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Preopen sets and prelocally closed sets in generalised topology and minimal structure spaces

Anjalu Albis Basumatary
Central Institute of Technology, Kokrajhar,
Assam-783370, India
and
Tripura University, Suryamaninagar, Agartala,
Tripura-799022, India
email: albisbasu@gmail.com

Diganta Jyoti Sarma
Central Institute of Technology,
Kokrajhar, Assam-783370, India
email: dj.sarma@cit.ac.in

Binod Chandra Tripathy
Tripura University, Suryamaninagar,
Agartala, Tripura-799022, India
email: tripathybc@yahoo.com

Abstract. The intention of this article is to introduce and characterise the concept of preopen sets and prelocally closed sets in Generalised Topology and Minimal structure spaces.

1 Introduction

The notion of minimal structure space was introduced by Maki et al. [7] in 1999. The concepts of m-preopen sets and m-precontinous functions on minimal spaces was studied by Min and Kim [9]. Boonpok [1] introduced the concept of m-preopen sets and studied the notion of M-continuous and weakly M-continuous functions in Biminimal spaces. Carpintero et al.[4] also studied

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and characterised the concepts of m-preopen sets and their related notions in Biminimal spaces. The concept of Generalised topologies was introduced by Csaszar [5] in 2002. He also introduced the concepts of continuous functions and associated interior and closure operators on generalised neighborhood systems and generalised topological spaces. In particular, he investigated characterisations for the generalised continuous functions by using a closure operator defined on generalised neighborhood systems. The concept of generalised topology and minimal structure(GTMS) spaces was introduced by Buadong et al.[3] in 2011, which is a space with a generalised topology and a minimal structure. In 2013, Zakari [11] studied on some generalisations for closed sets in generalised topology and minimal structure spaces. He also studied *gm*-continuous functions between GTMS spaces in [12]. The idea of a locally closed set in topological space was defined by Kuratowski and Sierpinski [6]. Bourbaki [2] defined this notion in a way that a subset of a space is locally closed if it is the intersection of an open set and a closed set in X. Minimal structure in fuzzy topological spaces has been studied by Tripathy and Debnath [10].

2 Preliminaries

Definition 1 [7] *Let* $P(X)$ *be the power set of a nonempty set* X. *A subfamily* M_X *of* $P(X)$ *is called a minimal structure (briefly* $\mathfrak{m}$*-structure) on* X *if* $\emptyset \in M_X$ *and* $X \in M_X$. *A set* X *with an* $\mathfrak{m}$*-structure* M_X *is called an* $\mathfrak{m}$*-space and is denoted by* (X, M_X). *Each member of* M_X *is said to be an* M_X*-open set and the complement of an* M_X*-open set is said to be* M_X*-closed set.*

Definition 2 [7] *Let* (X, M_X) *be an* $\mathfrak{m}$*-space where,* X *is a nonempty set and* M_X *an* $\mathfrak{m}$*-structure on* X. *Let* S *be a subset of* X, *then the* M_X*-closure of* S *and the* M_X*-interior of* S *are defined as follows*

(a) $M_X\text{-}Cl(S) = \cap\{F : S \subset F, X \setminus F \in M_X\}$.

(b) $M_X\text{-}Int(S) = \cup\{G : G \subset S, G \in M_X\}$.

Lemma 1 [7] *Let* (X, M_X) *be an* $\mathfrak{m}$*-space where,* X *is a nonempty set and* M_X *an* $\mathfrak{m}$*-structure on* X. *Let* S *and* T *be subsets of* X, *then the following properties hold:*

(a) $M_X\text{-}Cl(X \setminus S) = X \setminus M_X\text{-}Int(S)$ *and* $M_X\text{-}Int(X \setminus S) = X \setminus M_X\text{-}Cl(S)$.

(b) *If* $(X \setminus S) \in M_X$, *then* $M_X\text{-}Cl(S) = S$ *and if* $S \in M_X$, *then* $M_X\text{-}Int(S) = S$.

(c) $M_X\text{-}Cl(\emptyset) = \emptyset$, $M_X\text{-}Cl(X) = X$, $M_X\text{-}Int(\emptyset) = \emptyset$ *and* $M_X\text{-}Int(X) = X$.

(d) *If* $S \subset T$*, then* $M_X\text{-}Cl(S) \subset M_X\text{-}Cl(T)$ *and* $M_X\text{-}Int(S) \subset M_X\text{-}Int(T)$.

(e) $M_X\text{-}Int(S) \subset S \subset M_X\text{-}Cl(S)$.

(f) $M_X\text{-}Cl(M_X\text{-}Cl(S)) = M_X\text{-}Cl(S)$ *and* $M_X\text{-}Int(M_X\text{-}Int(S)) = M_X\text{-}Int(S)$.

Definition 3 [9] *Let* (X, M_X) *be an* m*-space where,* X *is a nonempty set and* M_X *an* m*-structure on* X*. A subset* S *of* X *is said to be* M_X*-preopen set if* $S \subset M_X\text{-}Int(M_X\text{-}Cl(S))$*. The complement of an* M_X*-preopen set is called an* M_X*-preclosed set.*

Definition 4 [5] *Let* X *be a nonempty set and* G_X *a collection of subsets of* X*. Then* G_X *is called a generalised topology (briefly GT) on* X *if and only if* $\emptyset \in G_X$ *and* $G_i \in G_X$ *for* $i \in I \neq \emptyset$ *implies* $\cup_{(i \in I)} G_i \in G_X$*. The pair* (X, G_X) *is called a generalised topological space (briefly GTS) on* X*. The elements of* G_X *are called* G_X*-open sets and the complements are called* G_X*-closed sets.*

The closure of a subset S *in a generalised topological space* (X, G_X)*, denoted by* $G_X\text{-}Cl(S)$ *is the intersection of generalised closed sets including* S *and the interior of* S*, denoted by* $G_X\text{-}Int(S)$*, is the union of generalised open sets contained in* S*.*

Theorem 1 [3] *Let* (X, G_X) *be a generalised topological space and* $S \subseteq X$*. Then*

(a) $G_X\text{-}Cl(S) = X \setminus G_X\text{-}Int(X \setminus S)$.

(b) $G_X\text{-}Int(S) = X \setminus G_X\text{-}Cl(X \setminus S)$.

Proposition 1 [8] *Let* (X, G_X) *be a generalised topological space and* $S \subseteq X$*. Then*

(a) $x \in G_X\text{-}Int(S)$ *if and only if there exists* $V \in G_X$ *such that* $x \in V \subseteq S$*;*

(b) $x \in G_X\text{-}Cl(S)$ *if and only if* $V \cap S \neq \emptyset$ *for every* G_X*-open set* V *containing* x*.*

Proposition 2 [8] *Let* (X, G_X) *be a generalised topological space. Let* S *and* T *be subsets of* X*, then the following properties hold:*

(a) $G_X\text{-}Cl(X \setminus S) = X \setminus G_X\text{-}Int(S)$ *and* $G_X\text{-}Int(X \setminus S) = X \setminus G_X\text{-}Cl(S)$.

(b) *If* $X \setminus S \in G_X$*, then* $G_X\text{-}Cl(S) = S$ *and if* $S \in G_X$*, then* $G_X\text{-}Int(S) = S$.

(c) *If* $S \subseteq T$*, then* $G_X\text{-}Cl(S) \subseteq G_X\text{-}Cl(T)$ *and* $G_X\text{-}Int(S) \subseteq G_X\text{-}Int(T)$.

(d) $S \subseteq G_X\text{-}Cl(S)$ *and* $G_X\text{-}Int(S) \subseteq S$.

(e) $G_X\text{-}Cl(G_X\text{-}Cl(S)) = G_X\text{-}Cl(S)$ *and* $G_X\text{-}Int(G_X\text{-}Int(S)) = G_X\text{-}Int(S)$.

(One may refer to Buadong et al. [3] Proposition 2.4)

Definition 5 [3] *Let* X *be a nonempty set and let* $\mathsf{G_X}$ *be a generalised topology and* $\mathsf{M_X}$ *a minimal structure on* X*. A triple* $(\mathsf{X}, \mathsf{G_X}, \mathsf{M_X})$ *is called a generalised topology and minimal structure space (briefly GTMS space).*

Let $(\mathsf{X}, \mathsf{G_X}, \mathsf{M_X})$ *be a GTMS space and* S *be a subset of* X*. The closure and interior of* S *in* $\mathsf{G_X}$ *are denoted by* $\mathsf{G_X}\text{-}\mathsf{Cl}(\mathsf{S})$ *and* $\mathsf{G_X}\text{-}\mathsf{Int}(\mathsf{S})$*, respectively. The closure and interior of* S *in* $\mathsf{M_X}$ *are denoted by* $\mathsf{M_X}\text{-}\mathsf{Cl}(\mathsf{S})$ *and* $\mathsf{M_X}\text{-}\mathsf{Int}(\mathsf{S})$*, respectively.*

Definition 6 [3] *Let* $(\mathsf{X}, \mathsf{G_X}, \mathsf{M_X})$ *be a GTMS space. A subset* S *of* X *is said to be a* $(\mathsf{G_X}, \mathsf{M_X})$*-closed set if* $\mathsf{G_X}\text{-}\mathsf{Cl}(\mathsf{M_X}\text{-}\mathsf{Cl}(\mathsf{S})) = \mathsf{S}$ *and a subset* S *of* X *is said to be a* $(\mathsf{M_X}, \mathsf{G_X})$*-closed set if* $\mathsf{M_X}\text{-}\mathsf{Cl}(\mathsf{G_X}\text{-}\mathsf{Cl}(\mathsf{S})) = \mathsf{S}$*. The complement of a* $(\mathsf{G_X}, \mathsf{M_X})$*-closed (resp.* $(\mathsf{M_X}, \mathsf{G_X})$*-closed) set is said to be* $(\mathsf{G_X}, \mathsf{M_X})$*-open set (resp.* $(\mathsf{M_X}, \mathsf{G_X})$*-open set).*

Lemma 2 [3] *Let* $(\mathsf{X}, \mathsf{G_X}, \mathsf{M_X})$ *be a GTMS space and* $\mathsf{S} \subseteq \mathsf{X}$*. Then*

(a) S *is* $(\mathsf{G_X}, \mathsf{M_X})$*-closed if and only if* $\mathsf{M_X}\text{-}\mathsf{Cl}(\mathsf{S}) = \mathsf{S}$ *and* $\mathsf{G_X}\text{-}\mathsf{Cl}(\mathsf{S}) = \mathsf{S}$*.*

(b) S *is* $(\mathsf{M_X}, \mathsf{G_X})$*-closed if and only if* $\mathsf{M_X}\text{-}\mathsf{Cl}(\mathsf{S}) = \mathsf{S}$ *and* $\mathsf{G_X}\text{-}\mathsf{Cl}(\mathsf{S}) = \mathsf{S}$*.*

Proposition 3 [3] *Let* $(\mathsf{X}, \mathsf{G_X}, \mathsf{M_X})$ *be a GTMS space and* $\mathsf{S} \subseteq \mathsf{X}$*. Then* S *is* $(\mathsf{G_X}, \mathsf{M_X})$*-closed if and only if* S *is* $(\mathsf{M_X}, \mathsf{G_X})$*-closed.*

Definition 7 [3] *Let* $(\mathsf{X}, \mathsf{G_X}, \mathsf{M_X})$ *be a GTMS space and* S *be a subset of* X*. Then* S *is said to be a closed set if* S *is* $(\mathsf{G_X}, \mathsf{M_X})$*-closed. The complement of a closed set is an open set.*

Proposition 4 [3] *Let* $(\mathsf{X}, \mathsf{G_X}, \mathsf{M_X})$ *be a GTMS space. Then* S *is open if and only if* $\mathsf{S} = \mathsf{G_X}\text{-}\mathsf{Int}(\mathsf{M_X}\text{-}\mathsf{Int}(\mathsf{S}))$*.*

Proposition 5 [3] *Let* $(\mathsf{X}, \mathsf{G_X}, \mathsf{M_X})$ *be a GTMS space.*

(a) *If* S *and* T *are closed, then* $\mathsf{S} \cap \mathsf{T}$ *is closed.*

(b) *If* S *and* T *are open, then* $\mathsf{S} \cup \mathsf{T}$ *is open.*

Definition 8 [3] *Let* $(\mathsf{X}, \mathsf{G_X}, \mathsf{M_X})$ *be a GTMS space and* T *be a subset of* X*. Then* T *is said to be* s*-closed if* $\mathsf{G_X}\text{-}\mathsf{Cl}(\mathsf{T}) = \mathsf{M_X}\text{-}\mathsf{Cl}(\mathsf{T})$ *and* T *is said to be* c*-closed if* $\mathsf{G_X}\text{-}\mathsf{Cl}(\mathsf{M_X}\text{-}\mathsf{Cl}(\mathsf{T})) = \mathsf{M_X}\text{-}\mathsf{Cl}(\mathsf{G_X}\text{-}\mathsf{Cl}(\mathsf{T}))$*. The complement of a* s*-closed (resp.* c*-closed) set is called a* s*-open (resp.* c*-open) set.*

Proposition 6 [3] *Let* (X, G_X, M_X) *be a GTMS space and* $T \subseteq X$. *Then*

(a) T *is* s*-open if and only if* $G_X\text{-}Int(T) = M_X\text{-}Int(T)$.

(b) T *is* c*-open if and only if* $G_X\text{-}Int(M_X\text{-}Int(T)) = M_X\text{-}Int(G_X\text{-}Int(T))$

Proposition 7 [3] *Let* (X, G_X, M_X) *be a GTMS space and* $T \subseteq X$.

(a) *If* T *is closed, then* T *is* s*-closed.*

(b) *If* T *is* s*-closed, then* T *is* c*-closed.*

3 (G_X, M_X)-preopen sets

Definition 9 *Let* (X, G_X, M_X) *be a GTMS space. A subset* S *of* X *is said to be a* (G_X, M_X)*-preopen set if* $S \subset G_X\text{-}Int(M_X\text{-}Cl(S))$ *and* (M_X, G_X)*-preopen set if* $S \subset M_X\text{-}Int(G_X\text{-}Cl(S))$. *A subset* S *of* X *is said to be a* (G_X, M_X)*-preclosed set (*(M_X, G_X)*-preclosed set) if the complement* $X \setminus S$ *of* S *is a* (G_X, M_X)*-preopen set (respectively,* (M_X, G_X)*-preopen set). The set of all* (G_X, M_X)*-preopen sets of* (X, G_X, M_X) *is denoted by* $(G_X, M_X)\text{-}PO(X)$ *and the set of all* (G_X, M_X)*-preclosed sets of* (X, G_X, M_X) *is denoted by* $(G_X, M_X)\text{-}PC(X)$.

Remark 1 *A set which is* (G_X, M_X)*-preopen need not be* (M_X, G_X)*-preopen in general as can be seen from the following example.*

Example 1 *Let* $X = \{a, b, c\}$. *We define generalised topology* G_X *and minimal structure space* M_X *on* X *as follows:* $G_X = \{\emptyset, \{a\}, \{a, c\}\}$ *and* $M_X = \{\emptyset, \{b\}, \{b, c\}, X\}$. *Let* $S = \{b\}$ *and* $T = \{c\}$ *then* S *is* (M_X, G_X)*-preopen but is not* (G_X, M_X)*-preopen set whereas* T *is both* (G_X, M_X)*-preopen and* (M_X, G_X)*-preopen.*

Remark 2 *The intersection of two* (G_X, M_X)*-preopen sets need not be a* (G_X, M_X)*-preopen set as the following example illustrates.*

Example 2 *Let* $X = \{a, b, c, d\}$. *We define generalised topology* G_X *and minimal structure space* M_X *on* X *by* $G_X = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}$ *and* $M_X = \{\emptyset, \{a\}, \{a, c\}, X\}$. *Let* $S = \{a, b\}$ *and* $T = \{b, c\}$. *Then* S *and* T *are* (G_X, M_X)*-preopen sets but* $S \cap T = \{b\}$ *is not a* (G_X, M_X)*-preopen set.*

Definition 10 *Let* (X, G_X, M_X) *be a GTMS space and* S *be a subset of* X. *Then*

(a) *the* $(\mathsf{G_X}, \mathsf{M_X})$*-preclosure of* S *(*$(\mathsf{M_X}, \mathsf{G_X})$*-preclosure of* S*) denoted by* $(\mathsf{G_X}, \mathsf{M_X})$-$\mathsf{Cl_p}(\mathsf{S})$ *(respectively,* $(\mathsf{M_X}, \mathsf{G_X})$-$\mathsf{Cl_p}(\mathsf{S})$*) is defined as the intersection of all* $(\mathsf{G_X}, \mathsf{M_X})$*-preclosed (respectively,* $(\mathsf{M_X}, \mathsf{G_X})$*-preclosed) sets containing* S*.*

b) *the* $(\mathsf{G_X}, \mathsf{M_X})$*-preinterior of* S *(*$(\mathsf{M_X}, \mathsf{G_X})$*-preinterior of* S*) denoted by* $(\mathsf{G_X}, \mathsf{M_X})$-$\mathsf{Int_p}(\mathsf{S})$ *(respectively,* $(\mathsf{M_X}, \mathsf{G_X})$-$\mathsf{Int_p}(\mathsf{S})$*) is defined as the union of all* $(\mathsf{G_X}, \mathsf{M_X})$*-preopen (respectively,* $(\mathsf{M_X}, \mathsf{G_X})$*-preopen) sets contained in* S*.*

Theorem 2 *Let* $(\mathsf{X}, \mathsf{G_X}, \mathsf{M_X})$ *be a GTMS space and* T *be a subset of* X*. Then*

(a) *if* T *is* s*-closed,* $(\mathsf{G_X}, \mathsf{M_X})$*-preopen set* T *is a* $\mathsf{G_X}$*-preopen set and* $(\mathsf{M_X}, \mathsf{G_X})$*-preopen set* T *is a* $\mathsf{M_X}$*-preopen set.*

(b) *if* T *is* s*-open,* $(\mathsf{G_X}, \mathsf{M_X})$*-preopen set* T *is a* $\mathsf{M_X}$*-preopen set and* $(\mathsf{M_X}, \mathsf{G_X})$*-preopen set* T *is a* $\mathsf{G_X}$*-preopen set.*

Proof.

(a) Let T be a $(\mathsf{G_X}, \mathsf{M_X})$-preopen set. Then $\mathsf{T} \subset \mathsf{G_X}\text{-}\mathsf{Int}(\mathsf{M_X}\text{-}\mathsf{Cl}(\mathsf{T}))$. But T is s-closed, so $\mathsf{G_X}\text{-}\mathsf{Cl}(\mathsf{T}) = \mathsf{M_X}\text{-}\mathsf{Cl}(\mathsf{T})$. Thus $\mathsf{T} \subset \mathsf{G_X}\text{-}\mathsf{Int}(\mathsf{G_X}\text{-}\mathsf{Cl}(\mathsf{T}))$. Hence T is a $\mathsf{G_X}$-preopen set. Similarly, if T is a $(\mathsf{M_X}, \mathsf{G_X})$-preopen set then T is a $\mathsf{M_X}$-preopen set.

(b) Let T be a $(\mathsf{G_X}, \mathsf{M_X})$-preopen set. Then $\mathsf{T} \subset \mathsf{G_X}\text{-}\mathsf{Int}(\mathsf{M_X}\text{-}\mathsf{Cl}(\mathsf{T}))$. But T is s-open, so $\mathsf{G_X}\text{-}\mathsf{Int}(\mathsf{T}) = \mathsf{M_X}\text{-}\mathsf{Int}(\mathsf{T})$, by Proposition 6. Thus $\mathsf{T} \subset \mathsf{M_X}\text{-}\mathsf{Int}(\mathsf{M_X}\text{-}\mathsf{Cl}(\mathsf{T}))$. Hence T is a $\mathsf{M_X}$-preopen set. Similarly, if T is a $(\mathsf{M_X}, \mathsf{G_X})$-preopen set then T is a $\mathsf{G_X}$-preopen set.

□

Theorem 3 *Let* $(\mathsf{X}, \mathsf{G_X}, \mathsf{M_X})$ *be a GTMS space. Then the arbitrary union of* $(\mathsf{G_X}, \mathsf{M_X})$*-preopen (*$(\mathsf{M_X}, \mathsf{G_X})$*-preopen) sets is a* $(\mathsf{G_X}, \mathsf{M_X})$*-preopen (respectively,* $(\mathsf{M_X}, \mathsf{G_X})$*-preopen) set.*

Proof. Let $\{\mathsf{S}_\alpha\}_{\alpha\in\Lambda}$ be a family of $(\mathsf{G_X}, \mathsf{M_X})$-preopen sets in $(\mathsf{X}, \mathsf{G_X}, \mathsf{M_X})$. Since, $\mathsf{S}_\alpha \subset \mathsf{G_X}\text{-}\mathsf{Int}(\mathsf{M_X}\text{-}\mathsf{Cl}(\mathsf{S}_\alpha))\ \forall \alpha \in \Lambda$. So $\cup_{\alpha\in\Lambda}\mathsf{S}_\alpha \subset \cup_{\alpha\in\Lambda}\{\mathsf{G_X}\text{-}\mathsf{Int}(\mathsf{M_X}\text{-}\mathsf{Cl}(\mathsf{S}_\alpha))\} \subset \mathsf{G_X}\text{-}\mathsf{Int}(\cup_{\alpha\in\Lambda}\{\mathsf{M_X}\text{-}\mathsf{Cl}(\mathsf{S}_\alpha)\}) = \mathsf{G_X}\text{-}\mathsf{Int}(\mathsf{M_X}\text{-}\ \mathsf{Cl}\{\cup_{\alpha\in\Lambda}\mathsf{S}_\alpha\})$. Thus, $\cup_{\alpha\in\Lambda}\mathsf{S}_\alpha$ is a $(\mathsf{G_X}, \mathsf{M_X})$-preopen set in $(\mathsf{X}, \mathsf{G_X}, \mathsf{M_X})$. □

Theorem 4 *Let* $(\mathsf{X}, \mathsf{G_X}, \mathsf{M_X})$ *be GTMS space and* S *and* T *be subsets of* X*. Then the following properties hold:*

(a) $(G_X, M_X)\text{-}Int_p(S) = \cup\{F : F \subset S$ *and* $F \in (G_X, M_X)\text{-}PO(X)\}$

(b) $(G_X, M_X)\text{-}Int_p(S)$ *is the largest* (G_X, M_X)*-preopen subset of* X *contained in* S.

(c) S *is* (G_X, M_X)*-preopen if and only if* $S = (G_X, M_X)\text{-}Int_p(S)$

(d) $(G_X, M_X)\text{-}Int_p((G_X, M_X)\text{-}Int_p(S)) = (G_X, M_X)\text{-}Int_p(S)$

(e) *If* $S \subset T$, *then* $(G_X, M_X)\text{-}Int_p(S) \subset (G_X, M_X)\text{-}Int_p(T)$.

(f) $(G_X, M_X)\text{-}Int_p(S) \cup (G_X, M_X)\text{-}Int_p(T) \subset (G_X, M_X)\text{-}Int_p(S \cup T)$

(g) $(G_X, M_X)\text{-}Int_p(S \cap T) \subset (G_X, M_X)\text{-}Int_p(S) \cap (G_X, M_X)\text{-}Int_p(T)$

Proof.

(a) Let $x \in (G_X, M_X)\text{-}Int_p(S)$. Then there exists $F \in (G_X, M_X)\text{-}PO(X)$ containing x such that $x \in F \subset S$. So, $x \in \cup\{F : F \subset S$ and $F \in (G_X, M_X)\text{-}PO(X)\}$ showing that $(G_X, M_X)\text{-}Int_p(S) \subset \cup\{F : F \subset S$ and $F \in (G_X, M_X)\text{-}PO(X)\}$. Let $x \in \cup\{F : F \subset S$ and $F \in (G_X, M_X)\text{-}PO(X)\}$. Then there exists a $F \in (G_X, M_X)\text{-}PO(X)$ containing x such that $x \in F \subset S$. Thus $x \in (G_X, M_X)\text{-}Int_p(S)$ showing that $\cup\{F : F \subset S$ and $F \in (G_X, M_X)\text{-}PO(X)\} \subset (G_X, M_X)\text{-}Int_p(S)$. Hence, $(G_X, M_X)\text{-}Int_p(S) = \cup\{F : F \subset S$ and $F \in (G_X, M_X)\text{-}PO(X)\}$.

The proofs of (b) – (e) are evident.

(f) $(G_X, M_X)\text{-}Int_p(S) \subset (G_X, M_X)\text{-}Int_p(S \cup T)$ and $(G_X, M_X)\text{-}Int_p(T) \subset (G_X, M_X)\text{-}Int_p(S \cup T)$. Then by (e) we obtain $(G_X, M_X)\text{-}Int_p(S) \cup (G_X, M_X)\text{-}Int_p(T) \subset (G_X, M_X)\text{-}Int_p(S \cup T)$.

(g) Since $S \cap T \subset S$ and $S \cap T \subset T$, by (e) we get $(G_X, M_X)\text{-}Int_p(S \cap T) \subset (G_X, M_X)\text{-}Int_p(S)$ and $(G_X, M_X)\text{-}Int_p(S \cap T) \subset (G_X, M_X)\text{-}Int_p(T)$. Thus by (e), $(G_X, M_X)\text{-}Int_p(S \cap T) \subset (G_X, M_X)\text{-}Int_p(S) \cap (G_X, M_X)\text{-}Int_p(T)$.

□

Theorem 5 *Let* S *and* T *be subsets of* (X, G_X, M_X). *Then the following properties hold:*

(a) $(G_X, M_X)\text{-}Cl_p(S) = \bigcap\{G : S \subset G$ *and* $G \in (G_X, M_X)\text{-}PC(X)\}$.

(b) $(G_X, M_X)\text{-}Cl_p(S)$ *is the smallest* (G_X, M_X)*-preclosed subset of* X *containing* S.

(c) S *is* (G_X, M_X)*-preclosed if and only if* $S = (G_X, M_X)\text{-}Cl_p(S)$.

(d) $(G_X, M_X)\text{-}Cl_p((G_X, M_X)\text{-}Cl_p(S)) = (G_X, M_X)\text{-}Cl_p(S)$.

(e) *If* $\mathsf{S} \subset \mathsf{T}$ *then* $(\mathsf{G_X}, \mathsf{M_X})\text{-}\mathsf{Cl_p}(\mathsf{S}) \subset (\mathsf{G_X}, \mathsf{M_X})\text{-}\mathsf{Cl_p}(\mathsf{T})$.

(f) $(\mathsf{G_X}, \mathsf{M_X})\text{-}\mathsf{Cl_p}(\mathsf{S}) \cup (\mathsf{G_X}, \mathsf{M_X})\text{-}\mathsf{Cl_p}(\mathsf{T}) \subset (\mathsf{G_X}, \mathsf{M_X})\text{-}\mathsf{Cl_p}(\mathsf{S} \cup \mathsf{T})$.

(g) $(\mathsf{G_X}, \mathsf{M_X})\text{-}\mathsf{Cl_p}(\mathsf{S} \cap \mathsf{T}) \subset (\mathsf{G_X}, \mathsf{M_X})\text{-}\mathsf{Cl_p}(\mathsf{S}) \cap (\mathsf{G_X}, \mathsf{M_X})\text{-}\mathsf{Cl_p}(\mathsf{T})$.

Proof.

(a) Suppose that $x \notin (\mathsf{G_X}, \mathsf{M_X})\text{-}\mathsf{Cl_p}(\mathsf{S})$. Then there exists $\mathsf{V} \in (\mathsf{G_X}, \mathsf{M_X})$-$\mathsf{PO}(\mathsf{X})$ containing x such that $\mathsf{V} \cap \mathsf{S} = \emptyset$. Since $\mathsf{X} \setminus \mathsf{V}$ is a $(\mathsf{G_X}, \mathsf{M_X})$-preclosed set containing S and $x \notin \mathsf{X} \setminus \mathsf{V}$, we get $x \notin \cap\{\mathsf{G} : \mathsf{S} \subset \mathsf{G}$ and $\mathsf{G} \in (\mathsf{G_X}, \mathsf{M_X})\text{-}\mathsf{PC}(\mathsf{X})\}$. Conversely, suppose that $x \notin \cap\{\mathsf{G} : \mathsf{S} \subset \mathsf{G}$ and $\mathsf{G} \in (\mathsf{G_X}, \mathsf{M_X})\text{-}\mathsf{PC}(\mathsf{X})\}$. Then there exists $\mathsf{G} \in (\mathsf{G_X}, \mathsf{M_X})\text{-}\mathsf{PC}(\mathsf{X})$ such that $\mathsf{S} \subset \mathsf{G}$ and $x \notin \mathsf{G}$. Since $\mathsf{X} \setminus \mathsf{G}$ is a $(\mathsf{G_X}, \mathsf{M_X})$-preopen set containing x, we get$(\mathsf{X} \setminus \mathsf{G}) \cap \mathsf{S} = \emptyset$. This shows that $x \notin (\mathsf{G_X}, \mathsf{M_X})\text{-}\mathsf{Cl_p}(\mathsf{S})$. Thus we get $(\mathsf{G_X}, \mathsf{M_X})\text{-}\mathsf{Cl_p}(\mathsf{S})=\cap\{\mathsf{G} : \mathsf{S} \subset \mathsf{G}$ and $\mathsf{G} \in (\mathsf{G_X}, \mathsf{M_X})\text{-}\mathsf{PC}(\mathsf{X})\}$.

The proofs of the other parts can be established similarly. □

We formulate the following result, which can be established easily.

Theorem 6 *Let* $(\mathsf{X}, \mathsf{G_X}, \mathsf{M_X})$ *be a GTMS space and* $\mathsf{S} \subset \mathsf{X}$. *Then the following properties hold:*

(a) $(\mathsf{G_X}, \mathsf{M_X})\text{-}\mathsf{Int_p}(\mathsf{X} \setminus \mathsf{S}) = \mathsf{X} \setminus (\mathsf{G_X}, \mathsf{M_X})\text{-}\mathsf{Cl_p}(\mathsf{S})$

(b) $(\mathsf{G_X}, \mathsf{M_X})\text{-}\mathsf{Cl_p}(\mathsf{X} \setminus \mathsf{S}) = \mathsf{X} \setminus (\mathsf{G_X}, \mathsf{M_X})\text{-}\mathsf{Int_p}(\mathsf{S})$.

4 $(\mathsf{G_X}, \mathsf{M_X})$-prelocally closed sets

Definition 11 *A subset* Q *of a GTMS* $(\mathsf{X}, \mathsf{G_X}, \mathsf{M_X})$ *is said to be*

(a) $(\mathsf{G_X}, \mathsf{M_X})$-*locally closed (in short* $(\mathsf{G_X}, \mathsf{M_X})\text{-}\mathsf{L_c}$*) if* $\mathsf{Q} = \mathsf{R} \cap \mathsf{S}$, *where* R *is* $\mathsf{G_X}$-*open and* S *is* $\mathsf{M_X}$-*closed.*

(b) $(\mathsf{G_X}, \mathsf{M_X})$-*Prelocally closed (in short* $(\mathsf{G_X}, \mathsf{M_X})\text{-}\mathsf{PL_c}$*) if* $\mathsf{Q} = \mathsf{R} \cap \mathsf{S}$, *where* R *is* $\mathsf{G_X}$-*preopen and* S *is* $\mathsf{M_X}$-*preclosed.*

(c) $(\mathsf{G_X}, \mathsf{M_X})$-*Prelocally closed* (in short* $(\mathsf{G_X}, \mathsf{M_X})\text{-}\mathsf{PL_c^*}$*) if* $\mathsf{Q} = \mathsf{R} \cap \mathsf{S}$, *where* R *is* $\mathsf{G_X}$-*preopen and* S *is* $\mathsf{M_X}$-*closed.*

(d) $(\mathsf{G_X}, \mathsf{M_X})$-*Prelocally closed** (in short* $(\mathsf{G_X}, \mathsf{M_X})\text{-}\mathsf{PL_c^{**}}$*) if* $\mathsf{Q} = \mathsf{R} \cap \mathsf{S}$, *where* R *is* $\mathsf{G_X}$-*open and* S *is* $\mathsf{M_X}$-*preclosed.*

Here we denote the collection of all $(\mathsf{G_X}, \mathsf{M_X})\text{-}\mathsf{L_c}$, $(\mathsf{G_X}, \mathsf{M_X})\text{-}\mathsf{PL_c}$, $(\mathsf{G_X}, \mathsf{M_X})\text{-}\mathsf{PL_c^*}$ *and* $(\mathsf{G_X}, \mathsf{M_X})\text{-}\mathsf{PL_c^{**}}$ *sets in* $(\mathsf{X}, \mathsf{G_X}, \mathsf{M_X})$ *by* $(\mathsf{G_X}, \mathsf{M_X})\text{-}\mathsf{L_c}(\mathsf{X})$, $(\mathsf{G_X}, \mathsf{M_X})\text{-}\mathsf{PL_c}(\mathsf{X})$, $(\mathsf{G_X}, \mathsf{M_X})\text{-}\mathsf{PL_c^*}(\mathsf{X})$ *and* $(\mathsf{G_X}, \mathsf{M_X})\text{-}\mathsf{PL_c^{**}}(\mathsf{X})$ *respectively.*

Theorem 7 *If* $R, S \in (G_X, M_X)\text{-}PL_c(X)$*, then* $R \cap S \in (G_X, M_X)\text{-}PL_c(X)$*.*

Proof. Since $R, S \in (G_X, M_X)\text{-}PL_c(X)$, so $R = V \cap W$ where V is G_X-preopen and W is M_X-preclosed, and $S = A \cap B$ where A is G_X-preopen and B is M_X-preclosed. Now, $R \cap S = (V \cap W) \cap (A \cap B) = (V \cap A) \cap (W \cap B)$. Since $V \cap A$ is G_X-preopen and $W \cap B$ is M_X-preclosed, so $R \cap S \in (G_X, M_X)\text{-}PL_c(X)$. □

Remark 3 *The converse of the Theorem 7 is not necessarily true as shown in the example below.*

Example 3 *Let* $X = \{a, b, c\}$*,* $G_X = \{\emptyset, \{a\}, \{a, c\}\}$ *and* $M_X = \{\emptyset, \{b\}, \{b, c\}, X\}$*. Then* $(G_X, M_X)\text{-}PL_c(X) = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$*. We have* $\{a, b\} \cap \{a, c\} = \{a\} \in (G_X, M_X)\text{-}PL_c(X)$ *but* $\{a, b\} \notin (G_X, M_X)\text{-}PL_c(X)$*.*

Theorem 8 *Let* S *be a subset of* (X, G_X, M_X)*. Then*

(a) *If* $S \in (G_X, M_X)\text{-}L_c(X)$*, then* $S \in (G_X, M_X)\text{-}PL_c(X)$*,* $(G_X, M_X)\text{-}PL_c^*(X)$ *and* $(G_X, M_X)\text{-}PL_c^{**}(X)$*.*

(b) *If* $S \in (G_X, M_X)\text{-}PL_c^*(X)$*, then* $S \in (G_X, M_X)\text{-}PL_c(X)$*.*

(c) *If* $S \in (G_X, M_X)\text{-}PL_c^{**}(X)$*, then* $S \in (G_X, M_X)\text{-}PL_c(X)$*.*

Proof. The above stated results are true in the sense that every G_X-open set is G_X-preopen and every M_X-closed set is M_X-preclosed. □

Remark 4 *The converse of the Theorem 8 is not true in general as the following example illustrates.*

Example 4 *Let* $X = \{a, b, c\}$*,* $G_X = \{\emptyset, \{a\}, X\}$ *and* $M_X = \{\emptyset, \{b\}, X\}$*. Then*
$(G_X, M_X)\text{-}L_c(X) = \{\emptyset, \{a\}, \{a, c\}, X\}$*,*
$(G_X, M_X)\text{-}PL_c(X) = \{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, X\}$*,*
$(G_X, M_X)\text{-}PL_c^*(X) = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$ *and*
$(G_X, M_X)\text{-}PL_c^{**}(X) = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$*.*
It is clearly seen that

(a) $\{c\} \in (G_X, M_X)\text{-}PL_c(X)$ *but* $\{c\} \notin (G_X, M_X)\text{-}L_c(X)$*;* $\{a, b\} \in (G_X, M_X)\text{-}PL_c^*(X)$ *but* $\{a, b\} \notin (G_X, M_X)\text{-}L_c(X)$*;* $\{c\} \in (G_X, M_X)\text{-}PL_c^{**}(X)$ *but* $\{c\} \notin (G_X, M_X)\text{-}L_c(X)$*.*

(b) $\{c\} \in (G_X, M_X)\text{-}PL_c(X)$ *but* $\{c\} \notin (G_X, M_X)\text{-}PL_c^*(X)$*.*

(c) $\{a, b\} \in (G_X, M_X)\text{-}PL_c(X)$ *but* $\{a, b\} \notin (G_X, M_X)\text{-}PL_c^{**}(X)$*.*

Theorem 9 *Let* T *be a subset of* (X, G_X, M_X), *then the following statements are equivalent:*

(a) $T \in (G_X, M_X)\text{-}PL_c(X)$.
(b) $T = Q \cap M_X\text{-}Cl_p(T)$ *for some* G_X*-preopen set* Q.
(c) $M_X\text{-}Cl_p(T) \setminus T$ *is* G_X*-preclosed.*
(d) $T \cup (X \setminus M_X\text{-}Cl_p(T))$ *is* G_X*-preopen*
(e) $T \subset G_X\text{-}Int_p(T \cup (X \setminus M_X\text{-}Cl_p(T)))$.

Proof. (a) ⇒ (b) Let $T \in (G_X, M_X)\text{-}PL_c(X)$. Then $T = Q \cap R$, where Q is G_X-preopen and R is M_X-preclosed. Since $T \subset R$ and $M_X\text{-}Cl_p(T)$ is the smallest M_X-preclosed set containing T, so $M_X\text{-}Cl_p(T) \subset R$. Now,

$$T = Q \cap R \supset Q \cap M_X - Cl_p(T) \tag{1}$$

Since $T \subset Q$ and $T \subset M_X\text{-}Cl_p(T)$, so

$$T \subset Q \cap M_X - Cl_p(T) \tag{2}$$

Hence, from (1) and (2), $T = Q \cap M_X\text{-}Cl_p(T)$.

(b) ⇒ (c) Let $T = Q \cap M_X\text{-}Cl_p(T)$ for some G_X-preopen set Q. Now, $M_X\text{-}Cl_p(T) \setminus T = M_X\text{-}Cl_p(T) \setminus (Q \cap M_X\text{-}Cl_p(T)) = (M_X\text{-}Cl_p(T) \setminus Q) \cup (M_X\text{-}Cl_p(T) \setminus M_X\text{-}Cl_p(T)) = M_X\text{-}Cl_p(T) \setminus Q = M_X\text{-}Cl_p(T) \cap (X \setminus Q)$, which is M_X-preclosed, since $X \setminus Q$ is M_X-preclosed.

(c) ⇒ (d) $T \cup (X \setminus M_X\text{-}Cl_p(T)) = X \setminus (M_X\text{-}Cl_p(T) \setminus T)$. By (c), $(M_X\text{-}Cl_p(T) \setminus T$ is G_X-preclosed, so $X \setminus M_X\text{-}C_p(T) \setminus T)$ is G_X-preopen. Hence, $T \cup (X \setminus M_X\text{-}Cl_p(T))$ is G_X-preopen.

(d) ⇒ (e) Let $T \cup (X \setminus M_X\text{-}Cl_p(T))$ be G_X-preopen. Then $T \cup (X \setminus M_X\text{-}Cl_p(T)) = G_X\text{-}Int_p(T \cup (X \setminus M_X\text{-}Cl_p(T)))$. Hence $T \subset G_X\text{-}Int_p(T \cup (X \setminus M_X\text{-}Cl_p(T)))$.

(e) ⇒ (a) By (e) we have $T \subset G_X\text{-}Int_p(T \cup (X \setminus M_X\text{-}Cl_p(T)))$. Since, $T \subset M_X\text{-}Cl_p(T)$, so $T \subset G_X\text{-}Int_p(T \cup (X \setminus M_X\text{-}Cl_p(T))) \cap M_X\text{-}Cl_p(T)$. Further, $\{G_X\text{-}Int_p(T \cup (X \setminus M_X\text{-}Cl_p(T))) \cap M_X\text{-}Cl_p(T)\} \subset \{T \cup (X \setminus M_X\text{-}Cl_p(T))) \cap M_X\text{-}Cl_p(T)\} = \{M_X\text{-}Cl_p(T) \cap T\} \cup \{M_X\text{-}Cl_p(T) \cap (X \setminus M_X\text{-}Cl_p(T))\} = T$. Consequently, $T = G_X\text{-}Int_p(T \cup (X \setminus G_X\text{-}Cl_p(T))) \cap M_X\text{-}Cl_p(T)$. Since, $G_X\text{-}Int_p(T \cup (X \setminus M_X - Cl_p(T)))$ is G_X-preopen and $M_X\text{-}Cl_p(T)$ is M_X-preclosed, so $T \in (G_X, M_X)\text{-}PL_c(X)$. □

Theorem 10 *If* $S \subset T \subset (X, G_X, M_X)$ *and* $T \in (G_X, M_X)\text{-}PL_c(X)$, *then there exists* $R \in (G_X, M_X)\text{-}PL_c(X)$ *such that* $S \subset R \subset T$.

Proof. Since $\mathsf{T} \in (\mathsf{G_X}, \mathsf{M_X})$-$\mathsf{PL_c}(\mathsf{X})$, by Theorem 9., we have $\mathsf{T} = \mathsf{Q} \cap \mathsf{M_X}$-$\mathsf{Cl_p}(\mathsf{T})$ where Q is $\mathsf{G_X}$-preopen. As $\mathsf{S} \subset \mathsf{T}$ and $\mathsf{T} \subset \mathsf{Q}$, so $\mathsf{S} \subset \mathsf{Q}$. Also, $\mathsf{S} \subset \mathsf{M_X}$-$\mathsf{Cl_p}(\mathsf{S})$. Therefore, $\mathsf{S} \subset \mathsf{Q} \cap \mathsf{M_X}$-$\mathsf{Cl_p}(\mathsf{S})$. Now, $\mathsf{R} = \mathsf{Q} \cap \mathsf{M_X}$-$\mathsf{Cl_p}(\mathsf{S}) \subset \mathsf{Q} \cap \mathsf{M_X}$-$\mathsf{Cl_p}(\mathsf{T}) = \mathsf{T}$ Since, Q is $\mathsf{G_X}$-preopen and $\mathsf{M_X}$-$\mathsf{Cl_p}(\mathsf{S})$ is $\mathsf{M_X}$-preclosed, so $\mathsf{R} \in (\mathsf{G_X}, \mathsf{M_X})$-$\mathsf{PL_c}(\mathsf{X})$ such that $\mathsf{S} \subset \mathsf{R} \subset \mathsf{T}$. □

Theorem 11 *Let* S *be a subset of* $(\mathsf{X}, \mathsf{G_X}, \mathsf{M_X})$. *Then*

(a) $\mathsf{S} \in (\mathsf{G_X}, \mathsf{M_X})$-$\mathsf{PL}^*_\mathsf{c}(\mathsf{X})$ *if and only if* $\mathsf{S} = \mathsf{T} \cap \mathsf{M_X}$-$\mathsf{Cl}(\mathsf{S})$ *for some* $\mathsf{G_X}$-*preopen set* T.

(b) $\mathsf{M_X}$-$\mathsf{Cl}(\mathsf{S}) \setminus \mathsf{S}$ *is* $\mathsf{G_X}$-*preclosed if and only if* $\mathsf{S} \cup (\mathsf{X} \setminus \mathsf{M_X}$-$\mathsf{Cl}(\mathsf{S}))$ *is* $\mathsf{G_X}$-*preopen.*

Proof.

(a) Let $\mathsf{S} \in (\mathsf{G_X}, \mathsf{M_X})$-$\mathsf{PL}^*_\mathsf{c}(\mathsf{X})$. Then there exists a $\mathsf{G_X}$-preopen set T and a $\mathsf{M_X}$-closed set N such that $\mathsf{S} = \mathsf{T} \cap \mathsf{N}$. Therefore, $\mathsf{S} \subset \mathsf{T}$. Since $\mathsf{S} \subset \mathsf{M_X}$-$\mathsf{Cl}(\mathsf{S})$, so $\mathsf{S} \subset \mathsf{T} \cap \mathsf{M_X}$-$\mathsf{Cl}(\mathsf{S})$. Also, since $\mathsf{S} \subset \mathsf{N}$ where N is $\mathsf{M_X}$-closed set and $\mathsf{M_X}$-$\mathsf{Cl}(\mathsf{S})$ is the smallest $\mathsf{M_X}$-closed set containing S, so $\mathsf{M_X}$-$\mathsf{Cl}(\mathsf{S}) \subset \mathsf{N}$. This implies $\mathsf{T} \cap \mathsf{M_X}$-$\mathsf{Cl}(\mathsf{S}) \subset \mathsf{T} \cap \mathsf{N} = \mathsf{S}$. Hence, $\mathsf{S} = \mathsf{T} \cap \mathsf{M_X}$-$\mathsf{Cl}(\mathsf{S})$. Conversely, let $\mathsf{S} = \mathsf{T} \cap \mathsf{M_X}$-$\mathsf{Cl}(\mathsf{S})$ where T is $\mathsf{G_X}$-preopen set. Since $\mathsf{M_X}$-$\mathsf{Cl}(\mathsf{S})$ is $\mathsf{M_X}$-closed, so by definition, $\mathsf{S} \in (\mathsf{G_X}, \mathsf{M_X})$-$\mathsf{PL}^*_\mathsf{c}(\mathsf{X})$.

(b) Let $\mathsf{M_X}$-$\mathsf{Cl}(\mathsf{S}) \setminus \mathsf{S}$ be $\mathsf{G_X}$-preclosed. Then $\mathsf{X} \setminus \mathsf{M_X} - \mathsf{Cl}(\mathsf{S}) \setminus \mathsf{S}$ is $\mathsf{G_X}$-preopen. Now, $\mathsf{X} \setminus \mathsf{M_X}$-$\mathsf{Cl}(\mathsf{S}) \setminus \mathsf{S}) = \mathsf{S} \cup (\mathsf{X} \setminus \mathsf{M_X}$-$\mathsf{Cl}(\mathsf{S}))$. Hence, $\mathsf{S} \cup (\mathsf{X} \setminus \mathsf{M_X}$-$\mathsf{Cl}(\mathsf{S}))$ is $\mathsf{G_X}$-preopen. Conversely, let $\mathsf{S} \cup (\mathsf{X} \setminus \mathsf{M_X}$-$\mathsf{Cl}(\mathsf{S}))$ be $\mathsf{G_X}$-preopen. We have $\mathsf{X} \setminus (\mathsf{S} \cup (\mathsf{X} \setminus \mathsf{M_X}$-$\mathsf{Cl}(\mathsf{S})))$ is $\mathsf{G_X}$-preclosed. Now, $\mathsf{X} \setminus (\mathsf{S} \cup (\mathsf{X} \setminus \mathsf{M_X}$-$\mathsf{Cl}(\mathsf{S}))) = \mathsf{M_X}$-$\mathsf{Cl}(\mathsf{S}) \setminus \mathsf{S}$. Hence $\mathsf{M_X}$-$\mathsf{Cl}(\mathsf{S}) \setminus \mathsf{S}$ is $\mathsf{G_X}$-preclosed. □

Theorem 12 *Let* S, T *be two subsets of* $(\mathsf{X}, \mathsf{G_X}, \mathsf{M_X})$. *If* $\mathsf{S} \in (\mathsf{G_X}, \mathsf{M_X})$-$\mathsf{PL_c}(\mathsf{X})$ *and* T *is* $\mathsf{G_X}$-*preopen or* $\mathsf{M_X}$-*preclosed then* $\mathsf{S} \cap \mathsf{T} \in (\mathsf{G_X}, \mathsf{M_X})$-$\mathsf{PL_c}(\mathsf{X})$.

Proof. Let $\mathsf{S} \in (\mathsf{G_X}, \mathsf{M_X})$-$\mathsf{PL_c}(\mathsf{X})$. Then $\mathsf{S} = \mathsf{Q} \cap \mathsf{R}$, where Q is $\mathsf{G_X}$-preopen and R is $\mathsf{M_X}$-preclosed. Now, if T is $\mathsf{G_X}$-preopen, then $\mathsf{Q} \cap \mathsf{T}$ is also $\mathsf{G_X}$-preopen. Also, R is $\mathsf{M_X}$-preclosed and $\mathsf{S} \cap \mathsf{T} = (\mathsf{Q} \cap \mathsf{R}) \cap \mathsf{T} = (\mathsf{Q} \cap \mathsf{T}) \cap \mathsf{R}$. Hence, $\mathsf{S} \cap \mathsf{T} \in (\mathsf{G_X}, \mathsf{M_X})$-$\mathsf{PL_c}(\mathsf{X})$. Again if T is $\mathsf{M_X}$-preclosed, then $\mathsf{T} \cap \mathsf{R}$ is $\mathsf{M_X}$-preclosed. Now, $\mathsf{S} \cap \mathsf{T} = (\mathsf{Q} \cap \mathsf{R}) \cap \mathsf{T} = \mathsf{Q} \cap (\mathsf{T} \cap \mathsf{R})$. Hence, $\mathsf{S} \cap \mathsf{T} \in (\mathsf{G_X}, \mathsf{M_X})$-$\mathsf{PL_c}(\mathsf{X})$. □

Theorem 13 *Let* S *and* T *be two subsets of* $(\mathsf{X}, \mathsf{G_X}, \mathsf{M_X})$. *Then*

(a) *If* $S, T \in (G_X, M_X)\text{-}PL_c^*(X)$ *then* $S \cap T \in (G_X, M_X)\text{-}PL_c^*(X)$.

(b) *If* $S, T \in (G_X, M_X)\text{-}PL_c^{**}(X)$ *then* $S \cap T \in (G_X, M_X)\text{-}PL_c^{**}(X)$.

Proof.

(a) Let $S, T \in (G_X, M_X)\text{-}PL_c^*(X)$. Then $S = A \cap B$, where A is G_X-preopen and B is M_X-closed and $T = C \cap D$, where C is G_X-preopen and D is M_X-closed. Now, $S \cap T = (A \cap B) \cap (C \cap D) = (A \cap C) \cap (B \cap D)$. Since, $A \cap C$ is G_X-preopen and $B \cap D$ is M_X-closed, so $S \cap T \in (G_X, M_X)\text{-}PL_c^*(X)$.

(b) Let $S, T \in (G_X, M_X)\text{-}PL_c^{**}(X)$. Then $S = A \cap B$, where A is G_X-open and B is M_X-preclosed and $T = C \cap D$, where C is G_X-open and D is M_X-preclosed. Now, $S \cap T = (A \cap B) \cap (C \cap D) = (A \cap C) \cap (B \cap D)$. Since, $A \cap C$ is G_X-open and $B \cap D$ is M_X-preclosed, so $S \cap T \in (G_X, M_X)\text{-}PL_c^{**}(X)$.

□

Remark 5 *The union of any two* $(G_X, M_X)\text{-}PL_c^*$ *(respectively,* $(G_X, M_X)\text{-}PL_c^{**}$*) sets is not necessarily a* $(G_X, M_X)\text{-}PL_c^*$ *(respectively,* $(G_X, M_X)\text{-}PL_c^{**}$*) set as shown in the following examples.*

Example 5

(a) *Let* $X = \{a, b, c\}$*,* $G_X = \{\emptyset, \{a\}, X\}$ *and* $M_X = \{\emptyset, \{a, b\}, \{a, c\}, X\}$*. Then,* $\{b\}, \{c\} \in (G_X, M_X)\text{-}PL_c^*(X)$ *but* $\{b\} \cup \{c\} = \{b, c\} \notin (G_X, M_X)\text{-}PL_c^*(X)$.

(b) *Let* $X = \{a, b, c\}$*,* $G_X = \{\emptyset, \{a\}, X\}$ *and* $M_X = \{\emptyset, \{b, c\}, X\}$*. Then,* $\{b\}, \{c\} \in (G_X, M_X)\text{-}PL_c^{**}(X)$*, but* $\{b\} \cup \{c\} = \{b, c\} \notin (G_X, M_X)\text{-}PL_c^*(X)$.

Remark 6 $(G_X, M_X)\text{-}PL_c^*(X)$ *and* $(G_X, M_X)\text{-}PL_c^{**}(X)$ *are independent as the following example exhibits.*

Example 6 *Let* $X = \{a, b, c\}$*,* $G_X = \{\emptyset, \{a\}, X\}$ *and* $M_X = \{\emptyset, \{b, c\}, X\}$*. Then*
$(G_X, M_X)\text{-}PL_c^*(X) = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$ *and*
$(G_X, M_X)\text{-}PL_c^{**}(X) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, X\}$.
So, $(G_X, M_X)\text{-}PL_c^*(X) \neq (G_X, M_X)\text{-}PL_c^{**}(X)$.

We formulate the following result without proof which can be established easily.

Theorem 14 *If* S *is* $(G_X, M_X)\text{-}PL_c^{**}$ *and* T *is either* M_X*-closed or* G_X*-open, then* $S \cap T$ *is* $(G_X, M_X)\text{-}PL_c^{**}$.

5 Conclusion

In this paper, the rudimentary concepts of preopen sets and prelocally closed sets in GTMS spaces have been examined. With the help of these notions, researchers can turn their attention towards generalisations of various types of continuous functions as well as to develop the notion of connectedness of generalised topology by considering minimal structure spaces.

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Declaration

The authors declare that the article does not have any competing interest involved.

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