = x_n$ we have

$$F\left(\begin{array}{c} S(fx_{n-1}, fx_{n-1}, fx_n), S(gx_{n-1}, gx_{n-1}, gx_n), S(gx_{n-1}, gx_{n-1}, fx_{n-1}), \\ S(gx_n, gx_n, fx_n), S(gx_n, gx_n, fx_{n-1}), S(gx_{n-1}, gx_{n-1}, fx_n) \end{array}\right) \le 0$$

$$F\left(\begin{array}{c} S(gx_n, gx_n, gx_{n+1}), S(gx_{n-1}, gx_{n-1}, gx_n), S(gx_{n-1}, gx_{n-1}, gx_n), \\ S(gx_n, gx_n, gx_{n+1}), 0, S(gx_{n-1}, gx_{n-1}, gx_{n+1}) \end{array}\right) \le 0 \tag{5}$$

By (S_2) and Lemma 1 we have

$$S(gx_{n-1}, gx_{n-1}, gx_{n+1}) \leq 2S(gx_n, gx_n, gx_{n+1}) + S(gx_{n-1}, gx_{n-1}, gx_n).$$

By (5) and (F_1) we obtain

$$F\left(\begin{array}{c} S(gx_n, gx_n, gx_{n+1}), S(gx_{n-1}, gx_{n-1}, gx_n), S(gx_{n-1}, gx_{n-1}, gx_n), \\ S(gx_n, gx_n, gx_{n+1}), 0, 2S(gx_n, gx_n, gx_{n+1}) + S(gx_{n-1}, gx_{n-1}, gx_n) \end{array}\right) \leq 0.$$

By (F_2) we obtain

$$S(gx_n, gx_n, gx_{n+1}) \leq \phi(S(gx_{n-1}, gx_{n-1}, gx_n)), \text{ for } n = 1, 2, ...$$

which implies

$$S(gx_n, gx_n, gx_{n+1}) \leq \phi^n(S(gx_0, gx_0, gx_1)).$$

Letting n tend to infinity we obtain

$$\lim_{n\to\infty} S(gx_n, gx_n, gx_{n+1}) = 0.$$

We prove that $\{gx_n\}$ is a Cauchy sequence in $g(X)$. Suppose that $\{gx_n\}$ is not a Cauchy sequence. Then, by Lemma 4, there exists an $\varepsilon > 0$ and two sequences m_k and n_k with $n_k > m_k > k$ and $S(x_{m_k}, x_{m_k}, x_{n_k}) \geq \varepsilon$ and $S(x_{m_k-1}, x_{m_k-1}, x_{n_k}) < \varepsilon$ and satisfying the inequalities (i) - (iv) by Lemma 4.

By (4) for $x = x_{m_k-1}$ and $y = x_{n_k-1}$ we have

$$F\left(\begin{array}{c} S(fx_{m_k-1}, fx_{m_k-1}, fx_{n_k-1}), S(gx_{m_k-1}, gx_{m_k-1}, gx_{n_k-1}), \\ S(gx_{m_k-1}, gx_{m_k-1}, fx_{m_k-1}), S(gx_{n_k-1}, gx_{n_k-1}, fx_{n_k-1}), \\ S(gx_{n_k-1}, gx_{n_k-1}, fx_{m_k-1}), S(gx_{m_k-1}, gx_{m_k-1}, fx_{n_k-1}) \end{array}\right) \leq 0$$

$$F\left(\begin{array}{c} S(gx_{m_k}, gx_{m_k}, gx_{n_k}), S(gx_{m_k-1}, gx_{m_k-1}, gx_{n_k-1}), \\ S(gx_{m_k-1}, gx_{m_k-1}, gx_{m_k}), S(gx_{n_k-1}, gx_{n_k-1}, gx_{n_k}), \\ S(gx_{n_k-1}, gx_{n_k-1}, gx_{m_k}), S(gx_{m_k-1}, gx_{m_k-1}, gx_{n_k}) \end{array}\right) \leq 0. \tag{6}$$

By Lemma 1,

$$S(gx_{m_k-1}, gx_{m_k-1}, gx_{n_k}) = S(gx_{n_k}, gx_{n_k}, gx_{m_k-1})$$

and

$$S(gx_{n_k-1}, gx_{n_k-1}, gx_{m_k}) = S(gx_{m_k}, gx_{m_k}, gx_{n_k-1}).$$

Letting n tend to infinity in (6) we obtain

$$F(\varepsilon, \varepsilon, 0, 0, \varepsilon, \varepsilon) \leq 0,$$

a contradiction of (F_3).

Hence, $\{gx_n\}$ is a Cauchy sequence in $g(X)$. Since $g(X)$ is complete, then $\{gx_n\}$ is convergent to a point $t \in g(X)$. Hence, there exists $p \in X$ such that $gp = t$ and $\lim_{n\to\infty} gx_n = gp$. We prove that $fp = gp$.

By (4) for $x = x_n$ and $y = p$ we have

$$F\left(\begin{array}{c} S(gx_n, gx_n, fp), S(gx_n, gx_n, gp), S(gx_n, gx_n, fx_n), \\ S(gp, gp, fp), S(gp, gp, fx_n), S(gx_n, gx_n, fp) \end{array}\right) \leq 0.$$

Letting n tend to infinity we obtain

$$F(S(gp, gp, fp), 0, 0, S(gp, gp, fp), 0, S(gp, gp, fp)) \leq 0.$$

By (F_1) we have

$$F(S(gp, gp, fp), 0, 0, S(gp, gp, fp), 0, 2S(gp, gp, fp)) \leq 0,$$

which implies $S(gp, gp, fp) = 0$. Hence $gp = fp = t$.

We prove that t is the unique point of coincidence of f and g. Suppose that there exists $z = fw = gw$. By (4) we obtain

$$F\left(\begin{array}{c} S(fp, fp, fw), S(gp, gp, gw), S(gp, gp, fp), \\ S(gw, gw, fw), S(gw, gw, fp), S(gp, gp, fw) \end{array}\right) \leq 0,$$

$$F(S(t, t, z), S(t, t, z), 0, 0, S(z, z, t), S(t, t, z)) \leq 0.$$

By Lemma 1 we have

$$F(S(t, t, z), S(t, t, z), 0, 0, S(t, t, z), S(t, t, z)) \leq 0,$$

a contradiction of (F_3) if $S(t, t, z) > 0$. Hence, $z = t$ and t is the unique point of coincidence of f and g.

Moreover, if f and g are weakly compatible, then by Lemma 5, f and g have a unique common fixed point t. $\square$

If $\phi(t) = kt$, $k \in [0, 1)$, by Example 8 and Theorem 4 we obtain

Corollary 1 *Let (X, S) be a S - metric space and $f, g : X \to X$ such that*

$$S(fx, fx, fy) \leq k \max \left\{ \begin{array}{c} S(gx, gx, gy), S(gx, gx, fx), S(gy, gy, fy), \\ \frac{S(gy, gy, fx) + S(gx, gx, fy)}{3} \end{array} \right\} \quad (7)$$

where $k \in [0, 1)$.

If $f(X) \subset g(X)$ *(or* $g(X) \subset f(X)$*) and* $g(X)$ *(or* $f(X)$*) is a complete subspace of* (X, S)*, then* f *and* g *have a unique point of coincidence. Moreover, if* f *and* g *are weakly compatible, then* f *and* g *have a unique common fixed point.*

Example 20 *Let* $X = \mathbb{R}$ *and* $S(x, y, z) = |x - z| + |y - z|$*. Then* $S(X)$ *is a complete* S *- metric space. Let* $fx = 2x - 2$*,* $gx = 3x - 4$*. Then* $f(X) = \mathbb{R}$*,* $g(X) = \mathbb{R}$ *and* $f(X) \subset g(X)$*. If* $fx = gx$*, then* $x = 2$ *which implies* $\mathcal{C}(f, g) = \{2\}$ *and* $fg2 = gf2 = 2$ *and* $x = 2$ *is the unique point of coincidence of* f *and* g *and* f *and* g *are weakly compatible. On the other hand,* $S(fx, fx, fy) = 4|x - z|$ *and* $S(gx, gx, gy) = 6|x - y|$*. Hence,* $S(fx, fx, fy) \leq kS(gx, gx, gy)$*, for* $k \in \left[\frac{2}{3}, 1\right)$*. This implies*

$$S(fx, fx, fy) \leq k \max \left\{ \begin{array}{c} S(gx, gx, gy), S(gx, gx, fx), S(gy, gy, fy), \\ \dfrac{S(gy, gy, fx) + S(gx, gx, fy)}{3} \end{array} \right\}$$

for $k \in \left[\frac{2}{3}, 1\right)$*. By Corollary 1,* f *and* g *have a unique common fixed point* $x = 2$*.*

If $g(x) = x$, then by Theorem 4 we obtain

Theorem 5 *Let* (X, S) *be a complete* S *- metric space and* $f : X \to X$ *such that*

$$F(S(fx, fx, fy), S(x, x, y), S(x, x, fx), S(y, y, fy), S(y, y, fx), S(x, x, fy)) \leq 0,$$

for all $x, y \in X$ *and some* $F \in \mathcal{F}_\phi$*.*

Then f *has a unique fixed point.*

Corollary 2 *Let* (X, S) *be a complete* S *- metric space and* $f : X \to X$ *such that*

$$S(fx, fx, fy) \leq k \max\{S(x, x, y), S(x, x, fx), S(y, y, fy), S(x, x, fy), S(x, x, fy)\},$$

for all $x, y \in X$ *and* $k \in \left[0, \frac{1}{3}\right]$*. Then* f *has a unique fixed point.*

Proof. The proof follows by Theorem 5 and Example 4. $\square$

Remark 2 1) *By Examples 13 - 19 and Theorem 4 we obtain Theorems 1-7* [13].
2) *By Example 4 and Theorem 4 we obtain Corollary 2.19* [13].
3) *By Example 5 and Theorem 4 we obtain Theorems 2.2, 2.4* [19] *and Theorems 3.2, 3.3, 3.4* [20].

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