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Analysis of a batch arrival multi-server queueing system with waiting servers, synchronous working vacations and impatient customers

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Abstract. This paper is concerned with the analysis of an infinite-capacity batch arrival multi-server queueing system with Bernoulli feedback, synchronous multiple and single working vacation policies, waiting servers, reneging and retention of reneged customers. The steady-state solution of the queueing system is obtained by using probability generating function (PGF). In addition, important performance measures of the queueing system are derived. Then, a cost model is formulated in order to carry out the parameter optimization using genetic algorithm (GA). Finally, numerical study is presented in which various system performance measures are evaluated based on supposed numerical values given to the system parameters.

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1 Introduction

In recent past, there has been a growing interest in the analysis of queueing models with working vacation, where during the vacation period, the server serves the arrivals with slower service rate rather than completely stopping the service. This interesting research area has shown a noticeable effect on queueing applications, especially in call centers, computer networks, manufacturing, production systems, etc. Excellent research work on the subject can be found in Servi and Finn [20], Baba [5], Jain and Jain [14], Sudhesh and Raj [23], Kempa and Kobielnik [17], Zhang and Zhou [31] and the references therein.

In queueing models cited above, it is supposed that customers arrive to the system one by one at a time, wherein there are many situations where customers arrive in a group, such examples can be found in digital communication systems, data traffic segmented as packets, and so forth. These queueing systems are known as batch arrival queues. For a comprehensive review of related models, the readers can be referred to Khalaf et al. [16], Baba [6], Baruah et al. [7], Singh et al. [21], Bhagat and Jain [8], Ayyappan and Udayageetha [4], Zhang [30] and the references therein.

Working vacation queueing models with impatient customers have been investigated extensively because of the large application in many real-life problems (cf. Yue et al. [28], Vijaya Laxmi and Jyothsna [24], Sudhesh et al. [22], Bouchentouf and Yahiaoui [12], Vijaya Laxmi and Rajesh [25], and Jain et al. [15]. The analysis of customers' impatience in multi-server vacation queueing models is more complex compared to single-server vacation queueing systems with impatient customers, where the servers may either take the same vacation together (synchronous vacation) or take individual vacations (asynchronous vacations) independently. Thus, a very limited literature is available for these models. The readers can be referred to Altman and Yechiali [1], Yue et al. [27], Altman and Yechiali [2], Yue et al. [29], Majid and Manoharan [18], Yahiaoui et al.[26], and Bouchentouf and Guendouzi [10].

The concept of vacation queues with a waiting server was first introduced by Boxma et al. [13], where once the system is empty, the server waits for a random amount of time before going on vacation. This situation reflects many real life queueing systems, particularly when dealing with human behaviour. For recent research works on the subject, the reader can refer to Padmavathy et al. [19], Ammar [3], Bouchentouf and Guendouzi [9], and Bouchentouf et al. [11].

In this paper, we deal with an infinite-space multi-server queueing system with batch arrival, waiting servers, synchronous multiple and single working vacation policies, Bernoulli feedback, reneging and retention of reneged customers. Our investigation has a great application in many practical life situations, especially when we deal with a human behavior, examples can be found in post offices, banks, hospitals, etc.

The rest of the paper is organized in the following manner. In Section 2, we describe the model. In Section 3, we develop the equations of the steady state probabilities of the model and derive their steady-state solutions, using the probability generating functions (PGFs). Section 4 is devoted to derive various performance measures. In Section 5, we formulate a cost model. Section 6 is consecrated to the numerical analysis.

2 The model

Consider a $M^X/M/c$ queueing system with feedback, waiting servers, both multiple and single working vacation policies, reneging, and retention of reneged customers.

– Customers arrive into the system according to a Poisson process with arrival rate λ. The sizes of successive arriving batches are i.i.d. random variables X_1, X_2,... distributed with probability mass function $P(X = l) = b_l$; $l = 1, 2, 3, ...$

– The service discipline is FCFS, and the system capacity is supposed to be infinite.

– The service time during normal busy period is assumed to be exponentially distributed with mean $1/\mu_1$.

– When the busy period is finished, the servers wait a random duration of time before they switch to a working vacation. This waiting duration is exponentially distributed with mean $1/\varpi$.

– The period of working vacation has an exponential distribution with mean $1/\vartheta$.

– During vacation, servers can provide service to new arrival. The service time during this period is assumed to be exponentially distributed with mean $1/\mu_2$, with $\mu_2 < \mu_1$.

– A synchronous working vacation is considered; once the system is empty, the servers, all together, go on working vacation, and they also return to the system as one at the same time.

– Both single and multiple working vacation are taking into consideration:

• Multiple working vacation policy (MWV); once the system is still empty at the end of a working vacation period, the servers begin another working vacation period. Otherwise, a normal busy period begins.

• Single working vacation policy (SWV); the servers take a working vacation all together and they comeback to the system as one, then wait passively for new arrivals. Otherwise, they start a new normal busy period.

– During working vacation period, the new arrival activates an impatience timer T, which is exponentially distributed with parameter χ. If the customer's service has not been completed before the customer's timer expires, the customer may abandon the queue. We suppose that the customers timers are independent and identically distributed random variables and independent of the number of waiting customers. Each reneged customer may leave the system with probability α and may be retained with probability $\alpha' = 1 - \alpha$.

– The inter-arrival times, vacation periods and service times during busy and working vacation periods are mutually independent.

– If the customer is unsatisfied with the quality of the service or if he requires another one, he can join the end of the queue with probability β'. Otherwise, he leaves the system definitively with probability β, where $\beta + \beta' = 1$.

It is worth noting that the system is stable under the condition $\lambda E(X) < c\beta\mu_1$, such that $E(X)$ is the mean of a batch of arrivals.

3 Steady-state solution

We present the steady-state probabilities of the system under both single and multiple working vacation policies. Let δ denote the indicator function:

$$\delta = \begin{cases} 1, & \text{for the single working vacation model;} \\ 0, & \text{for the multiple working vacation model.} \end{cases}$$

Let $L(t)$ be the number of customers in the system at time t. Let $J(t)$ denote the state of the servers at time t such that

$$J(t) = \begin{cases} 1, & \text{when the servers are on a normal busy period;} \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, the process $\{(J(t), L(t)), t \geq 0\}$ is a continuous-time Markov process with state space

$$\Omega = \{(j, n) : j = 0, 1, \;\; n = 0, 1, ...\}.$$

Let $P_{j,n} = \lim_{t\to\infty} P\{J(t) = j, L(t) = n\}, (j, n) \in \Omega$, denote the system state probabilities. The state transition diagram corresponding to our queueing system is illustrated in Figure 1.

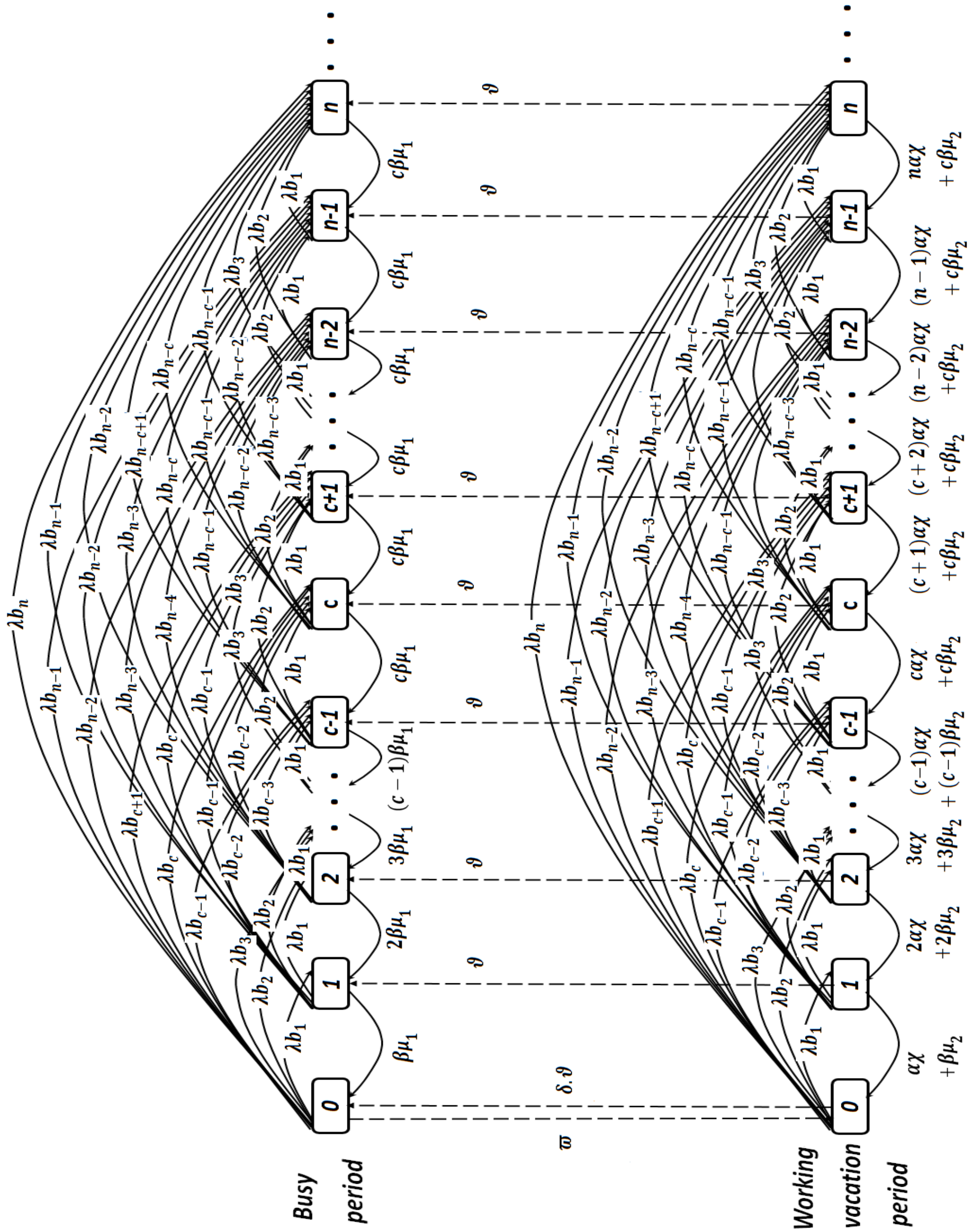

Figure 1: State-transition-rate diagram for SWV ($\delta = 1$) and MWV ($\delta = 0$).

Via Markov chain theory, we obtain the steady-state equations as follows:

$$(\lambda + \delta\vartheta)P_{0,0} = (\alpha\chi + \beta\mu_2)P_{0,1} + \varpi P_{1,0}, \;\; n = 0, \tag{1}$$

$$(\lambda + \vartheta + \beta\mu_2 + \alpha\chi)P_{0,1} = \lambda b_1 P_{0,0} + 2(\beta\mu_2 + \alpha\chi)P_{0,2}, \;\; n = 1, \tag{2}$$

$$(\lambda + \vartheta + n(\beta\mu_2 + \alpha\chi))P_{0,n} = \lambda \sum_{m=1}^{n} b_m P_{0,n-m} + (n+1)(\beta\mu_2 + \alpha\chi)P_{0,n+1}, \quad 2 \le n \le c-1, \tag{3}$$

$$(\lambda + \vartheta + c\beta\mu_2 + n\alpha\chi)P_{0,n} = \lambda\sum_{m=1}^{n} b_m P_{0,n-m} + (c\beta\mu_2 + (n+1)\alpha\chi)P_{0,n+1}, \quad n \ge c, \tag{4}$$

$$(\lambda + \varpi)P_{1,0} = \delta\vartheta P_{0,0} + \beta\mu_1 P_{1,1} \;\; n = 0, \tag{5}$$

$$(\lambda + \beta\mu_1)P_{1,1} = \lambda b_1 P_{1,0} + 2\beta\mu_1 P_{1,2} + \vartheta P_{0,1}, \;\; n = 1, \tag{6}$$

$$(\lambda + n\beta\mu_1)P_{1,n} = \lambda \sum_{m=1}^{n} b_m P_{1,n-m} + (n+1)\beta\mu_1 P_{1,n+1} + \vartheta P_{0,n}, \quad 2 \le n \le c-1, \tag{7}$$

$$(\lambda + c\beta\mu_1)P_{1,n} = \lambda\sum_{m=1}^{n} b_m P_{1,n-m} + c\beta\mu_1 P_{1,n+1} + \vartheta P_{0,n}, n \ge c. \tag{8}$$

The normalizing condition is defined as

$$\sum_{n=0}^{\infty} P_{0,n} + \sum_{n=0}^{\infty} P_{1,n} = 1. \tag{9}$$

Let the probability generating functions (PGFs) presented as

$$G_j(z) = \sum_{n=0}^{\infty} z^n P_{j,n}, \;\; j = 0, 1. \tag{10}$$

Define

$$G_j'(z) = \frac{d}{dz} G_j(z), \;\; j = 0, 1.$$

The probability generating function(PGF) of the batch size X is given by

$$G(z) = \sum_{i=1}^{\infty} b_i z^i, \;\; |z| \le 1, \;\; G(1) = \sum_{i=1}^{\infty} b_i = 1. \tag{11}$$

Multiplying equations (1)-(4) by z^n, and summing all possible values of n, we find

$$\begin{aligned}\alpha\chi z(1-z)G_0'(z) - [\lambda z(1-G(z)) + \vartheta z - c\beta\mu_2(1-z)]G_0(z) = \\ \beta\mu_2(1-z)\Phi_1(z) - \varpi z P_{1,0} - \vartheta(1-\delta)zP_{0,0}.\end{aligned} \tag{12}$$

In the same way, we get from equations (5)-(8),

$$\begin{aligned}(\lambda z(G(z)-1) + c\beta\mu_1(1-z))G_1(z) = -\vartheta z G_0(z) + \varpi z P_{1,0} \\ + \vartheta(1-\delta)zP_{0,0} + \beta\mu_1(1-z)\Phi_2(z),\end{aligned} \tag{13}$$

where

$$\Phi_1(z) = \sum_{n=0}^{c-1}(c-n)z^n P_{0,n}, \text{ and, } \Phi_2(z) = \sum_{n=0}^{c-1}(c-n)z^n P_{1,n}.$$

Next, putting $z = 1$ in equation (12) or equation (13), we find

$$\vartheta G_0(1) = \varpi P_{1,0} + \vartheta(1-\delta)P_{0,0}. \tag{14}$$

For $z \neq 1$, equation (12) can be given as

$$\begin{aligned}G_0'(z) - \left[\frac{\lambda\varphi'(z)}{\alpha\chi} + \frac{\vartheta + c\beta\mu_2}{\alpha\chi(1-z)} - \frac{c\beta\mu_2}{\alpha\chi z(1-z)}\right]G_0(z) = -\frac{\varpi}{\alpha\chi(1-z)}P_{1,0} \\ -\frac{\vartheta(1-\delta)}{\alpha\chi(1-z)}P_{0,0} + \frac{\beta\mu_2}{\alpha\chi z}\Phi_1(z),\end{aligned} \tag{15}$$

with

$$\varphi'(z) = \frac{1-G(z)}{1-z}.$$

Now, multiply both sides of Equation (15) by $e^{-\frac{\lambda\varphi(z)}{\alpha\chi}}(1-z)^{\frac{\vartheta}{\alpha\chi}}z^{\frac{c\beta\mu_2}{\alpha\chi}}$, we obtain

$$\begin{aligned}\frac{d}{dz}\left(e^{\frac{-\lambda\varphi(z)}{\alpha\chi}} z^{\frac{c\beta\mu_2}{\alpha\chi}}(1-z)^{\frac{\vartheta}{\alpha\chi}}G_0(z)\right) = e^{\frac{-\lambda\varphi(z)}{\alpha\chi}} z^{\frac{c\beta\mu_2}{\alpha\chi}}(1-z)^{\frac{\vartheta}{\alpha\chi}}\Bigg[-\frac{\varpi}{\alpha\chi(1-z)}P_{1,0} \\ +\frac{\beta\mu_2}{\alpha\chi z}\Phi_1(z) - \frac{\vartheta(1-\delta)}{\alpha\chi(1-z)}P_{0,0}\Bigg].\end{aligned} \tag{16}$$

Integrating the above equation from 0 to z, it yields

$$G_0(z) = e^{\frac{\lambda\varphi(z)}{\alpha\chi}}(1-z)^{-\frac{\vartheta}{\alpha\chi}}z^{-\frac{c\beta\mu_2}{\alpha\chi}}\left[\beta\mu_2 K_2(z)-\left(\vartheta(1-\delta)P_{0,0}+\varpi P_{1,0}\right)K_1(z)\right], \quad (17)$$

where

$$K_1(z) = \frac{1}{\alpha\chi}\int_0^z e^{-\frac{\lambda\varphi(s)}{\alpha\chi}}(1-s)^{\frac{\vartheta}{\alpha\chi}-1}s^{\frac{c\beta\mu_2}{\alpha\chi}}ds,$$

and

$$K_2(z) = \frac{1}{\alpha\chi}\int_0^z e^{-\frac{\lambda\varphi(s)}{\alpha\chi}}(1-s)^{\frac{\vartheta}{\alpha\chi}}s^{\frac{c\beta\mu_2}{\alpha\chi}-1}\Phi_1(s)ds.$$

To solve the differential equation (12), we must find $\Phi_1(z)$. Recursively, from equations (1)-(3), we obtain

$$P_{0,n} = \gamma_n P_{0,0} + \eta_n P_{1,0}, \quad (18)$$

with

$$\gamma_n = \begin{cases} 1, \text{ n=0}; \\ \dfrac{\lambda+\vartheta\delta}{\beta\mu_2+\alpha\chi}, \text{ n=1}; \\ \omega_{n-1}\gamma_{n-1} - \dfrac{M}{n}\displaystyle\sum_{i=1}^{n-1} b_i\gamma_{n-1-i}, \\ 2 \le n \le c-1, \end{cases} \qquad \eta_n = \begin{cases} 0, \text{ n=0}; \\ \dfrac{-\varpi}{\beta\mu_2+\alpha\chi}, \text{ n=1}; \\ \omega_{n-1}\eta_{n-1} - \dfrac{M}{n}\displaystyle\sum_{i=1}^{n-1} b_i\eta_{n-1-i}, \\ 2 \le n \le c-1, \end{cases}$$

where

$$\omega_n = \frac{\lambda+\vartheta+n(\beta\mu_2+\alpha\chi)}{(n+1)(\beta\mu_2+\alpha\chi)} \quad \text{and} \quad M = \frac{\lambda}{\beta\mu_2+\alpha\chi}.$$

Consequently,

$$\begin{aligned} G_0(z) = e^{\frac{\lambda\varphi(z)}{\alpha\chi}}(1-z)^{-\frac{\vartheta}{\alpha\chi}}z^{-\frac{c\beta\mu_2}{\alpha\chi}}\Bigg[&\left(\beta\mu_2 K_5(z) - \vartheta(1-\delta)K_1(z)\right)P_{0,0} \\ &-\left(\varpi K_1(z) - \beta\mu_2 K_4(z)\right)P_{1,0}\Bigg], \end{aligned} \quad (19)$$

with

$$K_4(z) = \frac{1}{\alpha\chi}\int_0^z e^{-\frac{\lambda\varphi(s)}{\alpha\chi}}(1-s)^{\frac{\vartheta}{\alpha\chi}}s^{\frac{c\beta\mu_2}{\alpha\chi}-1}\sum_{n=0}^{c-1}(c-n)s^n\eta_n ds,$$

and

$$K_5(z) = \frac{1}{\alpha\chi}\int_0^z e^{-\frac{\lambda\varphi(s)}{\alpha\chi}}(1-s)^{\frac{\vartheta}{\alpha\chi}}s^{\frac{c\beta\mu_2}{\alpha\chi}-1}\sum_{n=0}^{c-1}(c-n)s^n\gamma_n ds.$$

Since $G_0(1) = \sum_{n=0}^{\infty} P_{0,n} > 0$ and $z = 1$ is the root of denominator of the right hand side of Equation (15). Thus, from equation (19), we get

$$P_{1,0} = \rho_0 P_{0,0}, \tag{20}$$

where

$$\rho_0 = \left[\frac{\beta\mu_2 K_5(1) - \vartheta(1-\delta)K_1(1)}{\varpi K_1(1) - \beta\mu_2 K_4(1)}\right].$$

Substituting equation (20) into equation (19), we obtain

$$\begin{aligned} G_0(z) = e^{\frac{\lambda\varphi(z)}{\alpha\chi}}(1-z)^{-\frac{\vartheta}{\alpha\chi}}z^{-\frac{c\beta\mu_2}{\alpha\chi}}\Bigg[\beta\mu_2\Bigg(K_5(z) + K_4(z)\rho_0\Bigg) \\ -\Bigg(\vartheta(1-\delta) + \varpi\rho_0\Bigg)K_1(z)\Bigg]P_{0,0}. \end{aligned} \tag{21}$$

Since $P_{0,.} = G_0(1) = \sum_{n=0}^{\infty} P_{0,n}$, by substituting equation (20) into (14), we get

$$P_{0,.} = \frac{\varpi\rho_0 + \vartheta(1-\delta)}{\vartheta}P_{0,0}. \tag{22}$$

Now, equation (13) can be written as

$$G_1(z) = \frac{-\vartheta z G_0(z) + \varpi z P_{1,0} + \vartheta(1-\delta)zP_{0,0} + \beta\mu_1(1-z)\Phi_2(z)}{\lambda z(G(z)-1) + c\beta\mu_1(1-z)}. \tag{23}$$

Next, in order to define $G_1(z)$ in terms of $P_{0,0}$, we need to express $P_{1,n}$ in terms of $P_{0,0}$. To this end, we employ the recursive method, then from equations (5)-(7) using equation 18, we get

$$P_{1,n} = \rho_n P_{0,0}, \tag{24}$$

where

$$\rho_n = \begin{cases} \rho_0, \text{ n=0}; \\ \dfrac{(\lambda+\varpi)\rho_0 - \vartheta\delta}{\beta\mu_1}, \text{ n=1}; \\ \omega_{n-1}\rho_{n-1} - \dfrac{\Delta_1}{n}\displaystyle\sum_{i=1}^{n-1} b_i\rho_{n-1-i} - \dfrac{\Delta_2}{n}\xi_{n-1}, \; 2 \leq n \leq c-1, \end{cases}$$

with

$$\xi_n = \gamma_n + \rho_0\eta_n, \;\; \omega_n = \frac{\lambda + (n-1)\beta\mu_1}{n(\beta\mu_1)}, \;\; \Delta_1 = \frac{\lambda}{\beta\mu_1}, \;\text{ and }\; \Delta_2 = \frac{\vartheta}{\beta\mu_1}.$$

Next, substituting equation (14) into (23), we have

$$G_1(z) = \frac{\beta\mu_1(1-z)\Phi_2(z) - z\vartheta(G_0(z) - G_0(1))}{\lambda z(G(z)-1) + c\beta\mu_1(1-z)}. \tag{25}$$

Via equation (25), applying l'hospital rule, we get

$$\lim_{z\to 1} G_1(z) = G_1(1) = \frac{\beta\mu_1\Phi_2(1) + \vartheta G_0'(1)}{c\beta\mu_1 - \lambda G'(1)}, \tag{26}$$

where

$$\Phi_2(1) = \sum_{n=0}^{c-1}(c-n)\rho_n P_{0,0}.$$

Now, via equation (15), applying l'hospital rule, it yields

$$\lim_{z\to 1} G_0'(z) = G_0'(1) = \frac{(\lambda G'(1) - c\beta\mu_2)G_0(1) + \beta\mu_2\Phi_1(1)}{\alpha\chi + \vartheta}, \tag{27}$$

where

$$\Phi_1(1) = \sum_{n=0}^{c-1}(c-n)(\gamma_n + \eta_n\rho_0)P_{0,0}.$$

Next, substituting equation (22) in equation (27), it yields

$$G_0'(1) = \frac{(\lambda G'(1) - c\beta\mu_2)(\varpi\rho_0 + \vartheta(1-\delta)) + \beta\mu_2\vartheta H_1(1)}{\vartheta(\alpha\chi + \vartheta)} P_{0,0}, \tag{28}$$

with

$$H_1(1) = \sum_{n=0}^{c-1} (c-n)(\gamma_n + \rho_0 \eta_n).$$

Then, substituting equation (28) into (26), we get $G_1(1)$ in terms of $P_{0,0}$.

Since $P_{1,.} = G_1(1) = \sum_{n=0}^{\infty} P_{1,n} > 0$, we obtain

$$P_{1,.} = R(1)P_{0,0}, \tag{29}$$

where

$$R(1) = \frac{\beta\mu_1 H_2(1)(\alpha\chi + \vartheta) + (\lambda G'(1) - c\beta\mu_2)(\varpi\rho_0 + \vartheta(1-\delta)) + \beta\mu_2\vartheta H_1(1)}{(c\beta\mu_1 - \lambda G'(1))(\alpha\chi + \vartheta)},$$

and

$$H_2(1) = \sum_{n=0}^{c-1} (c-n)\rho_n.$$

Finally, by substituting equations (22) and (29) into (9), we get

$$P_{0,0} = \left(\frac{\varpi\rho_0 + \vartheta(1-\delta)}{\vartheta} + R(1) \right)^{-1}.$$

4 Performance measures

– Let L_{wv} be the system size when the servers are in working vacation period. Then, the mean system size when the servers are in working vacation period is given by

$$\begin{aligned} \mathbb{E}(L_{wv}) &= G_0'(1) = \lim_{z \to 1} G_0'(z) \\ &= \frac{(\lambda G'(1) - c\beta\mu_2)(\varpi\rho_0 + \vartheta(1-\delta)) + \beta\mu_2\vartheta H_1(1)}{\vartheta(\vartheta + \alpha\chi)} P_{0,0}. \end{aligned} \tag{30}$$

– Let L_1 be the system size when the servers are in busy period. Therefore, the mean system size when the servers are in this period is as follows

$$\mathbb{E}(L_1) = G_1'(1) = \lim_{z \to 1} G_1'(z).$$

From equation (25), we get

$$\mathbb{E}(L_1) = \frac{\vartheta}{2(c\beta\mu_1 - \lambda G'(1))} G_0''(1) + \frac{\vartheta(\lambda G''(1) + 2c\beta\mu_1)}{2(c\beta\mu_1 - \lambda G'(1))^2} G_0'(1)$$
$$+ \left[\frac{\beta\mu_1}{c\beta\mu_1 - \lambda G'(1)} H_2'(1) + \frac{\lambda\beta\mu_1(2G'(1) + G''(1))}{2(c\beta\mu_1 - \lambda G'(1))^2} H_2(1) \right] P_{0,0}, \tag{31}$$

with $H_2'(1) = \sum_{n=0}^{c-1} n(c-n)\rho_n$, and $G_0''(1)$ is obtained by differentiating twice $G_0(z)$ at $z = 1$. Via equation (12), we have

$$G_0''(1) = \frac{(2\lambda G'(1) - 2c\beta\mu_2)G_0'(1) + (\lambda G''(1) + 2c\beta\mu_2)G_0(1)}{\alpha\chi + \vartheta}. \tag{32}$$

Then, substituting equation(32) into (31), we find

$$\mathbb{E}(L_1) = \left[\frac{\vartheta(2\lambda G'(1) - 2c\beta\mu_2)}{2(c\beta\mu_1 - \lambda G'(1))(\alpha\chi + \vartheta)} + \frac{\vartheta(\lambda G''(1) + 2c\beta\mu_1)}{2(c\beta\mu_1 - \lambda G'(1))^2} \right] \mathbb{E}(L_{wv})$$
$$+ \left[\frac{\beta\mu_1}{c\beta\mu_1 - \lambda G'(1)} H_2'(1) + \frac{\lambda\beta\mu_1(2G'(1) + G''(1))}{2(c\beta\mu_1 - \lambda G'(1))^2} H_2(1) \right] P_{0,0} \tag{33}$$
$$+ \frac{\vartheta(\lambda G''(1) + 2c\beta\mu_2)}{2(c\beta\mu_1 - \lambda G'(1))(\alpha\chi + \vartheta)} P_{0,\cdot}.$$

– The mean system size: Let L denote the number of customers in the system. Thus

$$\mathbb{E}(L) = \mathbb{E}(L_{wv}) + \mathbb{E}(L_1).$$

– The mean queue length:

$$\mathbb{E}(L_q) = \sum_{n=c+1}^{\infty} (n-c)P_{n,0} + \sum_{n=c+1}^{\infty} (n-c)P_{n,1} = \mathbb{E}(L) - c + \Big(H_1(1) + H_2(1) \Big) P_{0,0}.$$

– The probability that the servers are idle during busy period: From (19), we get

$$P_I = \left[\frac{\beta\mu_2 K_5(1) - \vartheta(1-\delta)K_1(1)}{\varpi K_1(1) - \beta\mu_2 K_4(1)} \right] P_{0,0}.$$

– The probability that the servers are on working vacation period:

$$P_{wv} = G_0(1) = \frac{\varpi\rho_0 + \vartheta(1-\delta)}{\vartheta} P_{0,0}.$$

– The probability that the servers are working (serving customers) during normal busy period:

$$P_B = 1 - P_{wv} - P_I.$$

– The average rate of reneging:

$$R_{ren} = \alpha\xi \sum_{n=1}^{\infty} n P_{0,n} = \alpha\chi\mathbb{E}(L_{wv}).$$

– The average rate of retention of impatient customers:

$$R_{ret} = \alpha'\chi \sum_{n=1}^{\infty} n P_{0,n} = \alpha'\chi\mathbb{E}(L_{wv}).$$

5 Cost model

We present a cost model in order to develop a cost-optimum analysis of the queueing model under consideration. The following cost elements are needed:

- C_1 : Cost per unit time when the servers are working during busy period.
- C_2 : Cost per unit time when the servers are on working vacation.
- C_3 : Cost per unit time when the servers are idle during busy period.
- C_4 : Cost per unit time when a customer joins the queue and waits for service.
- C_5 : Cost per unit time when a customer reneges.
- C_6 : Cost per unit time when a customer is retained.
- C_7 : Cost per service per unit time.
- C_8 : Cost per unit time when a customer returns to the system as a feedback customer.
- C_9 : Fixed server purchase cost per unit.

Let F be the total expected cost per unit time of the system:

$$\begin{aligned} F = {} & C_1 P_B + C_2 P_{wv} + C_3 P_I + C_4\mathbb{E}(L_q) + C_5 R_{ren} + C_6 R_{ret} + c\mu_2(C_7 + \beta' C_8) \\ & + c\mu_1(C_7 + \beta' C_8) + cC_9. \end{aligned}$$

We consider in this investigation the cost optimization problem under a given cost structure via genetic algorithm (GA). A total expected cost function has been developed in order to determine an optimum regular and working service rates μ_1^* μ_2^*, the number of servers in the system c^* as well as the optimum expected cost $F(\mu_1^*, \mu_2^*, c^*)$.

The optimization problem may be illustrated mathematically as:

$$\text{Minimize: } F(\mu_1, \mu_2, c).$$

6 Numerical analysis

This section presents a numerical results conducted by coding computer program in R software in order to show the applicability of the theoretical analysis. We perform an analysis on the optimum values μ_1^*, μ_2^* and c^* based on changes in specific values of the system parameters λ, r, ϖ, ϑ, χ, α and β. For computational aim, we assume that the arrival batch size X follows a geometric distribution with parameter r;

$$P(X = l) = b_l = (1-r)^{l-1} r, \ 0 < r < 1 \ (l = 1, 2, ...),$$

with

$$B(z) = \frac{rz}{1-(1-r)z}, \quad E(X) = B'(1) = \frac{1}{r}, \quad \text{and} \quad E(X^2) = B''(1) = \frac{2(1-r)}{r^2}.$$

The different cost elements are taken as $C_1 = 15$, $C_2 = 10$, $C_3 = 5$, $C_4 = 15$, $C_5 = 25$, $C_6 = 5$, $C_7 = 15$, $C_8 = 10$, and $C_9 = 3$.

The total cost function is presented in Tables 1-4 and plotted (using GA) in Figures 2-3 by varying values of the system parameters. Further, Tables 1-4 depict the optimum values of μ_1, μ_2, c, and the minimum expected cost F^* along with the corresponding performance measures P_{wv}^*, P_I^*, P_B^*, $\mathbb{E}(L_{wv})^*$, $\mathbb{E}(L_1)^*$, R_{ren}^*, and R_{ret}^* for different values of λ, r, ϑ, ϖ, β, χ, and α, where

- Figure 2: $\lambda = 2.0$, $\chi = 1.50$, $\vartheta = 0.40$, $\varpi = 2.8$, $c = 4$, $r = 0.70$, $\beta = 0.75$, and $\alpha = 0.60$.
- Figure 3: $\lambda = 2.0$, $\chi = 1.50$, $\vartheta = 0.40$, $\varpi = 2.8$, $c = 3$, $r = 0.70$, $\beta = 0.75$, and $\alpha = 0.60$.
- Table 1: $\chi = 1.80$, $\vartheta = 0.80$, $\varpi = 1.50$, $\beta = 0.70$, and $\alpha = 0.60$.
- Table 2: $\chi = 1.80$, $\varpi = 1.50$, $r = 0.75$, $\beta = 0.70$, and $\alpha = 0.60$.
- Table 3: $\lambda = 2.10$, $\chi = 1.80$, $\vartheta = 1.70$, $r = 0.75$, and $\alpha = 0.60$.
- Table 4: $\lambda = 2.10$, $\vartheta = 1.20$, $\varpi = 1.40$, $r = 0.75$, and $\beta = 0.8$.

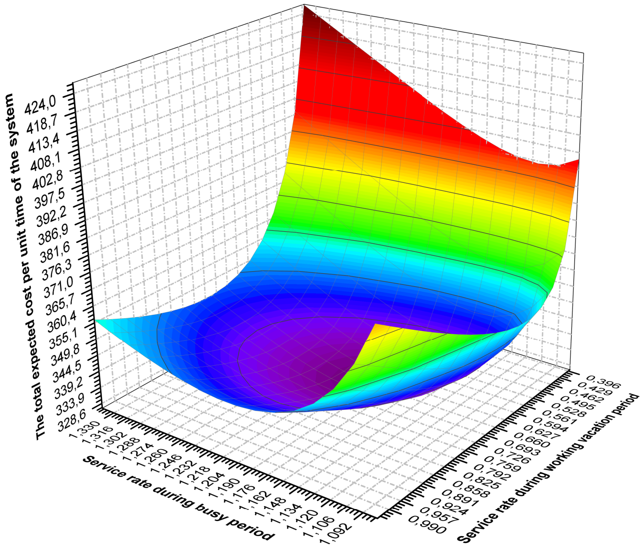

Figure 2: Optimal cost vs. μ_1 and μ_2 in multiple working vacation policy.

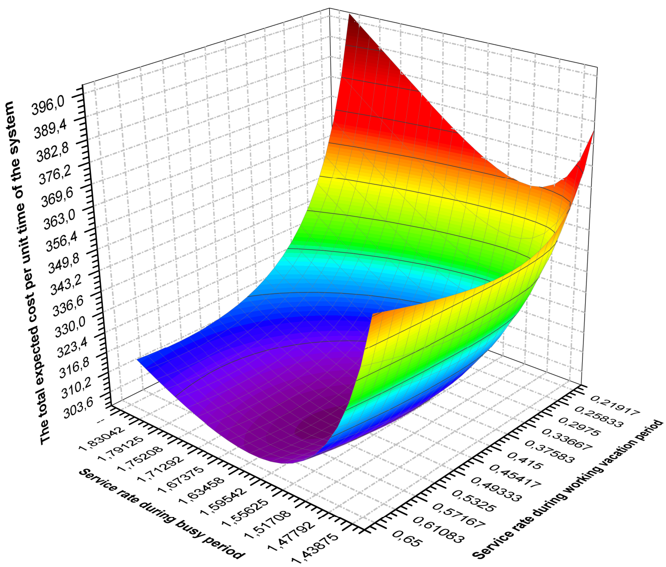

Figure 3: Optimal cost vs. μ_1 and μ_2 in single working vacation policy.

Table 1: Optimum performance measures for different values of λ and r.

	(λ, r)	(2.00,0.40)	(2.50,0.40)	(3.00,0.40)	(2.00,0.80)	(2.50,0.80)	(3.00,0.80)
MWV	μ_1^*	2.502229	2.814931	3.327466	2.308868	2.406292	2.410777
	μ_2^*	1.030319	0.994439	0.992243	0.966170	0.906214	0.895353
	c^*	4	4	4	4	4	4
	F^*	557.2287	629.3045	706.5886	456.5071	470.2396	472.7053
	P_{wv}^*	0.392421	0.272212	0.230084	0.599018	0.456898	0.322744
	P_I^*	0.008850	0.010206	0.013398	0.039825	0.043037	0.038241
	P_B^*	0.598729	0.717582	0.756518	0.361157	0.500066	0.639014
	$\mathbb{E}(L_{wv})^*$	1.991646	1.745720	1.787051	1.748974	1.680051	1.431087
	$\mathbb{E}(L_1)^*$	5.724183	9.696449	11.07218	0.664982	1.236184	2.069782
	R_{ren}^*	2.150977	1.885377	1.930015	1.888891	1.814455	1.545574
	R_{ret}^*	1.433985	1.256918	1.286677	1.259261	1.209637	1.030383
SWV	μ_1^*	2.568105	2.940578	3.443902	2.327490	2.415154	2.477015
	μ_2^*	0.810703	0.807034	0.805419	0.807520	0.804545	0.803916
	c^*	3	4	4	3	3	3
	F^*	466.4389	553.3533	645.1619	315.42420	329.8097	342.7443
	P_{wv}^*	0.168896	0.115348	0.086045	0.255817	0.184708	0.123709
	P_I^*	0.090078	0.061519	0.045891	0.136436	0.098511	0.065978
	P_B^*	0.741026	0.823133	0.868064	0.607747	0.716780	0.810313
	$\mathbb{E}(L_{wv})^*$	0.777926	0.736848	0.668306	0.673716	0.616176	0.499193
	$\mathbb{E}(L_1)^*$	7.314611	8.614643	11.298461	1.667467	2.546185	3.969401
	R_{ren}^*	0.840160	0.795796	0.721771	0.727614	0.665470	0.539129
	R_{ret}^*	0.560107	0.530530	0.481180	0.485076	0.443647	0.359419

Table 2: Optimum performance measures for different values of ϑ and λ.

	(ϑ, λ)	(0.60,2.40)	(1.20,2.40)	(1.80,2.40)	(0.60,2.80)	(1.20,2.80)	(1.80,2.80)
MWV	μ_1^*	2.190517	2.017012	2.015180	2.207845	2.110887	2.105012
	μ_2^*	0.909674	0.808033	0.821407	0.891618	0.847985	0.846099
	c^*	4	4	4	4	4	4
	F^*	431.3379	390.9331	376.1093	456.1155	401.9582	386.6201
	P_{wv}^*	0.418996	0.334364	0.284606	0.353305	0.232393	0.190844
	P_I^*	0.027096	0.037466	0.046395	0.027895	0.033732	0.039143
	P_B^*	0.553908	0.628170	0.668999	0.618799	0.733875	0.770013
	$\mathbb{E}(L_{wv})^*$	1.717204	1.052859	0.743039	1.697097	0.857633	0.583564
	$\mathbb{E}(L_1)^*$	1.840412	1.972379	1.994956	2.371337	2.861829	2.894786
	R_{ren}^*	1.854580	1.137087	0.802482	1.832864	0.926243	0.630249
	R_{ret}^*	1.236387	0.758058	0.534988	1.221910	0.617495	0.420166
SWV	μ_1^*	2.030994	2.083993	2.095927	2.302819	2.328809	2.358547
	μ_2^*	0.803204	0.810854	0.821965	0.819441	0.811570	0.829329
	c^*	3	3	3	3	3	3
	F^*	316.0318	301.9229	295.5010	348.8217	331.6997	326.7999
	P_{wv}^*	0.143504	0.090891	0.064578	0.127889	0.077189	0.057959
	P_I^*	0.057401	0.072713	0.077494	0.051156	0.061751	0.069551
	P_B^*	0.799095	0.836396	0.857928	0.820955	0.861059	0.872489
	$\mathbb{E}(L_{wv})^*$	0.538211	0.253632	0.144440	0.564638	0.253497	0.152763
	$\mathbb{E}(L_1)^*$	4.548054	4.317499	4.412283	5.172256	5.082254	4.873308
	R_{ren}^*	0.581268	0.273922	0.155995	0.609809	0.273777	0.164984
	R_{ret}^*	0.387512	0.182615	0.103996	0.406539	0.182518	0.109990

Table 3: Optimum performance measures for different values of ϖ and β.

	(ϖ, β)	(0.70,0.40)	(1.40,0.40)	(2.10,0.40)	(0.70,0.80)	(1.40,0.80)	(2.10,0.80)
MWV	μ_1^*	2.522527	2.519482	2.507588	2.520041	2.500059	2.494978
	μ_2^*	1.043003	1.074571	1.080151	0.877730	0.822386	0.815934
	c^*	4	4	4	4	4	4
	F^*	473.9346	484.3575	488.3804	429.9572	439.7665	445.7521
	P_{wv}^*	0.238491	0.329456	0.386641	0.388090	0.507123	0.559101
	P_I^*	0.038605	0.028735	0.022758	0.126880	0.079166	0.057835
	P_B^*	0.722904	0.641809	0.590601	0.485030	0.413712	0.383064
	$\mathbb{E}(L_{wv})^*$	0.557301	0.770098	0.903807	0.903750	1.182511	1.303904
	$\mathbb{E}(L_1)^*$	2.857388	2.240101	1.856173	0.874812	0.676071	0.589290
	R_{ren}^*	0.601885	0.831706	0.976112	0.976050	1.277112	1.408216
	R_{ret}^*	0.401257	0.554471	0.650741	0.650700	0.851408	0.938811
SWV	μ_1^*	3.053832	2.996650	2.963749	2.496449	2.487807	2.485160
	μ_2^*	0.822178	0.814544	0.813022	0.837259	0.844172	0.852589
	c^*	3	3	3	3	3	3
	F^*	397.0467	399.3800	402.0786	299.0660	306.4026	312.8086
	P_{wv}^*	0.031943	0.056947	0.078803	0.095088	0.162837	0.214488
	P_I^*	0.077577	0.069150	0.063793	0.230929	0.197730	0.173633
	P_B^*	0.890480	0.873903	0.857405	0.673983	0.639433	0.611879
	$\mathbb{E}(L_{wv})^*$	0.061687	0.109877	0.152020	0.192472	0.329719	0.434485
	$\mathbb{E}(L_1)^*$	5.108708	5.373007	5.512663	1.681255	1.625559	1.573625
	R_{ren}^*	0.066622	0.118667	0.164181	0.207870	0.356096	0.469244
	R_{ret}^*	0.044415	0.079111	0.109454	0.138580	0.237398	0.312829

Table 4: Optimum performance measures for different values of χ and α.

	(χ, α)	(1.40,0.40)	(2.00,0.40)	(2.60,0.40)	(1.40,0.80)	(2.00,0.80)	(2.60,0.80)
MWV	μ_1^*	2.035823	2.006344	2.003822	2.002408	2.001909	2.001728
	μ_2^*	0.652905	0.762774	0.778089	0.786285	0.958438	1.147763
	c^*	4	4	4	4	4	4
	F^*	365.2232	379.3121	397.5228	383.8521	414.1506	439.5431
	P_{wv}^*	0.299538	0.361594	0.456885	0.489400	0.661038	0.795944
	P_I^*	0.073968	0.064365	0.054125	0.049638	0.027214	0.008487
	P_B^*	0.626494	0.574041	0.488990	0.460962	0.311747	0.195570
	$\mathbb{E}(L_{wv})^*$	1.138056	1.166085	1.280413	1.312554	1.406371	1.409206
	$\mathbb{E}(L_1)^*$	1.531777	1.341754	1.068388	0.983563	0.526212	0.206863
	R_{ren}^*	0.637311	0.932868	1.331630	1.470061	2.250193	2.918669
	R_{ret}^*	0.955967	1.399302	1.997445	0.367515	0.562548	0.729667
SWV	μ_1^*	2.012715	2.008421	2.006876	2.003986	2.001375	2.000632
	μ_2^*	0.418530	0.454064	0.528121	0.473229	0.501516	0.556894
	c^*	3	3	3	3	3	3
	F^*	252.1570	256.7518	261.3502	254.0936	257.2435	264.9412
	P_{wv}^*	0.144164	0.151797	0.159100	0.169835	0.183949	0.190602
	P_I^*	0.123569	0.130112	0.136371	0.145573	0.157671	0.163373
	P_B^*	0.732268	0.718091	0.704529	0.684591	0.658381	0.646026
	$\mathbb{E}(L_{wv})^*$	0.481415	0.429367	0.387929	0.391589	0.328053	0.276519
	$\mathbb{E}(L_1)^*$	2.540087	2.433606	2.352216	2.232054	2.075916	1.995080
	R_{ren}^*	0.269592	0.343494	0.403446	0.438580	0.524884	0.575158
	R_{ret}^*	0.404389	0.515241	0.605170	0.109645	0.131221	0.143790

Discussion

1. Figures 2 and 3 describe the impact of μ_1 and μ_2 on the optimal expected cost, for multiple and single working vacation policies, respectively. We clearly see the convexity of the curves, which shows that there exist certain values of the service rates μ_1 and μ_2 that minimize the total expected cost function for the chosen set of model parameters. Further, the optimal expected cost per unit time converges to the solution $F = 327.783374$ at $\mu_1^* = 1.184418$, $\mu_2^* = 0.650655$, and $c^* = 4$, under multiple working vacation, and converges to $F = 299.378265$ at $\mu_1^* = 1.615057$, $\mu_2^* = 0.505152$, and $c^* = 3$, under single working vacation.

2. From Figures 2-3 and Tables 1-4, it is clearly observed that the optimum service rate μ_1^* of multiple vacation model is smaller than that of single vacation model, whereas, the optimum service rate μ_2^*, the minimum expected cost F^*, and the optimum value of the optimum number of servers c^* of multiple vacation model is bigger than that of single vacation model, as intuitively expected.

3. For both SWV and MWV, μ_1^* increases (resp. decrease) with λ (resp. with r), while μ_2^* decreases with λ and r. In view of the stability of the system, this results are quite reasonable. We remark from Table 2 that, μ_2^* increases with λ, this can be due to the choice of ϑ.

4. The parameters μ_1^* and μ_2^* decrease with ϑ in MWV and increase along the increasing of ϑ in SWV. Moreover, a decreasing trend is seen in μ_1^* with ϖ, χ, α and β in both MWV and SWV. While μ_2^* increases with χ and α, under both policies, it decreases with β in MWV and significantly increases along the increasing of β in SWV. These results match with our intuition. Further, μ_2^* is not monotone with ϖ; this is due to the choice of the system parameters.

5. For both SWV and MWV, the optimum expected cost F^* increases with λ, ϖ, χ, and α, while it decreases with ϑ, r and β. This is quite reasonable, λ (resp. r and β) increases (resp. decrease) the mean system size, this results in the increasing (resp. the decreasing) of the minimum expected cost. On the other side, with the increasing of vacation rate, the servers rapidly switch to the busy period at which the customers are served with a large service rate, this implies a decreases in F^*. In addition, the higher the waiting server rate, the greater the probability that the servers go on working vacation and the bigger the average

rate of impatience which results in the increasing of optimal expected cost F^*.

6. For both MWV and SWV models, along the increasing of λ, the characteristics $\mathbb{E}(L_1)^*$ and P_B^* increase, while $\mathbb{E}(L_{wv})^*$, P_{wv}^*, R_{ren}^*, and R_{ret}^* decrease; obviously, the arrival rate increases the system size which, in returns, increases the probability that the servers are serving customers during normal busy servers. On the other side, this implies a decrease in the mean number of customers in the system during vacation period which results in the decreasing of P_{wv}^*, R_{ren}^* and R_{ret}^*. Further, we observe from Table 2 that P_I^* decreases with λ, as it should be, while from Table 1, it increases with the increasing of the arrival rate λ, this can be due to the choice of the system parameters ϑ and r.

7. For both policies, with the increasing of r, $\mathbb{E}(L_1)^*$, $\mathbb{E}(L_{wv})^*$, P_B^*, R_{ren}^*, and R_{ret}^* decrease significantly, whereas P_{wv}^* increases with r, which is coherent with the fact that increasing the batch size r decreases the system size when the servers are in working vacation. Consequently, the probability of working vacation decreases. Further, the probability that the servers are idle during normal busy period P_I^* decreases with r under MWV and increases with the increasing of the batch size r under SWV, as intuitively expected.

8. For both MWV and SWV policies, an increasing trend is observed in $\mathbb{E}(L_1)^*$, P_B^*, and P_I^* with ϑ and a decreasing trend is seen in $\mathbb{E}(L_1)^*$, P_{wv}^*, $\mathbb{E}(L_{wv})^*$, R_{ren}^*, and R_{ret}^* along the increasing of the vacation rate ϑ. This is quite explicable, the higher the vacation rate, the greater the mean system size during normal busy period, and the smaller the mean number of customers in the system during working vacation period, which lead to the decreasing of average rates of reneging and retention.

9. For both MWV and SWV models, the characteristics P_B^* and $\mathbb{E}(L_1)^*$ decrease with χ and α, while P_{wv}^* and R_{ren}^* increase with the increasing of χ and α. Evidently, the impatience rate increases the probability of working vacation, thus significant customers may leave the system which results in the decreasing of the mean number of customers in the system during normal busy period. Consequently, the probability that the servers work during normal busy period decreases significantly. Further, R_{ret}^* increases (resp. decreases) with χ (resp. α) under single and multiple working vacation policies, as it should be. Then, obviously, $\mathbb{E}(L_{wv})^*$

(resp. P_I^*) decreases (resp. increases) with χ and α in SWV model and increases (resp. decreases) along the increasing of χ and α in MWV model.

10. Both $\mathbb{E}(L_{wv})^*$, P_{wv}^*, R_{ren}^*, and R_{ret}^* increase with the increasing of ϖ and β, as intuitively expected. It is quite clear that the probabilities of non-feedback decreases the mean system size. Therefore, the servers switch to working vacation period at which the customers may leave the system because of the impatience phenomenon. Consequently, the average reneging and retention rates increase with β and ϖ. In addition, as the mean waiting time of the servers decreases, the working vacation period increases at which the impatience phenomenon may take place. Thus, R_{ren}^* and R_{ret}^* increase with ϖ. Further, to keep the system size under control and to avoid more reneging of customers, the firm may employ some strategies, which can be increasing the service rates, or engaging some additional service channels. Therefore, the average retention rate increases. Moreover, P_B^* decreases with ϖ and β, while P_I^* increases with β and decreases with ϖ, as intuitively expected. In addition, $\mathbb{E}(L_1)^*$ decreases significantly with β, while it increases with ϖ for $\beta = 0.4$ and decreases along the increasing of ϖ for $\beta = 0.80$. This can be due to the choice of the system parameters.

11. By comparing the two policies, multiple and single working vacations, we observe that

$$\begin{aligned}
P_{wv}^*(\text{single working vacation}) &< P_{wv}^*(\text{multiple working vacation}),\\
R_{ren}^*(\text{single working vacation}) &< R_{ren}^*(\text{multiple working vacation}),\\
R_{ret}^*(\text{single working vacation}) &< R_{ret}^*(\text{multiple working vacation}),\\
\mathbb{E}(L_{wv})^*(\text{single working vacation}) &< \mathbb{E}(L_{wv})^*(\text{multiple working vacation}),
\end{aligned}$$

while

$$\begin{aligned}
\mathbb{E}(L_1)^*(\text{multiple working vacation}) &< \mathbb{E}(L_1)^*(\text{single working vacation}),\\
P_B^*(\text{multiple working vacation}) &< P_B^*(\text{single working vacation}),\\
P_I^*(\text{multiple working vacation}) &< P_I^*(\text{single working vacation}).
\end{aligned}$$

Thus, it can be concluded that the single working vacation model has better performance measures than the multiple working vacations model. This perfectly matches with our expected intuition.

7 Conclusion

In this paper, we considered an infinite-space multi-server queueing system with batch arrival, waiting servers, synchronous multiple and single working vacation policies, Bernoulli feedback, reneging and retention of reneged customers. We developed the equations of the steady state probabilities of the model and obtained their steady-state solutions, using the probability generating functions (PGFs). Further, we derived various performance measures. A cost model has been formulated. In addition, the cost optimization problem under a given cost structure via genetic algorithm (GA) has been done. For further works, it will be interesting to deal with more realistic models including $\mathsf{G/G/c}$ and $\mathsf{G^X/G/c}$ queues with waiting servers, multiple and single working vacation policies, reneging, and retention of reneged customers.

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