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On tridiagonal matrices associated with Jordan blocks

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Abstract. This paper aims to show how some standard general results can be used to uncover the spectral theory of tridiagonal and related matrices more elegantly and simply than existing approaches. As a typical example, we apply the theory to the special tridiagonal matrices in recent papers on orthogonal polynomials arising from Jordan blocks. Consequently, we find that the polynomials and spectral theory of the special matrices are expressible in terms of the Chebyshev polynomials of second kind, whose properties yield interesting results. For special cases, we obtain results in terms of the Fibonacci numbers and Legendre polynomials.

1 Introduction

In the recent survey [9], we presented important modern applications involving tridiagonal matrices in applied mathematics, physics, and engineering and

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showed how the solutions/spectral theory could be easily obtained from Losonczi's work [13] of nearly thirty years ago. The latter work seems not only to have been neglected, but has also been re-cast or re-discovered in alternative guises since. In fact, many of these papers cite less general work that followed Losonczi a decade or so later. Interestingly, as described in [10], Losonczi's work grew out of the 1928 seminal paper by Egerváry and Szász [5].

In this note we uncover most of the important material in an existing publication in a simple and more elegant manner. Specifically, we aim to study the work of Capparelli and Maroscia [2] on orthogonal polynomials arising from Jordan blocks, which appeared in 2013, more than 20 years after Losonczi's paper. Before doing so, however, we need to present a summary of the main results of [13] in the following section.

2 A general perturbed tridiagonal Toeplitz matrix

The spectral characterization of an n-dimensional tridiagonal Toeplitz matrix possessing perturbations of the form

$$M_n(a,b,c,d) = \begin{pmatrix} a & c & & & & \\ d & 0 & c & & & \\ & d & \ddots & \ddots & & \\ & & \ddots & \ddots & c & \\ & & & d & 0 & c \\ & & & & d & b \end{pmatrix}_{n\times n}$$

has become the focus of much interest or activity over the past thirty years. On this issue the interested reader is advised to consult [9], which also provides an extensive recent list of references to the applications stemming from this activity.

The first major theoretical study dealing with this type of matrix was Losonczi's 1992 paper [13], which, in turn, was apparently motivated by the Rutherford's seminal paper [16] (cf. also [6]). From [13, Theorem 2] we find that the characteristic polynomial of M_n is given by

$$p_n(x) = (\sqrt{cd})^n \left(U_n\left(\frac{x}{2\sqrt{cd}}\right) - \frac{(a+b)}{\sqrt{cd}}\, U_{n-1}\left(\frac{x}{2\sqrt{cd}}\right) + \frac{ab}{cd}\, U_{n-2}\left(\frac{x}{2\sqrt{cd}}\right)\right). \tag{1}$$

The Chebyshev polynomials of the second kind have been studied extensively, e.g. [3], and are known to satisfy the following three-term recurrence relation:

$$2xU_n(x) = U_{n+1}(x) + U_{n-1}(x)\,, \quad \text{for all } n = 1, 2, \ldots\,, \tag{2}$$

with the initial values being $U_0(x) = 1$ and $U_1(x) = 2x$. Arguably, the most commonly-used form for the Chebyshev polynomials is

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad 0 \leqslant \theta < \pi,$$

where $x = \cos\theta$ and n is a non-negative integer. Other standard representations are

$$U_n(x) = \frac{\left(x+\sqrt{x^2-1}\right)^{n+1} - \left(x-\sqrt{x^2-1}\right)^{n+1}}{2\sqrt{x^2-1}} \tag{3}$$

$$= \sum_{k=0}^{n} (-2)^k \binom{n+k+1}{2k+1} (1-x)^k \tag{4}$$

Furthermore, the generating function of $U_n(x)$, i.e., where they appear as the coefficients of an infinite power series, is given by

$$\sum_{n=0}^{\infty} U_n(x)\, t^n = \frac{1}{1-2xt+t^2}\,. \tag{5}$$

For more details about these polynomials and their properties, the reader is referred to Chapter 22 of [1]. Note also that Losonczi's paper applies to more general matrices than the above tridiagonal Toeplitz matrix. Nevertheless, there is no loss of generality in computing the eigenvalues of M_n and its characteristic polynomial given by (1) using [13].

As far as the eigenvectors are concerned, if λ is an eigenvalue of $M_{n,1}$ assuming that $cd \neq 0$, then from Section 3 of [13], the eigenvectors corresponding to λ, which can be represented as $u = (u_0, u_1, \ldots, u_{n-1})^T$ with the superscript denoting the transpose, are given by

$$u_\ell = C\left(-\sqrt{\frac{c}{d}}\right)^{\ell} \left(\sin(\ell+1)\,\theta + \frac{a}{b}\sin\ell\theta\right)\,, \quad \text{for } \ell = 0, 1, \ldots, n-1, \tag{6}$$

provided $\lambda = -2\sqrt{cd}\cos\theta$, $\theta \neq m\pi$, $m \in Z$, and the arbitrary constant, $C \neq 0$. Since it is not required here, we do not present the $\theta = m\pi$ result. By presenting the above material, we aim in this note to derive shorter and thus, more elegant, proofs in the study of singular values of Jordan blocks than those presented in [2].

3 Jordan blocks

According to Definition 3.1.1 in [12], a Jordan block $J_n(r)$ is a $n \times n$ matrix composed of zeros everywhere except along the diagonal, which consists of a single number and the superdiagonal, where each entry or element is equal to unity. Thus, it has the following form:

$$J_n(r) = \begin{pmatrix} r & 1 & & & \\ & r & 1 & & \\ & & \ddots & \ddots & \\ & & & r & 1 \\ & & & & r \end{pmatrix}_{n\times n} .$$

The scalar r is generally a complex number, while the determinant of a Jordan block is r^n. However, since we are concerned the singular values of $J_n(r)$, i.e., the square root of the eigenvalues of their product with their transpose, r will be treated here mainly as a real number following [6], although some interesting results will be obtained when $r = i$.

Capparelli and Maroscia [2] study the product of the transpose of a Jordan block with itself, which yields

$$J_n^T(r)J_n(r) = \begin{pmatrix} r^2 & r & & & \\ r & 1+r^2 & r & & \\ & r & \ddots & \ddots & \\ & & \ddots & \ddots & r \\ & & & r & 1+r^2 \end{pmatrix}_{n\times n} . \tag{7}$$

Their primary aim is to determine the singular values of $J_n(r)$, although bounds for more general cases, where the entries are less uniform, can be found in [6]. Consequently, they focus on the following matrices:

$$\mathcal{T}_n(\alpha, r) = \begin{pmatrix} \alpha & r & & & \\ r & 1+\alpha & r & & \\ & r & \ddots & \ddots & \\ & & \ddots & \ddots & r \\ & & & r & 1+\alpha \end{pmatrix}_{n\times n} . \tag{8}$$

and

$$\mathcal{U}_n(\alpha, r) = \begin{pmatrix} \alpha & r & & & \\ r & \alpha & r & & \\ & r & \ddots & \ddots & \\ & & \ddots & \ddots & r \\ & & & r & \alpha \end{pmatrix}_{n\times n}. \tag{9}$$

Since the determinant of the transpose of a square matrix is equal to the determinant of the square matrix, from the above we see that $\det J_n(r) = (\det \mathcal{T}_n(r^2, r))^{1/2}$.

In the remainder of this section we shall consider in the next subsection general results involving α and r for the above tridiagonal matrices, and then special cases in the following subsection.

3.1 General cases

Our first observation is concerned with the relationship between the determinants of both matrices in addition to the derivation of their generating functions. These are encapsulated in the following theorem.

Theorem 1 *The determinant of the matrix $\mathcal{T}_n(\alpha, r)$ is related to the determinant of $\mathcal{U}_n(\alpha, r)$ according to*

$$\det \mathcal{T}_n(\alpha, r) = \det \mathcal{U}_n(1+\alpha, r) - \det \mathcal{U}_{n-1}(1+\alpha, r), \tag{10}$$

while their generating functions are given by

$$\sum_{n=0}^{\infty} \det \mathcal{U}_n(\alpha, r) t^n = \frac{1}{1-\alpha t + r^2 t^2}, \tag{11}$$

and

$$\sum_{n=0}^{\infty} \det \mathcal{T}_n(\alpha, r) t^n = \frac{1-t}{1-(1+\alpha)t + r^2 t^2}. \tag{12}$$

Proof. From Theorem 2 in [13], or specifically, (1) with $x = 1+\alpha$, $a = -1$, $b = 0$, and $c = d = r$, we arrive at

$$\det \mathcal{T}_n(\alpha, r) = r^{n-1}\left(r\, U_n\left(\frac{1+\alpha}{2r}\right) - U_{n-1}\left(\frac{1+\alpha}{2r}\right)\right). \tag{13}$$

On the other hand, the determinant of $\mathcal{U}_n(\alpha, r)$ is found using the same values except that $x = \alpha$ and $a = b = 0$. Then we find that

$$\det \mathcal{U}_n(\alpha, r) = r^n U_n\left(\frac{\alpha}{2r}\right). \tag{14}$$

Introducing (14) into (13) yields (10).

If one multiplies (14) by t^n and sums from $n = 0$ to ∞, then one obtains

$$\sum_{n=0}^{\infty} \det \mathcal{U}_n(\alpha, r) t^n = \sum_{n=0}^{\infty} U_n\Big(\frac{\alpha}{2r}\Big)(rt)^n.$$

The right-hand side of the above result is simply the generating function for the Chebyshev polynomials of the second kind. Therefore, putting t and x in (5) equal to rt and $\alpha/2r$, respectively, yields (11).

Adopting the same procedure to (14) yields

$$\sum_{n=0}^{\infty} \det \mathcal{T}_n(\alpha, r) t^n = \sum_{n=0}^{\infty} \Big(\det \mathcal{U}_n(1+\alpha, r) - \det \mathcal{U}_{n-1}(1+\alpha, r)\Big) t^n.$$

The first sum on the right-hand side of the above equation can be replaced by the right-hand side (rhs) of (11) with α replaced by $1+\alpha$, while in the second sum n needs to be replaced by $n+1$. Then the second sum is identical to (11) except α is replaced $\alpha + 1$ and it is multiplied by t. Combining the results for both sums yields (12). □

The generating function (11) has been derived in Proposition 1.2 of [2], but as a result of Theorem 2 from [13], which also appears in the previous section, the derivation here is much more succinct. If (3) is introduced into (14), then (1.7) in Proposition 1.3 of [2] follows, which is

$$\det \mathcal{U}_n(\alpha, r) = \frac{1}{\sqrt{\alpha^2 - 4r^2}} \left(\Big(\frac{\alpha + \sqrt{\alpha^2 - 4r^2}}{2}\Big)^{n+1} - \Big(\frac{\alpha - \sqrt{\alpha^2 - 4r^2}}{2}\Big)^{n+1} \right).$$

In this instance, however, the condition accompanying the proposition that $\alpha^2 - 4r^2 > 0$ in [2], becomes redundant. Moreover, by introducing the above result into (13), one obtains the corresponding form for $\det \mathcal{T}_n(\alpha, r)$. In addition, inserting (14) into the recurrence relation given by (2) produces the following recurrence relation:

$$\alpha \det \mathcal{U}_n(\alpha, r) = \det \mathcal{U}_{n+1}(\alpha, r) + r^2 \det \mathcal{U}_{n-1}(\alpha, r).$$

Finally, introducing (2) into (10) yields a recurrence relation for $\det \mathcal{T}_n(\alpha, r)$, which is

$$2(1+\alpha) \det \mathcal{T}_n(\alpha, r) = \det \mathcal{T}_{n+1}(\alpha, r) + r^2 \det \mathcal{T}_{n-1}(\alpha, r).$$

3.2 Special cases

By putting $\alpha = r^2$ in (13), we obtain

$$\det \mathcal{T}_n(r^2, r) = r^{n-1}\left(r\,U_n\left(\frac{r}{2}+\frac{1}{2r}\right) - U_{n-1}\left(\frac{r}{2}+\frac{1}{2r}\right)\right).$$

By noting that $\det \mathcal{T}_1(r^2, r) = r^2$ and the recurrence relation (2), one can prove by induction that

$$\det \mathcal{T}_n(r^2, r) = r^{2n}.$$

Hence, we find that $\det J_n(r) = r^n$. Furthermore, if λ denotes the eigenvalues of $\mathcal{T}_n(r^2, r)$, then we can modify the above result to obtain the characteristic equation for the matrix, which is

$$r\,U_n\left(\frac{r^2+1-\lambda}{2r}\right) - U_{n-1}\left(\frac{r^2+1-\lambda}{2r}\right) = 0.$$

The above equation can be solved by using the Solve routine in Mathematica. For example, when $n = 2$, one simply types
Solve[(r ChebyshevU[2, (r^2 + 1 - λ)/(2 r)] - ChebyshevU[2 - 1, (r^2 + 1 - λ)/(2 r)]) == 0, λ],
which yields

$$\lambda_1 = (1 + 2r^2 - \sqrt{1+4r^2})/2, \quad \text{and} \quad \lambda_2 = (1 + 2r^2 + \sqrt{1+4r^2})/2.$$

These results have also been obtained by Capparelli and Maroscia [2].

By following the same procedure for $n = 3$, we find that

$$\lambda_1 = \frac{1}{3}\left(2 + 3r^2 - \frac{2^{1/3}(1+6r^2)}{p(r)} - 2^{-1/3}p(r)\right),$$

$$\lambda_2 = \frac{1}{3}\left(2 + 3r^2 - \frac{2^{1/3}e^{i\pi/3}(1+6r^2)}{p(r)} + 2^{-1/3}e^{-i\pi/3}p(r)\right),$$

and

$$\lambda_3 = \frac{1}{3}\left(2 + 3r^2 + \frac{2^{1/3}e^{-i\pi/3}(1+6r^2)}{p(r)} + 2^{-1/3}e^{i\pi/3}p(r)\right),$$

where

$$p(r) = \left(2 - 9r^2 + 3\sqrt{3}\sqrt{-4r^2 - 13r^4 - 32r^6}\right)^{1/3}.$$

For $n = 4$, the characteristic polynomial becomes

$$\lambda^4 - (4r^2+3)\lambda^3 + (6r^4+6r^2+3)\lambda^2 - (4r^6+3r^4+2r^2+1)\lambda + r^8 = 0,$$

while for $n = 5$, it is given by

$$\lambda^5 - (5r^2+4)\lambda^4 + (10r^4+12r^2+6)\lambda^3 - (10r^6+12r^4+9r^2+4)\lambda^2 + (5r^8+4r^6+3r^4+2r^2+1)\lambda - r^{10} = 0.$$

Although the above equations can be solved for λ using the Solve routine in Mathematica, the solutions are too cumbersome to display here.

For $\alpha = 1$ and $r = \exp(i\pi/2)$ or i, the right-hand side of (11) becomes the right-hand side of the generating function for the Fibonacci numbers, F_n, divided by t. As a consequence, we observe that

$$\det \mathcal{U}_{n-1}(1,i) = F_n, \tag{15}$$

while from (13), we arrive at

$$F_n = \exp(i(n-1)\pi/2)U_{n-1}(-i/2). \tag{16}$$

See also [7]. This result can also be derived from (5) by setting $x = -i/2$ and $t = it$.

If we put $\alpha = 2x$ and $r = 1$, then we obtain

$$\sum_{n=0}^{\infty} \det \mathcal{U}_n(2x,1)t^n = \frac{1}{1-2xt+t^2} = \sum_{n=0}^{\infty} t^n \sum_{j=0}^{n} \mathcal{P}_j(x)\mathcal{P}_{n-j}(x),$$

where $\mathcal{P}_n(x)$ denotes the Legendre polynomial of degree n and is not be confused with $P_n(x)$ in [2] appearing here shortly. By equating like powers of t, we obtain

$$\det \mathcal{U}_n(2x,1) = \sum_{j=0}^{n} \mathcal{P}_j(x)\mathcal{P}_{n-j}(x). \tag{17}$$

Now introducing (14) into the above equation yields

$$\sum_{j=0}^{n} \mathcal{P}_j(x)\mathcal{P}_{n-j}(x) = U_n(x).$$

Alternatively, with the aid of (16), we arrive at

$$i^n \sum_{j=0}^{n} \mathcal{P}_j(-i/2)\mathcal{P}_{n-j}(-i/2) = F_n,$$

which is an interesting relationship between a sum of products of Legendre polynomials with imaginary arguments and the Fibonacci numbers.

In a similar manner the generating function of $\det \mathcal{T}_n(\alpha, r)$ is found to be

$$\sum_{n=0}^{\infty} \det \mathcal{T}_n(\alpha, r) t^n = \frac{1-t}{1-(1+\alpha)t + r^2t^2}. \tag{18}$$

By putting $\alpha = 0$ and $r = i$, we observe that the right-hand side of (12) becomes the difference of two forms of the generating function for the Fibonacci numbers, namely

$$\sum_{n=0}^{\infty} \det \mathcal{T}_n(0, i) t^n = \sum_{n=0}^{\infty} (F_{n+1} - F_n)\, t^n.$$

Equating like powers of t yields

$$\det \mathcal{T}_n(0, i) = F_{n-1},$$

while from (15), we see that $\det U_{n-1}(1, i) = \det T_{n+1}(0, i)$.

If we set $\alpha = 1 - x$ and $r = 1$ in (18), then we derive the generating function of the characteristic polynomial of $\mathcal{T}_n(1, 1)$, which is denoted by $P_n(x)$ in [2]. That is, $P_n(x) = \det \mathcal{T}_n(1 - x, 1)$. Note also for the case of $\mathcal{T}_n(1, 1)$, which we discuss in more detail later, the right-hand side of the generating function reduces to $1/(1-t)$. By expanding this term as the geometric series for $|t| < 1$ and equating like powers of t on both sides, one finds that $\det \mathcal{T}_n(1, 1) = 1$. On the other hand, if we replace α by $1 + \alpha$ in (11) and subtract (12) from the resulting equation, then we find that

$$\sum_{n=0}^{\infty} \big(\det \mathcal{U}_n(1 + \alpha, r) - \det \mathcal{T}_n(\alpha, r)\big) t^n = \frac{t}{1-(1+\alpha)t + r^2t^2}.$$

Consequently, from the preceding material, we arrive at

$$\det \mathcal{U}_n(1, i) - \det \mathcal{T}_n(0, i) = F_n.$$

Now if we set $\alpha = 1 - x$ and $r = 1$, then we obtain the generating function of the other set of polynomials appearing in Section 2 of [2], denoted by $Q_n(x)$. Thus, we arrrive at

$$Q_n(x) = \det \mathcal{U}_n(2 - x, 1) - \det \mathcal{T}_n(1 - x, 1) = U_{n-1}(1 - x/2).$$

Alternatively, this may be expressed as $Q_n(x)+P_n(x) = \det \mathcal{U}_n(2-x,1)$ or the sum of both polynomials with index n yields the characteristic polynomial of $\mathcal{U}_n(2,1)$. Then using (17), we can express the sum of the polynomials in terms of shifted Legendre polynomials as

$$Q_n(x) + P_n(x) = \sum_{j=0}^{n} \mathcal{P}_j(1-x)\mathcal{P}_{n-j}(1-x).$$

For the case of $\alpha = r = 1$ in $\mathcal{T}_n(\alpha, r)$ or

$$\mathcal{T}_n(1,1) = \begin{pmatrix} 1 & 1 & & & \\ 1 & 2 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 2 \end{pmatrix}_{n\times n}$$

we find that the characteristic polynomial reduces to

$$P_n(x) = U_n\left(\frac{2-x}{2}\right) - U_{n-1}\left(\frac{2-x}{2}\right), \tag{19}$$

using (1) (the reader is also referred to [8]). If we set $x = i+2$ into the above recurrence relation, then with the aid of (16) we find that

$$i^n P_n(i+2) = F_{n+1} - iF_n.$$

On the other hand, by considering (4) and Pascal's formula, which is

$$\binom{n+k+1}{2k+1} - \binom{n+k}{2k+1} = \binom{n+k}{2k},$$

we find that

$$P_n(x) = \sum_{k=0}^{n} (-1)^k \binom{n+k}{2k} x^k.$$

This is proved in Theorem 2.5 of [2]. On the other hand, the polynomials $Q_n(x)$, which are presented after Corollary 2.6, are even simpler to evaluate since we have already observed that they are equal to $U_{n-1}(1-x/2)$. Introducing (4) into this result yields

$$Q_{n-1}(x) = \sum_{k=0}^{n-1} (-1)^k \binom{n+k}{2k+1} x^k.$$

By combining (5) with (19), we obtain the generating function of $P_n(x)$ appearing in Proposition 2.1 of [2] or

$$\sum_{n=0}^{\infty} P_n(x)t^n = \frac{1-t}{1-(2-x)t+t^2}.$$

If we put $\alpha = 1-x$ and $r = 1$ in (12) and equate like powers of t, then we arrive at $P_n(x) = \det \mathcal{T}_n(1-x, 1)$, which was derived earlier. Furthermore, by setting $x = 2$, we find that

$$\begin{aligned} P_n(2) &= \sum_{k=0}^{n} (-1)^k \binom{n+k}{2k} 2^k \\ &= U_n(0) - U_{n-1}(0) \\ &= \begin{cases} 1 & \text{if } n \equiv 0, 3 \pmod 4, \\ -1 & \text{if } n \equiv 1, 2 \pmod 4. \end{cases} \end{aligned} \tag{20}$$

Hence it can be seen that

$$\det \mathcal{T}_n(-1, 1) = \begin{cases} 1 & \text{if } n \equiv 0, 3 \pmod 4, \\ -1 & \text{if } n \equiv 1, 2 \pmod 4. \end{cases}$$

With regard to the question posed after Corollary 2.6 in [2] concerning whether there exists a combinatorial interpretation of the above sequence, the answer is affirmative. To see this, we refer the reader to [17], where it is not only stated that Sequence A087960 is given by

$$P_n(2) = (-1)^{\binom{n+1}{2}},$$

but there are also numerous references to applications. In fact, from the site we observe that $P_n(2)$ represents the coefficient of x^{n+1} in the power series of $(1+x)/(1+x^2) - 1$ and is given by

$$P_n(2) = (-1)^{\binom{n+1}{2}} = \cos\big(n\pi/2\big) - \sin\big(n\pi/2\big).$$

A similar sequence, where each term is shifted by incrementing n by one, also appears as A057077 in [17], again with numerous applications.

It should also be noted that Capparelli and Maroscia were unaware that the properties of the $P_n(x)$ are well-known in the theory of orthogonal polynomials. In particular, these polynomials were first studied by Chihara [4], where they were referred to as co-recursive polynomials. More generally, they are now regarded as a particular case of anti-associated polynomials of a certain family of orthogonal polynomials derived by Ronveaux and van Assche [15].

Finally, we consider the singular values of the Jordan block with $r = 1$. As indicated before, the product of the Jordan block with its transpose yields $\mathcal{T}_n(r^2, r)$ as per (7). Since the transpose has the same characteristic polynomial as the Jordan block, the characteristic polynomial of $\mathcal{T}_n(1,1)$ is the square of the characteristic polynomial of the Jordan block. From (19), $P_n(x)$ represents the characteristic polynomial for $\mathcal{T}_n(1,1)$, whose eigenvalues are obtained by setting the right-hand side of (19) to zero. Thus, we require the solutions of

$$U_n\left(\frac{2-x}{2}\right) = U_{n-1}\left(\frac{2-x}{2}\right) .$$

The above equation is solved simply by replacing $1 - x/2$ by $\cos\theta$. After carrying out a little algebra using (2), one arrives at

$$\cos\big((n+1/2)\theta\big)\sin\big(\theta/2\big) = 0,$$

whose solutions are

$$\theta = (2k+1)\pi/(2n+1).$$

As a consequence, the eigenvalues of $\mathcal{T}_n(1,1)$ are given by

$$\begin{aligned} \lambda_k &= 2 - 2\cos\left(\frac{(2k+1)\pi}{2n+1}\right) \\ &= 4\sin^2\left(\frac{(2k+1)\pi}{4n+2}\right), \quad \text{for } k = 0, 1, \ldots, n-1. \end{aligned}$$

From (6) or Theorem 3 in [13], the eigenvectors of $\mathcal{T}_n(1,1)$ denoted by $u = (u_0, u_1, \ldots, u_{n-1})^T$ are simply given by

$$u_k = (-1)^k C\Big(\sin(k+1)\phi + \frac{1}{2}\sin k\phi\Big),$$

and

$$\phi = \arccos\Big(\cos\Big(\frac{(2k+1)\pi}{2n+1}\Big) - 1\Big).$$

Furthermore, taking the square root of the eigenvalues yields the singular values of $J_n(1)$. Hence we have arrived at (3.1) in Theorem 3.3 of [2] with little effort, while Capparelli and Maroscia had to introduce Lemma 3.2. It should also be mentioned that there was no need to prove this lemma since it is a well-known result that appears as No. 17.14.4 on p. 250 of Hansen [11] or as No. I.1.9 on p. 760 of Prudnikov et al. [14], both of which preceded [2] by several decades.

4 Conclusion

In this article we have shown how important theory in Losonczi's article [13] can be used to uncover the spectral theory of tridiagonal and related matrices. As an example, we have demonstrated that most of the interesting results in Capparelli and Maroscia's work [2] on orthogonal polynomials arising from Jordan blocks can be obtained in a simple and elegant manner, often requiring only a few lines. We have achieved this by applying the results in [13] to the matrix types, $\mathcal{T}_n(\alpha, r)$ and $\mathcal{U}_n(\alpha, r)$, given by (8) and (9), respectively. The first type of matrix is related to the product of a Jordan block and its tranpose, which also enabled us to derive the spectral theory of these matrices from [13]. Furthermore, we were able to show that the orthogonal polynomials, denoted by $P_n(x)$ and $Q_n(x)$ in [2], were expressible in terms of the second kind of Chebyshev polynomials. In carrying out this work, we also answered the question after Corollary 2.6 in [2] whether there is a natural combinatorial interpretation of (20) or (2.9) in [2]. This was shown to be minus one raised to a specific binomial coefficient. Finally, we found that the spectral theory for imaginary elements/entries in special cases of $\mathcal{U}_n(\alpha, r)$ and $\mathcal{T}_n(\alpha, r)$, can be expressed in terms of the Fibonacci numbers and Legendre polynomials.

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