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Fixed point theorems and equivalence results for classes of multivalued mappings in modular function spaces

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Abstract. We contribute to the development of equivalence of fixed point iterative sequences for multivalued mappings in modular function spaces, by proving the equivalence of convergence of implicit Mann, implicit Ishikawa, implicit Noor, implicit multistep iterative sequences for multivalued $\rho-$ quasi-contractive-like mapppings in modular function spaces. An example is provided to support the applicability of the results. This work is complementary to equivalence results on normed and metric spaces in the literature.

1 Introduction and preliminaries

The existence and approximation of fixed points for multivalued mappings in modular function spaces abound in the literature. Some of the notable authors

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whose research work are very important in this study are ([8], [9], [10], [11], [12], [14], [15] and [16]).

Let Ω be a nonempty set and $\sum$ be a nontrival $\sigma-$algebra of subsets of Ω. Let P be a $\delta-$ring of subsets of Ω such that $E \cap A \in P$ for any $E \in P$ and $A \in \sum$. Assume there exists an increasing sequence $K_n \in P$ such that $\Omega = \bigcup K_n$. Let I_A represent the characteristic function of the set A in Ω. Let ϵ represent the linear space of all simple functions with supports from P. Let M_∞ represent the space of all extended measurable functions, that is, all functions $f : \Omega \to [-\infty, \infty]$ such that there exist a sequence $\{g_n\} \subset \epsilon, |g_n| \leq |f|$ and $g(\omega) \to f(\omega)$ for all $\omega \in \Omega$.

Definition 1 [16] *Let $\rho : M_\infty \to [0, \infty]$ be a nontrivial, convex and even function. We say that ρ is a regular convex function pseudomodular if*

(1) $\rho(0) = 0$;

(2) ρ *is monotone, that is,* $|f(\omega)| \leq g|(\omega)|$ *for any* $\omega \in \Omega$ *implies* $\rho(f) \leq \rho(g)$, *where* $f, g \in M_\infty$;

(3) ρ *is orthogonally subadditive, that is,* $\rho(fI_{A\cup B}) \leq \rho(fI_A) + \rho(fI_B)$ *for any* $A, B \in \sum$ *such that* $A \cap B \neq \phi$, $f \in M_\infty$;

(4) ρ *has Fatou property, that is,* $|f_n(\omega)| \uparrow |f(\omega)|$ *for all* $\omega \in \Omega$ *implies* $\rho(f_n) \uparrow \rho(f)$, *where* $f \in M_\infty$;

(5) ρ *is order continuous in* ϵ, *that is,* $g_n \in \epsilon$ *and* $|g_n(\omega)| \downarrow 0$ *for all* $\omega \in \Omega$ *implies* $\rho(g_n) \downarrow 0$.

Definition 2 [8]. *Let ρ be a regular function pseudomodular;*

(a) *we say that ρ is a regular convex function modular if* $\rho(f) = 0$ *implies* $f = 0$ ρ *-a.e.*

(b) *we say that ρ is a regular convex function semimodular if* $\rho(\alpha f) = 0$ *for every* $\alpha > 0$ *implies* $f = 0$ ρ *-a.e.*

ρ *also satisfies the following properties* [10]*:*

(1) $\rho(0) = 0$ *iff* $f = 0$ ρ *-a.e.*

(2) $\rho(\alpha f) = \rho(f)$ *for every scalar* α *with* $|\alpha| = 1$ *and* $f \in M$.

(3) $\rho(\alpha f + \beta g) \leq \rho(f) + \rho(g)$ *if* $\alpha + \beta = 1$, $\alpha, \beta \geq 0$ *and* $f, g \in M$, ρ *is called a convex modular if, in addition, the following property is satisfied:*

(4) $\rho(\alpha f + \beta g) \leq \alpha\rho(f) + \beta\rho(g)$ *if* $\alpha + \beta = 1$, $\alpha, \beta \geq 0$ *and* $f, g \in M$.

The class of all nonzero regular convex function modulars on Ω *is denoted by* $\mathfrak{R}$.

Definition 3 [16]. *The modular function space* L_ρ *is defined as:* $L_\rho = \{f \in M : \rho(\lambda f) \to 0$ *as* $\lambda \to 0\}$.

In general terms, the modular ρ *is not subadditive and therefore does not behave as a norm or a distance. Nevertheless, the modular space* L_ρ *can be furnished with an* $F-$*norm defined thus:*

$$\|f\|_\rho = \inf\left\{\alpha > 0 : \rho\left(\frac{f}{\alpha}\right) \leq \alpha\right\}.$$

In the instance ρ *is convex modular,* $\|f\|_\rho = \inf\{\alpha > 0 : \rho(\frac{f}{\alpha}) \leq 1\}$ *defines a norm on the modular space* L_ρ. *This type of norm is known as the Luxemburg norm.*

Definition 4 [16]. *A nonzero regular convex function* ρ *is said to satisfy the* Δ_2- *condition, if* $\sup_{n\geq 1} \rho(2f_n, D_k) \to 0$ *as* $k \to \infty$ *whenever* $\{D_k\}$ *decreases to* $\emptyset$ *and* $sup_{n\geq 1}\rho(f_n, D_k) \to 0$ *as* $k \to \infty$. *If* ρ *is convex and satisfies* Δ_2-*condition, then* $L_\rho = E_\rho$.

Definition 5 [16] *Let* ρ *be a nonzero regular convex function modular defined on* Ω.

(i) *Let* $r > 0$, $\epsilon > 0$. *Define* $D_1(r,\epsilon) = \{(f,g) : f,g \in L_\rho, \rho(f), \rho(g) \leq r, \rho(f-g) \geq \epsilon r\}$. *Suppose,* $\delta_1(r,\epsilon) = \inf\{1 - \frac{1}{r}\rho(\frac{f+g}{2}) : (f,g) \in D_1(r,\epsilon)\}$, *if* $D_1(r,\epsilon) \neq \emptyset$ *and* $\delta_1(r,\epsilon) = 1$ *if* $D_1(r,\epsilon) = \emptyset$. *We say that* ρ *satisfies* $(UC1)$, *if for every* $r > 0$, $\epsilon > 0$, $\delta_1(r,\epsilon) > 0$. *Observe that for every* $r > 0$, $D_1(r,\epsilon) \neq \emptyset$, $\epsilon > 0$ *small enough.*

(ii) *We say that* ρ *satisfies* $(UUC1)$, *if for every* $s \geq 0$, $\epsilon > 0$, *there exists* $\eta_1(s,\epsilon) > 0$ *depending only on* s *and* ϵ *such that* $\delta_1(r,\epsilon) > \eta_1(s,\epsilon) > 0$ *for any* $r > s$.

(iii) *Let* $r > 0$, $\epsilon > 0$. *Define* $D_2(r,\epsilon) = \{(f,g) : f,g \in L_\rho, \rho(f), \rho(g) \leq r, \rho(\frac{f-g}{2}) \geq \epsilon r\}$. *Suppose,* $\delta_2(r,\epsilon) = inf\,\{1 - \frac{1}{r}\rho(\frac{f+g}{2}) : (f,g) \in D_2(r,\epsilon)\}$ *if* $D_2(r,\epsilon) \neq \emptyset$ *and* $\delta_2(r,\epsilon) = 1$ *if* $D_2(r,\epsilon) = \emptyset$. *We say that* ρ *satisfies* $(UC2)$, *if for every* $r > 0$, $\epsilon > 0$, $\delta_2(r,\epsilon) > 0$. *Observe that for every* $r > 0$, $D_2(r,\epsilon) \neq \emptyset$, $\epsilon > 0$ *small enough.*

(iv) *We say that* ρ *satisfies* $(UUC2)$, *if for every* $s \geq 0$, $\epsilon > 0$, *there exists* $\eta_2(s,\epsilon) > 0$ *depending only on* s *and* ϵ *such that* $\delta_2(r,\epsilon) > \eta_2(s,\epsilon) > 0$ *for any* $r > s$.

(v) *We say that* ρ *is strictly convex* (SC), *if for every* $f, g \in L_p$ *such that* $\rho(f) = \rho(g)$ *and* $\rho(\frac{f+g}{2}) = \frac{\rho(f)+\rho(g)}{2}$, *there holds* $f = g$.

Definition 6 [16]. *Let* L_ρ *be a modular space. The sequence* $\{f_n\} \in L_\rho$ *is called:*

(1) $\rho-$*convergent to* $f \in L_\rho$, *if* $\rho(f_n - f) \to 0$ *as* $n \to \infty$;

(2) $\rho-$*Cauchy, if* $\rho(f_n - f_m) \to 0$ *as* $n, m \to \infty$.

Remark 1 $\rho-$*convergent sequence implies* $\rho-$*Cauchy sequence, if and only if* $\rho-$ *satisfies the* Δ_2- *condition. However,* ρ *does not satisfy the triangle inequality.*

Definition 7 [16]. *Let* L_ρ *be a modular space. A subset* $D \subset L_\rho$ *is called:*

(1) $\rho-$*closed, if the* $\rho-$*limit of a* $\rho-$*convergent sequence of* D *always belongs to* D;

(2) $\rho-$*a.e. closed, if the* $\rho-$*a.e. limit of a* $\rho-$*a.e. convergent sequence of* D *always belongs to* D;

(3) $\rho-$*compact, if every sequence in* D *has a* $\rho-$*convergent subsequence in* D;

(4) $\rho-$*a.e. compact, if every sequence in* D *has a* $\rho-$*a.e. convergent subsequence in* D;

Definition 8 [16]. *Let* L_ρ *be a modular space. A function* $f \in L_\rho$ *is called a fixed point of a multivalued mapping* $T : L_\rho \to P_\rho(D)$ *if* $f \in Tf$. *The set of all fixed points of* T *is represented by* $F_\rho(T)$.

The following contractive defintions are useful in stating our definitions in terms of functions in modular function spaces. In 1972, Zamfirescu [22] proved a remarkable generalization of the Banach fixed point theorem by employing the following quasi-contractive mapping:

$$d(Tx, Ty) \leq h \max\{d(x, y), \frac{1}{2}[d(x, Tx) + d(y, Ty)], \frac{1}{2}[d(x, Ty) + d(y, Tx)]\}, \quad (1)$$

where $0 \leq h < 1$. In a Normed linear space setting, condition (1) implies

$$\|Tx - Ty\| \leq \delta\|x - y\| + 2\delta\|x - Tx\|, \quad (2)$$

where $0 \leq \delta < 1$ and $\delta = \max\left\{h, \frac{h}{2-h}\right\}$.

In [18], the following contractive definition was used. Let X be a Banach space, for each $x, y \in X$, there exists $\delta \in [0, 1)$ and $L \geq 0$ such that

$$\|Tx - Ty\| \leq \delta\|x - y\| + L\|x - Tx\|. \tag{3}$$

In [13], the following contractive definition was employed in proving stability results. Let X be a Banach space, for each $x, y \in X$, there exist $\delta \in [0, 1)$ and a monotone increasing function $\varphi : R^+ \to R^+$ with $\varphi(0) = 0$ such that

$$\|Tx - Ty\| \leq \delta\|x - y\| + \varphi(\|x - Tx\|). \tag{4}$$

It is important to remark that contractive condition (4) is a generalization of (3) and (2) for single valued map T.

The modified versions of contractive conditions (2)–(4), is hereby presented in a modular function space as follows.

Let L_ρ be a modular space. A set $D \subset L_\rho$ is called $\rho-$proximinal if for each $f \in L_\rho$ there exists an element $g \in D$ such that $\rho(f-g) = \mathtt{dist}_\rho(f, D)$. We represent the family of nonempty $\rho-$bounded $\rho-$proximinal subsets of D by $P_\rho(D)$, the family of nonempty $\rho-$closed $\rho-$bounded subsets of D by $C_\rho(D)$ and the family of $\rho-$compact subsets of D by $K_\rho(D)$. Let $H_\rho(.,.)$ be the $\rho-$Hausdorff distance on $C_\rho(L_\rho)$, that is, $H_\rho(A, B) = \max\{\sup_{f\in A} \mathrm{dist}_\rho(f, B), \sup_{g\in B} \mathrm{dist}_\rho(g, A)\}$, $A, B \in C_\rho(L_\rho)$.

A multivalued map $T : D \to C_\rho(L_\rho)$ is said to be:

(1) $\rho-$contraction mapping, if there exists a constant $\delta \in [0, 1)$ such that

$$H_\rho(Tf, Tg) \leq \delta\rho(f - g), \text{ for all } f, g \in D. \tag{5}$$

(2) $\rho-$Zamfirescu mapping if

$$H_\rho(Tf, Tg) \leq \delta\rho(f - g) + 2\delta\rho(Tf - f), \text{ for all } f, g \in D. \tag{6}$$

(3) $\rho-$quasi- contractive mapping if

$$H_\rho(Tf, Tg) \leq \delta\rho(f - g) + L\rho(Tf - f), \text{ for all } f, g \in D, L \geq 0. \tag{7}$$

(4) $\rho-$quasi-contractive-like mapping if

$$H_\rho(Tf, Tg) \leq \delta\rho(f - g) + \varphi_\rho(\rho(Tf - f)), \text{ for all } f, g \in D. \tag{8}$$

where $\varphi_\rho : R^+ \to R^+$ is a $\rho-$monotone increasing function with $\varphi_\rho(0) = 0$.

Implicit iterations exist in literature and have been proved to have advantage over explicit iterations for nonlinear problems as they provide better approximation of fixed points, and are widely used in many applications, when explicit iterations are ineficient. Approximation of fixed points in computer oriented programs using implicit iterations can reduce the computational cost of the fixed point problems (see [7]). The following implicit iterative sequences in the framework of modular function spaces are hereby presented:

Let L_ρ be a modular space, $D \subset L_\rho$ and $T : D \to P_\rho(D)$ be a multivalued mapping, then the implicit multistep iterative sequence $\{f_n\}_{n=0}^{\infty} \subset D$ is defined by:

$$\begin{cases} f_0 \in D \\ f_{n+1} = (1-\alpha_n)f_n^1 + \alpha_n u_{n+1}, \\ f_n^i = (1-\beta_n^i)f_n^{i+1} + \beta_n^i u_n^i, \ \ i = 1, 2, ..., k-2 \\ f_n^{k-1} = (1-\beta_n^{k-1})f_n + \beta_n^{k-1}u_n^{k-1}, \ \ n = 0, 1, 2, ..., \end{cases} \tag{9}$$

where $u_n \in P_\rho^T(f_n)$, $u_n^i \in P_\rho^T(f_n^i)$, $u_n^{k-1} \in P_\rho^T(f_n^{k-1})$, the sequences $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n^i\}_{n=0}^{\infty} \subset (0,1)(i = 1, 2, ..., k-1)$ such that $\sum_{n=0}^{\infty} \alpha_n = \infty$.

The implicit Noor iterative sequence $\{g_n\}_{n=0}^{\infty} \subset D$ is defined by:

$$\begin{cases} g_0 \in D \\ g_{n+1} = (1-\alpha_n)g_n^1 + \alpha_n v_{n+1}, \\ g_n^1 = (1-\beta_n^1)g_n^2 + \beta_n^1 v_n^1, \\ g_n^2 = (1-\beta_n^2)g_n + \beta_n^2 v_n^2, \ \ n = 0, 1, 2, ..., \end{cases} \tag{10}$$

where $v_{n+1} \in P_\rho^T(g_{n+1})$, $v_n^1 \in P_\rho^T(g_n)^1$, $v_n^2 \in P_\rho^T(g_n^2)$, the sequences $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n^1\}_{n=0}^{\infty}, \{\beta_n^2\}_{n=0}^{\infty} \subset (0,1)$, such that $\sum_{n=0}^{\infty} \alpha_n = \infty$.

The implicit Ishikawa iterative sequence $\{h_n\}_{n=0}^{\infty} \subset D$ is defined by:

$$\begin{cases} h_0 \in D \\ h_{n+1} = (1-\alpha_n)h_n^1 + \alpha_n s_{n+1}, \\ h_n^1 = (1-\beta_n^1)h_n + \beta_n^1 s_n^1, \ \ n = 0, 1, 2, ..., \end{cases} \tag{11}$$

where $s_{n+1} \in P_\rho^T(h_{n+1})$, $s_n^1 \in P_\rho^T(h_n^1)$, the sequences $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n^1\}_{n=0}^{\infty} \subset (0,1)$, such that $\sum_{n=0}^{\infty} \alpha_n = \infty$.

The implicit Mann iterative sequence $\{g_n\}_{n=0}^{\infty} \subset D$ is defined by:

$$\begin{cases} g_0 \in D \\ g_{n+1} = (1-\alpha_n)g_n + \alpha_n v_{n+1}, \ \ n = 0, 1, 2, ..., \end{cases} \tag{12}$$

where $v_{n+1} \in P_\rho^T(g_{n+1})$, the sequence $\{\alpha_n\}_{n=0}^{\infty} \subset (0,1)$, such that $\sum_{n=0}^{\infty} \alpha_n = \infty$.

The following Lemmas will be needed in proving the main results.

Lemma 1 [10]. *Let* $T : D \to P_\rho(D)$ *be a multivalued mapping and* $P_\rho^T(f) = \{g \in Tf : \rho(f - g) = dist_\rho(f, Tf)\}$. *Then the following are equivalent:*

(1) $f \in F_\rho(T)$, *that is,* $f \in Tf$.

(2) $P_\rho^T(f) = \{f\}$, *that is,* $f = g$ *for each* $g \in P_\rho^T(f)$.

(3) $f \in F(P_\rho^T(f))$, *that is,* $f \in P_\rho^T(f)$. *Further* $F_\rho(T) = F(P_\rho^T(f))$ *where* $F(P_\rho^T(f))$ *represent the set of fixed points of* $P_\rho^T(f)$.

Lemma 2 [6]. *Let* δ *be a real number satisfying* $0 \le \delta < 1$ *and* $\{\epsilon_n\}_{n=0}^{\infty}$ *a sequence of positive numbers such that* $\lim_{n\to\infty}\epsilon_n = 0$, *then for any sequence of positive numbers* $\{u_n\}_{n=0}^{\infty}$ *satisfying* $u_{n+1} \le \delta u_n + \epsilon_n$, *n=0,1,2,...,* *we have* $\lim_{n\to\infty} u_n = 0$.

Laudable papers have written by notable researchers on the convergence and the equivalence of convergence of various iterative sequences for single mapping T *on normed and metric spaces. That is, different authors have shown that the convergence of any of the iterative method to the unique fixed point of the contractive operator for single map* T *is equivalent to the convergence of the other iterative sequences. For a look at some of the fine works in this direction, see references:* [1], [2], [5], [6], [7], [19],[20], [21], and [23]. Some results also appear for pair of maps, for example, (see [3], [4], and [17] for details). The new version of equivalence results will now be proved for multivalued $\rho-$quasi-contractive-like mappings in modular function spaces in the following theorems.

2 Main results

Theorem 1 *Let* ρ *satisfy* $(UUC1)$ *and* Δ_2-*condition. Let D be a* $\rho-$*closed,* $\rho-$*bounded and convex subset of a* $\rho-$*complete modular space* L_ρ *and* $T : D \to P_\rho(D)$ *be a multivalued mapping such that* P_ρ^T *is a* $\rho-$*quasi-contractive-like mapping, satisfying the contractive condition*

$$H_\rho(Tf, Tg) \le \delta\rho(f - g) + \varphi_\rho(\rho(Tf - f)), \tag{13}$$

for all $f, g \in D$ *and* $F_\rho(T) \neq \emptyset$, *where* $\delta \in [0, 1)$ *and* $\varphi_\rho : R^+ \to R^+$ *is a* $\rho-$*monotone increasing function with* $\varphi_\rho(0) = 0$. *Let* $f_0, g_0 \in D$ *and* $\{f_n\}, \{g_n\} \subset D$ *be defined by the implicit multistep* (9) *and implicit Mann* (12) *iterative sequence respectively, where the sequences* $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n^i\}_{n=0}^{\infty} \subset (0, 1)$ *such that* $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=0}^{\infty} \beta_n^i = \infty$, *for* $i = 1, 2, ..., k - 1$. *Then the following are equivalent:*

(i) *the implicit Mann iterative sequence* (12) *converges strongly to the fixed point of the multivalued map* T;

(ii) *the implicit multistep iterative sequence* (9) *converges strongly to the fixed point of the multivalued map* T.

Proof. Let $p \in F_\rho(T)$, from Lemma 1, $P_\rho^T(p) = \{p\}$ and $F_\rho(T) = F(P_\rho^T)$.

We prove that (i) $\Rightarrow$ (ii). Assume $\lim_{n\to\infty} g_n = p$. Using $\rho-$quasi-contractive-like condition (13), implicit Mann (12) and implicit multistep iterative sequences (9), we obtain the following:

$$\rho(g_{n+1} - f_{n+1}) = \rho[(1-\alpha_n)(g_n - f_n^1) + \alpha_n(v_{n+1} - u_{n+1})]. \tag{14}$$

Using the convexity of ρ in equation (14), we have

$$\begin{aligned}\rho(g_{n+1} - f_{n+1}) &= (1-\alpha_n)\rho(g_n - f_n^1) + \alpha_n \rho(v_{n+1} - u_{n+1}) \\ &\le (1-\alpha_n)\rho(g_n - f_n^1) + \alpha_n(H_\rho(P_\rho^T(g_{n+1}), P_\rho^T(f_{n+1}))).\end{aligned} \tag{15}$$

Using (13), let $f = f_{n+1}$, $g = g_{n+1}$, then, from (15), we get the following:

$$H_\rho(P_\rho^T(g_{n+1}), P_\rho^T(f_{n+1})) \le \delta\rho(g_{n+1} - f_{n+1}) + (1+\delta)\varphi_\rho(\rho(g_{n+1} - p)). \tag{16}$$

Substituting inequality (16) in inequality (15), we obtain

$$\begin{aligned}\rho(g_{n+1} - f_{n+1}) \le\ & (1-\alpha_n)\rho(g_n - f_n^1) + \delta\alpha_n\rho(g_{n+1} - f_{n+1}) + \\ & (1+\delta)\alpha_n\varphi_\rho(\rho(g_{n+1} - p)).\end{aligned}$$

That is,

$$\rho(g_{n+1} - f_{n+1}) \le \left(\frac{1-\alpha_n}{1-\delta\alpha_n}\right)\rho(g_n - f_n^1) + \left(\frac{(1+\delta)\alpha_n}{1-\delta\alpha_n}\right)\varphi_\rho(\rho(g_{n+1} - p)). \tag{17}$$

$$\rho(g_n - f_n^1) = \rho(g_n - ((1-\beta_n^1)f_n^2 + \beta_n^1 u_n^1)). \tag{18}$$

Using the convexity of ρ in equation (18), we have

$$\begin{aligned}\rho(g_n - f_n^1) &\le (1-\beta_n^1)\rho(g_n - f_n^2) + \beta_n^1\rho(g_n - u_n^1) \\ &\le (1-\beta_n^1)\rho(g_n - f_n^2) + \beta_n^1\rho(g_n - v_n) + \beta_n^1\rho(v_n - u_n^1) \\ &\le (1-\beta_n^1)\rho(g_n - f_n^2) + \beta_n^1\rho(g_n - p) \\ &\quad + \beta_n^1\rho(v_n - p) + \beta_n^1\rho(v_n - u_n^1) \\ &\le (1-\beta_n^1)\rho(g_n - f_n^2) + \beta_n^1\rho(g_n - p) + \beta_n^1 H_\rho(P_\rho^T(g_n), P_\rho^T(p)) \\ &\quad + \beta_n^1 H_\rho(P_\rho^T(g_n), P_\rho^T(f_n^1)).\end{aligned} \tag{19}$$

Using (13), let $f = p$, $g = g_n$, and also let $f = g_n$, $g = f_n^1$, then, from (19), we get the following:

$$\begin{aligned}\rho(g_n - f_n^1) \leq \left(\frac{1-\beta_n^1}{1-\delta\beta_n^1}\right)\rho(g_n - f_n^2) + \left(\frac{(1+\delta)\beta_n^1}{1-\delta\beta_n^1}\right)\rho(g_n - p)\\ + \left(\frac{(1+\delta)\beta_n^1}{1-\delta\beta_n^1}\right)\varphi_\rho(\rho(g_n - p)).\end{aligned} \tag{20}$$

Substituting inequality (20) in inequality (17), we obtain

$$\begin{aligned}\rho(g_{n+1} - f_{n+1}) \leq & \left(\frac{1-\alpha_n}{1-\delta\alpha_n}\right)\left(\frac{1-\beta_n^1}{1-\delta\beta_n^1}\right)\rho(g_n - f_n^2)\\ & + \left(\frac{1-\alpha_n}{1-\delta\alpha_n}\right)\left(\frac{(1+\delta)\beta_n^1}{1-\delta\beta_n^1}\right)\rho(g_n - p)\\ & + \left(\frac{1-\alpha_n}{1-\delta\alpha_n}\right)\left(\frac{(1+\delta)\beta_n^1}{1-\delta\beta_n^1}\right)\varphi_\rho(\rho(g_n - p))\\ & + \left(\frac{(1+\delta)\alpha_n}{1-\delta\alpha_n}\right)\varphi_\rho(\rho(g_{n+1} - p)).\end{aligned} \tag{21}$$

Similarly, an application of (13) and (9) and (12) give the following

$$\begin{aligned}\rho(g_n - f_n^2) \leq \left(\frac{1-\beta_n^2}{1-\delta\beta_n^2}\right)\rho(g_n - f_n^3) + \left(\frac{(1+\delta)\beta_n^2}{1-\delta\beta_n^2}\right)\rho(g_n - p)\\ + \left(\frac{(1+\delta)\beta_n^2}{1-\delta\beta_n^2}\right)\varphi_\rho(\rho(g_n - p)).\end{aligned} \tag{22}$$

$$\begin{aligned}\rho(g_n - f_n^3) \leq \left(\frac{1-\beta_n^3}{1-\delta\beta_n^3}\right)\rho(g_n - f_n^4) + \left(\frac{(1+\delta)\beta_n^3}{1-\delta\beta_n^3})\rho(g_n - p\right)\\ + \left(\frac{(1+\delta)\beta_n^3}{1-\delta\beta_n^3}\right)\varphi_\rho(\rho(g_n - p)).\end{aligned} \tag{23}$$

$$\vdots$$

$$\begin{aligned}\rho(g_n - f_n^{k-2}) \leq \left(\frac{1-\beta_n^{k-2}}{1-\delta\beta_n^{k-2}}\right)\rho(g_n - f_n^{k-1}) + \left(\frac{(1+\delta)\beta_n^{k-2}}{1-\delta\beta_n^{k-2}}\right)\rho(g_n - p)\\ + \left(\frac{(1+\delta)\beta_n^{k-2}}{1-\delta\beta_n^{k-2}}\right)\varphi_\rho(\rho(g_n - p)).\end{aligned} \tag{24}$$

$$
\begin{aligned}
\rho(g_n - f_n^{k-1}) \leq & \left(\frac{1-\beta_n^{k-1}}{1-\delta\beta_n^{k-1}}\right)\rho(g_n - f_n) + \left(\frac{(1+\delta)\beta_n^{k-1}}{1-\delta\beta_n^{k-1}}\right)\rho(g_n - p) \\
& + \left(\frac{(1+\delta)\beta_n^{k-1}}{1-\delta\beta_n^{k-1}}\right)\varphi_\rho(\rho(g_n - p)).
\end{aligned} \tag{25}
$$

Substituting inequalities (25), (24), (23), (22) in inequality (21) inductively and simplifying, we obtain

$$
\begin{aligned}
\rho(g_{n+1} - f_{n+1}) \leq & \left(\frac{1-\alpha_n}{1-\delta\alpha_n}\right)\left(\frac{1-\beta_n^1}{1-\delta\beta_n^1}\right)\left(\frac{1-\beta_n^2}{1-\delta\beta_n^2}\right)\left(\frac{1-\beta_n^3}{1-\delta\beta_n^3}\right)\ldots \\
& \left(\frac{1-\beta_n^{k-2}}{1-\delta\beta_n^{k-2}}\right)\left(\frac{1-\beta_n^{k-1}}{1-\delta\beta_n^{k-1}}\right)\rho(g_n - f_n) \\
& + \left(\frac{1-\alpha_n}{1-\delta\alpha_n}\right)\left(\frac{(1+\delta)\beta_n^1}{1-\delta\beta_n^1}\right)\left(\frac{(1+\delta)\beta_n^2}{1-\delta\beta_n^2}\right)\left(\frac{(1+\delta)\beta_n^3}{1-\delta\beta_n^3}\right)\ldots \\
& \left(\frac{(1+\delta)\beta_n^{k-2}}{1-\delta\beta_n^{k-2}}\right)\left(\frac{(1+\delta)\beta_n^{k-1}}{1-\delta\beta_n^{k-1}}\right)\rho(g_n - p) \\
& + \left(\frac{1-\alpha_n}{1-\delta\alpha_n}\right)\left(\frac{(1+\delta)\beta_n^1}{1-\delta\beta_n^1}\right)\left(\frac{(1+\delta)\beta_n^2}{1-\delta\beta_n^2}\right)\left(\frac{(1+\delta)\beta_n^3}{1-\delta\beta_n^3}\right)\ldots \\
& \left(\frac{(1+\delta)\beta_n^{k-2}}{1-\delta\beta_n^{k-2}}\right)\left(\frac{(1+\delta)\beta_n^{k-1}}{1-\delta\beta_n^{k-1}}\right)\varphi_\rho(\rho(g_n - p)) \\
& + \left(\frac{(1+\delta)\alpha_n}{1-\delta\alpha_n})\varphi_\rho(\rho(g_{n+1} - p)\right).
\end{aligned} \tag{26}
$$

Observe that

$$
\begin{aligned}
& \left[\frac{1-\alpha_n}{1-\delta\alpha_n}\right] \leq 1 - \alpha_n + \delta\alpha_n, \\
& \left[\frac{1-\beta_n^1}{1-\delta\beta_n^1}\right] \leq 1 - \beta_n^1 + \delta\beta_n^1, \\
& \left[\frac{1-\beta_n^2}{1-\delta\beta_n^2}\right] \leq 1 - \beta_n^2 + \delta\beta_n^2, \ldots, \\
& \left[\frac{1-\beta_n^{k-2}}{1-\delta\beta_n^{k-2}}\right] \leq 1 - \beta_n^{k-2} + \delta\beta_n^{k-2} \text{ and} \\
& \left[\frac{1-\beta_n^{k-1}}{1-\delta\beta_n^{k-1}}\right] \leq 1 - \beta_n^{k-1} + \delta\beta_n^{k-1}.
\end{aligned} \tag{27}
$$

Applying the inequality (27) in inequality (26) and simplifying, we obtain

$$\begin{aligned}\rho(g_{n+1}-f_{n+1}) &\leq (1-\alpha_n+\delta\alpha_n)(1-\beta_n^1+\delta\beta_n^1)(1-\beta_n^2+\delta\beta_n^2),\ldots,\\ &(1-\beta_n^{k-2}+\delta\beta_n^{k-2})(1-\beta_n^{k-1}+\delta\beta_n^{k-1})+e_n\\ &\leq [1-(1-\delta)\alpha_n]\rho(g_n-f_n)+e_n,\end{aligned} \tag{28}$$

where,

$$\begin{aligned}e_n &= [1-(1-\delta)\alpha_n][1-(1-\delta)\beta_n^1][1-(1-\delta)\beta_n^2][1-(1-\delta)\beta_n^3]\ldots\\ &[1-(1-\delta)\beta_n^{k-2}][1-(1-\delta)\beta_n^{k-1}]\rho(g_n-p)\\ &+\left(\frac{1-\alpha_n}{1-\delta\alpha_n}\right)\left(\frac{(1+\delta)\beta_n^1}{1-\delta\beta_n^1}\right)\left(\frac{(1+\delta)\beta_n^2}{1-\delta\beta_n^2}\right)\left(\frac{(1+\delta)\beta_n^3}{1-\delta\beta_n^3}\right)\ldots\\ &\left(\frac{(1+\delta)\beta_n^{k-2}}{1-\delta\beta_n^{k-2}}\right)\left(\frac{(1+\delta)\beta_n^{k-1}}{1-\delta\beta_n^{k-1}}\right)\varphi_\rho(\rho(g_n-p))\\ &+\left(\frac{(1+\delta)\alpha_n}{1-\delta\alpha_n})\varphi_\rho(\rho(g_{n+1}-p)\right).\end{aligned}$$

Using the fact that $0\leq\delta<1$ and the conditions $\{\alpha_n\}_{n=0}^\infty, \{\beta_n^i\}_{n=0}^\infty\subset(0,1)(i=1,2,\ldots,k-1)$ in iterative sequences (9) to (12) in (28), it follows that

$$\lim_{n\to\infty}\rho(g_n-f_n)=0.$$

Since by assumption $\lim_{n\to\infty}g_n=p$, then $\rho(f_n-p)\leq\rho(g_n-f_n)+\rho(g_n-p)\to 0$ as $n\to\infty$. That is, $\lim_{n\to\infty}f_n=p$.

Next we show that (ii) $\to$ (i). Assume $\lim_{n\to\infty}f_n=p$.

Then using $\rho-$quasi-contractive-like condition (13), implicit multistep (9) and implicit Mann iterative sequences (12), we obtain the following:

$$\rho(f_{n+1}-g_{n+1})=\rho[(1-\alpha_n)(f_n^1-g_n)+\alpha_n(u_{n+1}-v_{n+1})]. \tag{29}$$

Using the convexity of ρ in (29), we have

$$\begin{aligned}\rho(f_{n+1}-g_{n+1}) &= (1-\alpha_n)\rho(f_n^1-g_n)+\alpha_n\rho(u_{n+1}-v_{n+1})\\ &\leq (1-\alpha_n)\rho(f_n^1-g_n)+\alpha_n(H_\rho(P_\rho^T(f_{n+1}),P_\rho^T(g_{n+1}))).\end{aligned} \tag{30}$$

Using (13), let $f=f_{n+1}$, $g=g_{n+1}$, then, from (29), we get the following:

$$H_\rho(P_\rho^T(f_{n+1}),P_\rho^T(g_{n+1}))\leq\delta\rho(f_{n+1}-g_{n+1})+(1+\delta)\varphi_\rho(\rho(f_{n+1}-p)). \tag{31}$$

Substituting inequality (31) in inequality (30), we obtain

$$\begin{aligned}\rho(f_{n+1}-g_{n+1}) &\le (1-\alpha_n)\rho(f_n^1-g_n)+\delta\alpha_n\rho(f_{n+1}-g_{n+1})\\ &\quad +(1+\delta)\alpha_n\varphi_\rho(\rho(f_{n+1}-p)).\end{aligned}$$

That is,

$$\rho(f_{n+1}-g_{n+1}) \le \left(\frac{1-\alpha_n}{1-\delta\alpha_n}\right)\rho(f_n^1-g_n)+\left(\frac{(1+\delta)\alpha_n}{1-\delta\alpha_n}\right)\varphi_\rho(\rho(f_{n+1}-p)). \tag{32}$$

$$\begin{aligned}\rho(f_n^1-g_n) \le &\left(\frac{1-\beta_n^1}{1-\delta\beta_n^1}\right)\rho(f_n^2-g_n)+\left(\frac{(1+\delta)\beta_n^1}{1-\delta\beta_n^1}\right)\rho(f_n^1-p)+\\ &\left(\frac{(1+\delta)\beta_n^1}{1-\delta\beta_n^1}\right)\varphi_\rho(\rho(f_n^1-p)).\end{aligned} \tag{33}$$

Substituting inequality (33) in inequality (32), we obtain

$$\begin{aligned}\rho(f_{n+1}-g_{n+1}) \le &\left(\frac{1-\alpha_n}{1-\delta\alpha_n}\right)\left(\frac{1-\beta_n^1}{1-\delta\beta_n^1}\right)\rho(f_n^2-g_n)\\ &+\left(\frac{1-\alpha_n}{1-\delta\alpha_n}\right)\left(\frac{(1+\delta)\beta_n^1}{1-\delta\beta_n^1}\right)\rho(f_n^1-p)\\ &+\left(\frac{1-\alpha_n}{1-\delta\alpha_n}\right)\left(\frac{(1+\delta)\beta_n^1}{1-\delta\beta_n^1}\right)\varphi_\rho(\rho(f_n^1-p))\\ &+\left(\frac{(1+\delta)\alpha_n}{1-\delta\alpha_n}\right)\varphi_\rho(\rho(f_{n+1}-p)).\end{aligned} \tag{34}$$

Similarly, an application of (13) and (9) and (12) give the following

$$\begin{aligned}\rho(f_n^2-g_n) \le &\left(\frac{1-\beta_n^2}{1-\delta\beta_n^2}\right)\rho(f_n^3-g_n)+\left(\frac{(1+\delta)\beta_n^2}{1-\delta\beta_n^2}\right)\rho(f_n^2-p)\\ &+\left(\frac{(1+\delta)\beta_n^2}{1-\delta\beta_n^2}\right)\varphi_\rho(\rho(f_n^2-p)).\end{aligned} \tag{35}$$

$$\begin{aligned}\rho(f_n^3-g_n) \le &\left(\frac{1-\beta_n^3}{1-\delta\beta_n^3}\right)\rho(f_n^4-g_n)+\left(\frac{(1+\delta)\beta_n^3}{1-\delta\beta_n^3}\right)\rho(f_n^3-p)\\ &+\left(\frac{(1+\delta)\beta_n^3}{1-\delta\beta_n^3}\right)\varphi_\rho(\rho(f_n^3-p)).\end{aligned} \tag{36}$$

$$\vdots$$

$$\rho(f_n^{k-2}-g_n) \leq \left(\frac{1-\beta_n^{k-2}}{1-\delta\beta_n^{k-2}}\right)\rho(f_n^{k-1}-g_n) + \left(\frac{(1+\delta)\beta_n^{k-2}}{1-\delta\beta_n^{k-2}}\right)\rho(f_n^{k-2}-p) + \left(\frac{(1+\delta)\beta_n^{k-2}}{1-\delta\beta_n^{k-2}}\right)\varphi_\rho(\rho(f_n^{k-2}-p)). \tag{37}$$

$$\rho(f_n^{k-1}-g_n) \leq \left(\frac{1-\beta_n^{k-1}}{1-\delta\beta_n^{k-1}}\right)\rho(f_n-g_n) + \left(\frac{(1+\delta)\beta_n^{k-1}}{1-\delta\beta_n^{k-1}}\right)\rho(f_n^{k-1}-p) + \left(\frac{(1+\delta)\beta_n^{k-1}}{1-\delta\beta_n^{k-1}}\right)\varphi_\rho(\rho(f_n^{k-1}-p)). \tag{38}$$

Substituting inequalities (38), (37), (36), (35) in inequality (32) inductively and simplifying, we obtain

$$\rho(f_{n+1}-g_{n+1}) \leq [1-(1-\delta)\alpha_n]\rho(f_n-g_n)+b_n, \tag{39}$$

where,

$$\begin{aligned} b_n \;=\; & [1-(1-\delta)\alpha_n][1-(1-\delta)\beta_n^1][1-(1-\delta)\beta_n^2][1-(1-\delta)\beta_n^3]... \\ & [1-(1-\delta)\beta_n^{k-2}][1-(1-\delta)\beta_n^{k-1}]\rho(f_n^{k-1}-p) \\ & +\left(\frac{1-\alpha_n}{1-\delta\alpha_n}\right)\left(\frac{(1+\delta)\beta_n^1}{1-\delta\beta_n^1}\right)\left(\frac{(1+\delta)\beta_n^2}{1-\delta\beta_n^2}\right)\left(\frac{(1+\delta)\beta_n^3}{1-\delta\beta_n^3}\right)... \\ & \left(\frac{(1+\delta)\beta_n^{k-2}}{1-\delta\beta_n^{k-2}}\right)\left(\frac{(1+\delta)\beta_n^{k-1}}{1-\delta\beta_n^{k-1}}\right)\varphi_\rho(\rho(f_n^{k-1}-p)) \\ & +\left(\frac{(1+\delta)\alpha_n}{1-\delta\alpha_n}\right)\varphi_\rho(\rho(f_{n+1}-p)). \end{aligned}$$

Using the fact that $0 \leq \delta < 1$ and the conditions $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n^i\}_{n=0}^{\infty} \subset (0,1)(i = 1,2,...,k-1)$ in iterative sequences (9) to (12) in (39), it follows that

$$\lim_{n\to\infty} \rho(f_n-g_n)=0.$$

Since by assumption $\lim_{n\to\infty} f_n = p$, then
$\rho(g_n-p) \leq \rho(f_n-g_n)+\rho(f_n-p) \to 0$ as $n \to \infty$.
That is, $\lim_{n\to\infty} g_n = p$. This ends the proof. □

Since the implicit Noor (10), the implicit Ishikawa (11) and the implicit Mann (12) iterative sequences are special cases of the implicit multistep iterative sequence (9), then Theorem 1 leads to the following corollary:

Corollary 1 *Let ρ satisfy $(UUC1)$ and Δ_2-condition. Let D be a $\rho-$closed, $\rho-$bounded and convex subset of a $\rho-$complete modular space L_ρ and $T : D \to P_\rho(D)$ be a multivalued mapping such that P_ρ^T is a $\rho-$quasi-contractive-like mapping, satisfying contractive-like condition*

$$H_\rho(Tf, Tg) \leq \delta\rho(f-g) + \varphi_\rho(\rho(Tf-f)), \tag{40}$$

for all $g, h, g \in D$ and $F_\rho(T) \neq \emptyset$, where $\delta \in [0, 1)$ and $\varphi_\rho : R^+ \to R^+$ is a $\rho-$monotone increasing function with $\varphi_\rho(0) = 0$. Let $g_0, h_0, g_0 \in D$ and $\{g_n\}, \{h_n\}, \{g_n\} \subset D$ be defined by the implicit Mann (12), implicit Ishikawa (11) and implicit Noor iterative sequences (10) respectively, where the sequences $\{\alpha_n\}_{n=0}^\infty, \{\beta_n^1\}_{n=0}^\infty, \{\beta_n^2\}_{n=0}^\infty \subset (0, 1)$ such that $\sum_{n=0}^\infty \alpha_n = \infty$, $\sum_{n=0}^\infty \beta_n^i = \infty$, for $i = 1, 2$. Then the following are equivalent:

a. (i) *the implicit Mann iterative sequence (12) converges strongly to the fixed point of the multivalued map T;*
(ii) *the implicit Ishikawa iterative sequence (11) converges strongly to the fixed point of the multivalued map T.*

b. (i) *the implicit Mann iterative sequence (12) converges strongly to the fixed point of the multivalued map T;*
(ii) *the implicit Noor iterative sequence (10) converges strongly to the fixed point of the multivalued map T.*

Proof. The proof of Corollary 1 is similar to that of Theorem 1. □

Corollary 2 *Let ρ satisfy $(UUC1)$ and Δ_2-condition. Let D be a $\rho-$closed, $\rho-$bounded and convex subset of a $\rho-$complete modular space L_ρ and $T : D \to P_\rho(D)$ be a multivalued mapping such that P_ρ^T is a $\rho-$quasi-contractive-like mapping, satisfying the condition*

$$H_\rho(Tf, Tg) \leq \delta\rho(f-g) + \varphi_\rho(\rho(Tf-f)), \tag{41}$$

for all $g, h, g, f \in D$ and $F_\rho(T) \neq \emptyset$, where $\delta \in [0, 1)$ and $\varphi_\rho : R^+ \to R^+$ is a $\rho-$monotone increasing function with $\varphi_\rho(0) = 0$. Let $g_0, h_0, g_0, f_0 \in D$ and $\{g_n\}, \{h_n\}, \{g_n\}, \{f_n\} \subset D$ be defined by the implicit Mann (12), implicit Ishikawa (11), implicit Noor (10), implicit multistep (9) iterative sequences respectively, where the sequences $\{\alpha_n\}_{n=0}^\infty, \{\beta_n^i\}_{n=0}^\infty \subset (0, 1)$ such that $\sum_{n=0}^\infty \alpha_n = \infty$, $\sum_{n=0}^\infty \beta_n^i = \infty$ for $i = 1, 2, ..., k-1$. Then the following are equivalent:

(i) *the implicit Mann iterative sequence (12) converges strongly to the fixed point of the multivalued map T;*

(ii) *the implicit Ishikawa iterative sequence* (11) *converges strongly to the fixed point of the multivalued map* T;

(iii) *the implicit Noor iterative sequence* (10) *converges strongly to the fixed point of the multivalued map* T;

(iv) *the implicit multistep iterative sequence* (9) *converges strongly to the fixed point of the multivalued map* T.

Theorem 2 *Let* ρ *satisfy* $(UUC1)$ *and* Δ_2-*condition. Let D be a* $\rho-$*closed,* $\rho-$*bounded and convex subset of a* $\rho-$*complete modular space* L_ρ *and* $T : D \to P_\rho(D)$ *be a multivalued mapping such that* P_ρ^T *is a* $\rho-$*quasi-contractive mapping, satisfying the condition*

$$H_\rho(Tf, Tg) \leq \delta\rho(f - g) + J\rho(Tf - f), \tag{42}$$

for all $f, g \in D$ *and* $F_\rho(T) \neq \emptyset$*, where* $\delta \in [0, 1)$ *and* $J \geq 0$. *Let* $f_0, g_0 \in D$ *and* $\{f_n\}, \{g_n\} \subset D$ *be defined by the implicit multistep* (9) *and implicit Mann iterative sequences* (12) *respectively, where the sequences* $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n^i\}_{n=0}^{\infty} \subset (0, 1)$ *such that* $\sum_{n=0}^{\infty} \alpha_n = \infty, \sum_{n=0}^{\infty} \beta_n^i = \infty$ *for* $i = 1, 2, ..., k - 1$. *Then the following are equivalent:*

(i) *the implicit Mann iterative sequence* (12) *converges strongly to the fixed point of the multivalued map* T;

(ii) *the implicit multistep iterative sequence* (9) *converges strongly to the fixed point of the multivalued map* T.

Proof. The method of proof of Theorem 2 is similar to that of Theorem 1. The proof is complete. □

Theorem 2 leads to the following corollary:

Corollary 3 *Let* ρ *satisfy* $(UUC1)$ *and* Δ_2-*condition. Let D be a* $\rho-$*closed,* $\rho-$*bounded and convex subset of a* $\rho-$*complete modular space* L_ρ *and* $T : D \to P_\rho(D)$ *be a multivalued mapping such that* P_ρ^T *is a* $\rho-$*quasi-contractive mapping, satisfying the condition*

$$H_\rho(Tf, Tg) \leq \delta\rho(f - g) + J\rho(Tf - f), \tag{43}$$

for all $g, h, g \in D$ *and* $F_\rho(T) \neq \emptyset$*, where* $\delta \in [0, 1)$ *and* $J \geq 0$. *Let* $g_0, h_0, g_0 \in D$ *and* $\{g_n\}, \{h_n\}, \{g_n\} \subset D$ *be defined by the implicit Mann* (12)*, implicit Ishikawa* (11) *and implicit Noor* (10) *iterative sequence respectively, where the sequences* $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n^1\}_{n=0}^{\infty}, \{\beta_n^2\}_{n=0}^{\infty} \subset (0, 1)$ *such that* $\sum_{n=0}^{\infty} \alpha_n = \infty, \sum_{n=0}^{\infty} \beta_n^i = \infty$ *for*

$i = 1, 2$. *Then the following are equivalent:*

a. (i) *the implicit Mann iterative sequence* (12) *converges strongly to the fixed point of the multivalued map* T;
(ii) *the implicit Ishikawa iterative sequence* (11) *converges strongly to the fixed point of the multivalued map* T.

b. (i) *the implicit Mann iterative sequence* (12) *converges strongly to the fixed point of the multivalued map* T;
(ii) *the implicit Noor iterative sequence* (10) *converges strongly to the fixed point of the multivalued map* T.

Theorem 3 *Let* ρ *satisfy* $(UUC1)$ *and* Δ_2-*condition. Let D be a* $\rho-$*closed,* $\rho-$*bounded and convex subset of a* $\rho-$*complete modular space* L_ρ *and* $T : D \to P_\rho(D)$ *be a multivalued mapping such that,* P_ρ^T *is a* $\rho-$*Zamfirescu mapping, satisfying the condition*

$$H_\rho(Tf, Tg) \leq \delta\rho(f - g) + 2\delta\rho(Tf - f), \tag{44}$$

for all $f, g \in D$ *and* $F_\rho(T) \neq \emptyset$, *where* $\delta \in [0, 1)$. *Let* $f_0, g_0 \in D$ *and* $\{f_n\}, \{g_n\} \subset D$ *be defined by the implicit multistep* (9) *and implicit Mann iterative sequences* (12) *respectively, where the sequences* $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n^i\}_{n=0}^{\infty} \subset (0, 1)$ *such that* $\sum_{n=0}^{\infty} \alpha_n = \infty, \sum_{n=0}^{\infty} \beta_n^i = \infty$ *for* $i = 1, 2, ..., k - 1$. *Then the following are equivalent:*

(i) *the implicit Mann iterative sequence* (12) *converges strongly to the fixed point of the multivalued map* T;

(ii) *the implicit multistep iterative sequence* (9) *converges strongly to the fixed point of the multivalued map* T.

Proof. The method of proof of Theorem 3 is similar to that of Theorem 1. The proof is complete. □

Theorem 3 leads to the following corollary:

Corollary 4 *Let* ρ *satisfy* $(UUC1)$ *and* Δ_2-*condition. Let D be a* $\rho-$*closed,* $\rho-$*bounded and convex subset of a* $\rho-$*complete modular space* L_ρ *and* $T : D \to P_\rho(D)$ *be a multivalued mapping such that* P_ρ^T *is a* $\rho-$*Zamfirescu mapping, satisfying the condition*

$$H_\rho(Tf, Tg) \leq \delta\rho(f - g) + 2\delta\rho(Tf - f), \tag{45}$$

for all $g, h, g \in D$ *and* $F_\rho(T) \neq \emptyset$, *where* $\delta \in [0, 1)$. *Let* $g_0, h_0, g_0 \in D$ *and* $\{g_n\}, \{h_n\}, \{g_n\} \subset D$ *be defined by the implicit Mann* (12), *implicit Ishikawa*

(11) *and implicit Noor* (10) *iterative sequence respectively, where the sequences* $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n^1\}_{n=0}^{\infty}, \{\beta_n^2\}_{n=0}^{\infty} \subset (0,1)$ *such that,* $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=0}^{\infty} \beta_n^i = \infty$ *for* $i = 1, 2$. *Then the following are equivalent:*

a. (i) *the implicit Mann iterative sequence* (12) *converges strongly to the fixed point of the multivalued map* T;
(ii) *the implicit Ishikawa iterative sequence* (11) *converges strongly to the fixed point of the multivalued map* T.

b. (i) *the implicit Mann iterative sequence* (12) *converges strongly to the fixed point of the multivalued map* T;
(ii) *the implicit Noor iterative sequence* (10) *converges strongly to the fixed point of the multivalued map* T.

Theorem 4 *Let* ρ *satisfy* $(UUC1)$ *and* Δ_2-*condition. Let D be a* $\rho-$*closed,* $\rho-$*bounded and convex subset of a* $\rho-$*complete modular space* L_ρ *and* $T : D \to P_\rho(D)$ *be a multivalued mapping such that* P_ρ^T *is a* $\rho-$*contraction mapping, satisfying the condition*

$$H_\rho(Tf, Tg) \leq \delta\rho(f - g), \tag{46}$$

for all $f, g \in D$ *and* $F_\rho(T) \neq \emptyset$, *where* $\delta \in [0, 1)$. *Let* $f_0, g_0 \in D$ *and* $\{f_n\}, \{g_n\} \subset D$ *be defined by the implicit multistep* (9) *and implicit Mann iterative sequences* (12) *respectively, where the sequences* $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n^i\}_{n=0}^{\infty} \subset (0,1)$ *such that* $\sum_{n=0}^{\infty} \alpha_n = \infty, \sum_{n=0}^{\infty} \beta_n^i = \infty$ *for* $i = 1, 2, ..., k-1$. *Then the following are equivalent:*
(i) *the implicit Mann iterative sequence* (12) *converges strongly to the fixed point of the multivalued map* T;
(ii) *the implicit multistep iterative sequence* (9) *converges strongly to the fixed point of the multivalued map* T.

Proof. The method of proof of Theorem 4 is similar to that of Theorem 1. The proof is complete. □

Theorem 4 leads to the following corollary:

Corollary 5 *Let* ρ *satisfy* $(UUC1)$ *and* Δ_2-*condition. Let D be a* $\rho-$*closed,* $\rho-$*bounded and convex subset of a* $\rho-$*complete modular space* L_ρ *and* $T : D \to P_\rho(D)$ *be a multivalued mapping such that* P_ρ^T *is a* $\rho-$*contraction mapping, satisfying the condition*

$$H_\rho(Tf, Tg) \leq \delta\rho(f - g), \tag{47}$$

for all $g, h, g \in D$ *and* $F_\rho(T) \neq \emptyset$*, where* $\delta \in [0,1)$*. Let* $g_0, h_0, g_0 \in D$ *and* $\{g_n\}, \{h_n\}, \{g_n\} \subset D$ *be defined by the implicit Mann* (12)*, implicit Ishikawa* (11) *and implicit Noor* (10) *iterative sequence respectively, where the sequences* $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n^1\}_{n=0}^{\infty}, \{\beta_n^2\}_{n=0}^{\infty} \subset (0,1)$ *such that* $\sum_{n=0}^{\infty} \alpha_n = \infty$*,* $\sum_{n=0}^{\infty} \beta_n^i = \infty$ *for* $i = 1, 2$*. Then the following are equivalent:*
a. (i) *the implicit Mann iterative sequence* (12) *converges strongly to the fixed point of the multivalued map* T*;*
(ii) *the implicit Ishikawa iterative sequence* (11) *converges strongly to the fixed point of the multivalued map* T*.*

b. (i) *the implicit Mann iterative sequence* (12) *converges strongly to the fixed point of the multivalued map* T*;*
(ii) *the implicit Noor iterative sequence* (10) *converges strongly to the fixed point of the multivalued map* T*.*

3 Numerical example

Example 1 [3]*. Let* $M[0,1]$ *be the collection of all real-valued measurable functions on* $[0,1]$ *and* $\rho : M[0,1] \to R$ *a convex function modular defined by* $\rho(f) = \int_0^1 |f| \ \ \forall f \in M[0,1]$*. Let* $D = \{f \in L_\rho : \ 0 \leq f(x) \leq 2 \ \forall x \in [0,1]\}$ *be a subset of the modular function space* $L_\rho = M[0,1]$ *defined by* ρ*.* D *is nonempty, closed, and convex. Define map* $T : D \to P_\rho(D)$ *by* $Tf = \{\delta f\}$*, where* $\delta = 0.9$*.* T *satisfies property (I), has a unique fixed point* $f = 0$ *(since* $0 \in T(0)$*), and* P_ρ^T *is a* ρ*-contraction, with* $P_\rho^T(f) = \{Tf\} \ \forall f \in D$*. In fact,* P_ρ^T *is an* m*-strong* ρ*-strong contraction for all* $m \in N$*, since* $\rho(g) = m\rho(\frac{g}{m})$*.*

We present the results of convergence to $f = 0$ of implicit Mann iterative sequence (12), implicit Ishikawa iterative sequence (11), implicit Noor iterative sequence (10), and implicit multistep iterative sequence (9) using MATLAB. The parameters used are the following: $g_0(x) = h_0(x) = f_0(x) = 0.5x + 0.95 \ \forall x \in [0,1]$, $\alpha_n = \frac{1}{4} + \frac{1}{n+2}$, $\beta_n^i = \frac{1}{n+2}$ for $i = 1, 2, ..., k-1$, where $k = 11$ and $n = 1, 2, ..., 130$.