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On sums of monotone functions over smooth numbers

Gábor Román
Eötvös Loránd University, Hungary
email: romangabor@caesar.elte.hu

Abstract. In this article, we are going to look at the requirements regarding a monotone function $f \in \mathbb{R} \to \mathbb{R}_{\geq 0}$, and regarding the sets of natural numbers $(A_i)_{i=1}^{\infty} \subseteq \mathrm{dmn}(f)$, which requirements are sufficient for the asymptotic

$$\sum_{\substack{n \in A_N \\ P(n) \leq N^{\theta}}} f(n) \sim \rho(1/\theta) \sum_{n \in A_N} f(n)$$

to hold, where N is a positive integer, $\theta \in (0,1)$ is a constant, $P(n)$ denotes the largest prime factor of n, and ρ is the Dickman function.

1 Introduction

In his article [3], Croot gave a sufficient condition to express sums of non-negative functions over smooth natural numbers, using the Dickman function ρ. The result can be summarized as

$$\sum_{\substack{1 \leq n \leq N \\ P(n) \leq N^{\theta}}} f(n) \sim \rho(1/\theta) \sum_{1 \leq n \leq N} f(n) \tag{1}$$

where f is a non-negative function defined over $\mathbb{N}$, $\theta \in (0,1)$ is a constant, and $P(n)$ denotes the largest prime factor of n, with the convention that $P(1) = 1$.

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The Dickman function can be taken as the limit

$$\rho(1/\theta) = \lim_{N\to\infty} \frac{\Psi(N, N^\theta)}{N} \tag{2}$$

which limit exists if $\theta > 0$, see the article of Dickman [4]. Here $\Psi(x, y)$ is the count of y-smooth positive integers smaller than-, or equal to x. For a recollection about the behavior of the function ρ, and about smooth integers, see article [6], and chapter III.5 in [7].

The method of Croot is specialized for the problem tackled by him, and it is difficult to apply in more general situations. We are going to look at when we can say that the asymptotic equality (1) holds, based on properties of the examined function, which properties are easier to check.

Based on the properties of the function Ψ, it is easy to see that the idea works for functions $f(n) := c$, with any real constant c, as the equalities

$$\sum_{\substack{1\le n\le N\\ P(n)\le N^\theta}} c = c\Psi(N, N^\theta) = \frac{\Psi(N, N^\theta)}{N} \sum_{1\le n\le N} c$$

hold. We are expecting a similar result for more general functions. Concerning the basic properties of the examined functions, we expect them to be non-negative, monotone changing functions, which are not the constant zero function. As we are going to apply Abel's identity to handle certain sums, a heavier requirement arises, namely that the examined functions should be continuously differentiable.

A sufficient condition for (1) to hold is — informally — that f shouldn't change too fast. To introduce the concept in iterations, first we say that $f(x)$ should be in $o(x^\alpha) \cap \omega(x^{-\alpha})$ for every $\alpha > 0$, so f should be changing with at most the speed of the polylogarithmic functions or their reciprocals. As a second iteration, because we will bound the derivative of f, we will actually need a bit stronger requirement, namely that $f'(x)$ should be in $o(x^{\alpha-1})$ while $f(x) \in \omega(x^{-\alpha})$ for every $\alpha > 0$. (We need this, because differentiation doesn't preserve inequalities.) As a third, and final iteration, we can actually lighten these requirements a bit. Let

$$L_1 := \{f \in \mathbb{R} \to \mathbb{R} : \forall\alpha > 0, f'(x) \in \mathcal{O}(x^{\alpha-1}) \wedge f(x) \in \omega(x^{-\alpha})\}$$

and

$$L_2 := \{f \in \mathbb{R} \to \mathbb{R} : \forall\alpha > 0, f'(x) \in o(x^{\alpha-1}) \wedge f(x) \in \Omega(x^{-\alpha})\}$$

then let $L := L_1 \cup L_2$. We will show that $f \in L$ is a sufficient condition for (1) to hold. It's worth mentioning, that we cannot lighten both conditions at the same time. (Regarding the asymptotic notation, we refer to section 3.1 of [2], and to section 4.1.1 of [5]. Take note that we use these notations in the sense that they express a bound on the *absolute value* of the examined function.)

As a final generalization, instead of looking at the sum going from some initial positive value up until N, we will sum the examined function over some sets $(A_i)_{i=1}^{\infty} \subseteq \mathrm{dmn}(f)$. The only requirement concerning these sets is that they should be "dense" among the natural numbers, i. e. $|A_N| \sim N$ should hold.

Proposition 1 *Let $\theta \in (0,1)$, $m \in \mathbb{N}$, and let $f : [m, +\infty) \to \mathbb{R}_{\geq 0}$ be a monotone, continuously differentiable function which is in L. Take the sets $(A_i)_{i=1}^{\infty} \subseteq \{m, \dots, N\}$, where $N > m$ is an integer, which sets satisfy $|A_N| \sim N$. Then*

$$\sum_{\substack{n \in A_N \\ P(n) \leq N^{\theta}}} f(n) \sim \rho(1/\theta) \sum_{n \in A_N} f(n).$$

2 Proof of the proposition

First, we separately prove a lemma, which we are going to use after the application of Abel's identity, to bound the remaining integral term.

Lemma 1 *Let $m \in \mathbb{N}$, and let $f : [m, +\infty) \to \mathbb{R}_{\geq 0}$ be a monotone, continuously differentiable function which is in L. Then*

$$\frac{1}{f(x)} \int_m^x \lfloor t \rfloor |f'(t)| \, dt \in o(x).$$

Proof.

- Assume that $f \in L_1$, and take an arbitrary real $\alpha > 0$. Then because $f'(x) \in \mathcal{O}(x^{\alpha-1})$, there exists a real $c > 0$, and a real x_c, such that for every real $x > x_c$, we have that $|f'(x)| \leq c x^{\alpha-1}$ holds. So the inequality

$$\frac{1}{f(x)} \int_m^x \lfloor t \rfloor |f'(t)| \, dt < \frac{c}{f(x)} \int_m^x t^{\alpha} \, dt \tag{3}$$

holds when $x > \max(m, x_c)$. Because $f(x) \in \omega(x^{-\alpha})$, for every real $\varepsilon > 0$, there exists a real x_ε, such that for every real $x > x_\varepsilon$, we have that $|f(x)| > \varepsilon x^{-\alpha}$ holds. By this, when $x > \max(m, x_c, x_\varepsilon)$, the right hand side of inequality (3) is smaller than

$$\frac{c}{\varepsilon x^{-\alpha}} \int_m^x t^\alpha \, dt < \frac{c}{\varepsilon(\alpha+1)} x^{2\alpha+1} \to \frac{c}{\varepsilon} x$$

as α goes to zero. So for every real $\delta = c/\varepsilon > 0$, there exists a real $x_\delta = \max(m, x_c, x_\varepsilon)$, such that for every real $x > x_\delta$, we have the left hand side of (3) is smaller than δx.

- Assume that $f \in L_2$, and take an arbitrary real $\alpha > 0$. Then because $f(x) \in \Omega(x^{-\alpha})$, there exists a real $c > 0$, and a real x_c, such that for every real $x > x_c$, we have that $|f(x)| \geq c x^{-\alpha}$ holds. So the inequality

$$\frac{1}{f(x)} \int_m^x \lfloor t \rfloor |f'(t)| \, dt \leq \frac{1}{c x^{-\alpha}} \int_m^x \lfloor t \rfloor |f'(t)| \, dt \tag{4}$$

holds when $x > \max(m, x_c)$. Because $f'(x) \in o(x^{\alpha-1})$, for every real $\varepsilon > 0$, there exists a real x_ε, such that for every real $x > x_\varepsilon$, we have that $|f'(x)| < \varepsilon x^{\alpha-1}$ holds. By this, when $x > \max(m, x_c, x_\varepsilon)$, the right hand side of inequality (4) is smaller than

$$\frac{\varepsilon}{c x^{-\alpha}} \int_m^x t^\alpha \, dt < \frac{\varepsilon}{c(\alpha+1)} x^{2\alpha+1} \to \frac{\varepsilon}{c} x$$

as α goes to zero. So for every real $\delta = \varepsilon/c > 0$, there exists a real $x_\delta = \max(m, x_c, x_\varepsilon)$, such that for every real $x > x_\delta$, we have the left hand side of (4) is smaller than δx.

□

Now we are going to give an asymptotic for the sum of our examined function over the sets A_N by using Abel's identity.

Lemma 2 *Let* $m \in \mathbb{N}$, *and let* $f : [m, +\infty) \to \mathbb{R}_{\geq 0}$ *be a monotone, continuously differentiable function which is in* L. *Take the sets* $(A_i)_{i=1}^{\infty} \subseteq \{m, \ldots, N\}$, *where* $N > m$ *is an integer, which sets satisfy* $|A_N| \sim N$. *Then*

$$\sum_{n \in A_N} f(n) \sim N f(N).$$

Proof. First, we split the examined sum as

$$\sum_{n \in A_N} f(n) = \sum_{m \leq n \leq N} f(n) - \sum_{n \in \{m, \ldots, N\} \setminus A_N} f(n). \tag{5}$$

Because f has a continuous derivative on the interval $[m, +\infty)$, we can apply Abel's identity, see theorem 4.2 in section 4.3 of the book of Apostol [1], to get the equality

$$\sum_{m<n\leq N} f(n) = Nf(N) - mf(m) - \int_m^N \lfloor t \rfloor f'(t)\,dt. \tag{6}$$

- Assume that f is monotone increasing. Then

  $$\sum_{n\in\{m,\ldots,N\}\setminus A_N} f(n) \leq f(N)(N - m + 1 - |A_N|).$$

  Using this inequality, and equality (6), we get that the left hand side of equality (5) is greater than-, or equal to

  $$f(N)\left((1-m)\frac{f(m)}{f(N)} - \frac{1}{f(N)}\int_m^N \lfloor t \rfloor f'(t)\,dt + m - 1 + |A_N|\right)$$

  which, by lemma 1, is greater than-, or equal to $f(N)\,(|A_N| + o(N))$. By this, we have that the limit

  $$\lim_{N\to+\infty} \frac{\sum_{n\in A_N} f(n)}{Nf(N)} \geq \lim_{N\to+\infty}\left(\frac{|A_N|}{N} + o_N(1)\right) = 1$$

  because $|A_N| \sim N$. Regarding the upper bound of the limit, we have

  $$\lim_{N\to+\infty} \frac{\sum_{n\in A_N} f(n)}{Nf(N)} \leq \lim_{N\to+\infty} \frac{f(N)\sum_{n\in A_N} 1}{Nf(N)} = \lim_{N\to+\infty} \frac{|A_N|}{N} = 1$$

  because f is monotone increasing, and $|A_N| \sim N$.

- Assume that f is monotone decreasing. Then

  $$\sum_{n\in\{m,\ldots,N\}\setminus A_N} f(n) \geq f(N)(N - m + 1 - |A_N|).$$

  Using this inequality, and equality (6), we get that the left hand side of equality (5) is less than-, or equal to

  $$f(N)\left((1-m)\frac{f(m)}{f(N)} + \frac{1}{f(N)}\int_m^N \lfloor t \rfloor |f'(t)|\,dt + m - 1 + |A_N|\right)$$

where we could switch the sign of the integral, because f is monotone decreasing, so f' is non-positive on $[m, N]$. By lemma 1, this is less than-, or equal to $f(N)(|A_N| + o(N))$. Based on this, using the same reasoning as in the case when f was monotone increasing, we can show that

$$\lim_{N\to+\infty} \frac{\sum_{n\in A_N} f(n)}{Nf(N)} \leq 1$$

holds. Regarding the lower bound of the limit, we have

$$\lim_{N\to+\infty} \frac{\sum_{n\in A_N} f(n)}{Nf(N)} \geq \lim_{N\to+\infty} \frac{f(N)\sum_{n\in A_N} 1}{Nf(N)} = \lim_{N\to+\infty} \frac{|A_N|}{N} = 1$$

because f is monotone decreasing, and $|A_N| \sim N$.

□

Proof. (Proposition 1) Fix a smoothness $\theta \in (0, 1)$, and assume that we have a function f, and sets A_N satisfying the requirements mentioned in the proposition. We will show that the limit

$$\lim_{N\to+\infty} \frac{\sum_{\substack{n\in A_N \\ P(n)\leq N^\theta}} f(n)}{\rho(1/\theta)\sum_{n\in A_N} f(n)} \tag{7}$$

is equal to one, separately when f is monotone increasing, and when f is monotone decreasing. Assuming that N is big enough, we can guarantee that A_N is not empty, thus the sums in the numerator and the denominator are not zero.

- Assume that f is monotone increasing. Then the limit (7) is less than-, or equal to

$$\lim_{N\to+\infty} \frac{f(N)\Psi(N, N^\theta)}{\rho(1/\theta)\sum_{n\in A_N} f(n)}$$

because f is monotone increasing, and $A_N \subseteq \{m, \ldots, N\}$. Using lemma 2, this is equal to

$$\frac{1}{\rho(1/\theta)} \lim_{N\to+\infty} \frac{\Psi(N, N^\theta)}{N(1 + o_N(1))} = 1$$

based on the limit (2). Regarding the lower bound of the limit, first we note that

$$\sum_{\substack{n\in A_N \\ P(n)\leq N^\theta}} f(n) = \sum_{n\in A_n} f(n) - \sum_{\substack{n\in A_N \\ P(n)>N^\theta}} f(n) \tag{8}$$

where

$$\sum_{\substack{n\in A_N\\ P(n)>N^\theta}} f(n) \le f(N) \sum_{\substack{n\in A_N\\ P(n)>N^\theta}} 1 \le f(N)(N-\Psi(N,N^\theta))$$

because f is monotone increasing, and $A_N \subseteq \{m,\dots,N\}$. By these, we have that the limit (7) is greater than-, or equal to

$$\lim_{N\to+\infty} \frac{\sum_{n\in A_N} f(n) - f(N)(N-\Psi(N,N^\theta))}{\rho(1/\theta)\sum_{n\in A_N} f(n)}$$

where, by using lemma 2, we get

$$\frac{1}{\rho(1/\theta)}\left(1 - \lim_{N\to+\infty}\frac{1}{1+o_N(1)} + \lim_{N\to+\infty}\frac{\Psi(N,N^\theta)}{N(1+o_N(1))}\right) = 1$$

based on the limit (2).

- Assume that f is monotone decreasing. Because

$$\sum_{\substack{n\in A_N\\ P(n)\le N^\theta}} 1 = \sum_{\substack{1\le n\le N\\ P(n)\le N^\theta}} 1 - \sum_{\substack{n\in\{1,\dots,N\}\setminus A_N\\ P(n)\le N^\theta}} 1 \ge \Psi(N,N^\theta) - N + |A_N|$$

we have that the limit (7) is greater than-, or equal to

$$\lim_{N\to+\infty} \frac{f(N)(\Psi(N,N^\theta) - N + |A_N|)}{\rho(1/\theta)\sum_{n\in A_N} f(n)}$$

because f is monotone decreasing. Here, by using lemma 2, we get

$$\frac{1}{\rho(1/\theta)}\lim_{N\to+\infty}\left(\frac{\Psi(N,N^\theta)}{N(1+o_N(1))} - \frac{1}{1+o_N(1)} + \frac{|A_N|}{N(1+o_N(1))}\right) = 1$$

based on $|A_N| \sim N$, and on the limit (2). Regarding the upper bound of the limit, because

$$\sum_{\substack{n\in A_N\\ P(n)>N^\theta}} 1 = \sum_{\substack{1\le n\le N\\ P(n)>N^\theta}} 1 - \sum_{\substack{n\in\{1,\dots,N\}\setminus A_N\\ P(n)>N^\theta}} 1 \ge |A_N| - \Psi(N,N^\theta)$$

we have that the limit (7) is less than-, or equal to

$$\lim_{N\to+\infty} \frac{\sum_{n\in A_N} f(n) - f(N)(|A_N| - \Psi(N,N^\theta))}{\rho(1/\theta)\sum_{n\in A_N} f(n)}$$

based on equality (8). Here, by using lemma 2, we get

$$\frac{1}{\rho(1/\theta)}\left(1-\lim_{N\to+\infty}\frac{|A_N|}{N(1+o_N(1))}+\lim_{N\to+\infty}\frac{\Psi(N,N^{\theta})}{N(1+o_N(1))}\right)=1$$

based on $|A_N| \sim N$, and on the limit (2).

□

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