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Study of periodic solutions for third-order iterative differential equations via Krasnoselskii-Burton's theorem

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Abstract. This paper studies the existence of periodic solutions of a third order iterative differential equation. The main tool used here is Krasnoselskii-Burton's fixed point theorem dealing with a sum of two mappings, one is a large contraction and the other is compact.

1 Introduction

Delay or iterative differential equations have attracted considerable attention in mathematics during recent years since these equations have been showed to be valuable tools in the modeling of many phenomena in various fields of science, physics, chemistry and engineering, etc. In particular, periodicity, positivity and stability of solutions for delay or iterative differential equations has been studied extensively by many authors, see the references [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]. For example in [9], the third-order iterative differential equation

$$x'''(t) + p(t)\,x''(t) + q(t)\,x'(t) + r(t)\,x(t) = x(t)\sum_{k=1}^{n} c_k(t)\,x^{[k]}(t),$$

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has been investigated. By using Krasnoselskii's fixed point theorem and the contraction mapping principle, Bouakkaz et al. obtained the existence, uniqueness and continuous dependence of periodic solution. Inspired and motivated by the references [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20], we study the existence of periodic solutions for the third order iterative differential equation

$$x'''(t) + p(t)x''(t) + q(t)x'(t) + r(t)h(x(t)) = x(t)\sum_{k=1}^{n} c_k(t)x^{[k]}(t), \quad (1)$$

where $x^{[1]}(t) = x(t)$, $x^{[2]}(t) = x(x(t))$, ..., $x^{[n]}(t) = x^{[n-1]}(x(t))$, p, q, r and c_k, $k = \overline{1, n}$ are continuous real-valued functions. Our purpose here is to use Krasnoselskii-Burton's fixed point theorem to prove the existence of periodic solutions for (1). To prove the existence of periodic solutions, we transform (1) into an equivalent integral equation and then use Krasnoselskii-Burton's fixed point theorem. The obtained integral equation splits in the sum of two mappings, one is a large contraction and the other is compact.

2 Preliminaries

For $T > 0$, let P_T be the set of all continuous scalar functions x, periodic in t of period T. Then $(P_T, \|.\|)$ is a Banach space with the supremum norm

$$\|x\| = \sup_{t\in\mathbb{R}} |x(t)| = \sup_{t\in[0,T]} |x(t)|,$$

and for $N, K > 0$, let

$$P_T(N, K) = \{x \in P_T, \ \|x\| \le N, \ |x(t_2) - x(t_1)| \le K|t_2 - t_1|, \forall t_1, t_2 \in \mathbb{R}\},$$

be a closed convex and bounded subset of P_T.

Throughout this paper, we assume that

(H1) There exist two differentiable positive T-periodic functions a_1, a_2 and a positive real constant ρ such that

$$\begin{cases} a_1(t) + \rho = p(t), \\ a_1'(t) + a_2(t) + \rho a_1(t) = q(t), \\ a_2'(t) + \rho a_2(t) = r(t). \end{cases}$$

(H2) $p, q, r \in P_T$ and

$$\int_0^T p(s)\,ds > \rho T \text{ and } \int_0^T q(s)\,ds > 0.$$

Now, we consider the equation

$$x'''(t) + p(t)x''(t) + q(t)x'(t) + r(t)x(t) = e(t), \tag{2}$$

where e is a continuous T-periodic function. It is easy to see that by virtue of (H1) and (H2), the above equation can be transformed into the following system

$$\begin{cases} y'(t) + \rho y(t) = e(t), \\ x''(t) + a_1(t)x'(t) + a_2(t)x(t) = y(t). \end{cases}$$

Lemma 1 **([5])** *If* $y, e \in P_T$, *then* y *is a solution of the equation*

$$y'(t) + \rho y(t) = e(t),$$

if and only if

$$y(t) = \int_t^{t+T} G_1(t,s)\,e(s)\,ds, \tag{3}$$

where

$$G_1(t,s) = \frac{\exp(\rho(s-t))}{\exp(\rho T) - 1}. \tag{4}$$

Corollary 1 **([14])** *Green's function* G_1 *satisfies the following property*

$$m_1 \le G_1(t,s) \le M_1,$$

where

$$m_1 = \frac{1}{\exp(\rho T) - 1},\ M_1 = \frac{\exp(\rho T)}{\exp(\rho T) - 1}.$$

Lemma 2 **([13])** *Suppose that (H1), (H2) hold and*

$$\frac{R_1\left[\exp\left(\int_0^T a_1(v)\,dv\right) - 1\right]}{Q_1 T} \ge 1, \tag{5}$$

where

$$R_1 = \max_{t\in[0,T]}\left|\int_t^{t+T} \frac{\exp\left(\int_t^s a_1(v)\,dv\right)}{\exp\left(\int_0^T a_1(v)\,dv\right) - 1} a_2(s)\,ds\right|,$$

and

$$Q_1 = \left(1 + \exp\left(\int_0^T a_1(v)\,dv\right)\right)^2 R_1^2.$$

Then, there are continuous and T*-periodic functions* a *and* b *such that*

$$b(t) > 0, \quad \int_0^T a(v)dv > 0,$$

and

$$a(t) + b(t) = a_1(t), \quad \frac{d}{dt}b(t) + a(t)b(t) = a_2(t) \text{ for all } t \in \mathbb{R}.$$

Lemma 3 **([17])** *Suppose the conditions of Lemma 2 hold and* $y \in P_T$*. Then the equation*

$$\frac{d^2}{dt^2}x(t) + a_1(t)\frac{d}{dt}x(t) + a_2(t)x(t) = y(t),$$

has a T*-periodic solution. Moreover, the periodic solution can be expressed as*

$$x(t) = \int_t^{t+T} G_2(t,s)\,y(s)\,ds, \tag{6}$$

where

$$G_2(t,s) = \frac{\int_t^s \exp\left[\int_t^v b(u)\,du + \int_v^s a(u)\,du\right] dv}{\left[\exp\left(\int_0^T a(v)\,dv\right) - 1\right]\left[\exp\left(\int_0^T b(v)\,dv\right) - 1\right]} + \frac{\int_s^{t+T} \exp\left[\int_t^v b(u)\,du + \int_v^{s+T} a(u)\,du\right] dv}{\left[\exp\left(\int_0^T a(v)\,dv\right) - 1\right]\left[\exp\left(\int_0^T b(v)\,dv\right) - 1\right]}. \tag{7}$$

Lemma 4 **([13])** *Let* $A = \int_0^T a_1(v)\,dv$ *and* $B = T^2\exp\left(\frac{1}{T}\int_0^T \ln(a_2(v))\,dv\right)$*. If*

$$A^2 \geq 4B, \tag{8}$$

then

$$\min\left\{\int_0^T a(v)\,dv, \int_0^T b(v)\,dv\right\} \geq \frac{1}{2}\left(A - \sqrt{A^2 - 4B}\right) := l,$$

and

$$\max\left\{\int_0^T a(v)\,dv, \int_0^T b(v)\,dv\right\} \leq \frac{1}{2}\left(A + \sqrt{A^2 - 4B}\right) := L.$$

Corollary 2 **([14])** *Green's function* G_2 *satisfies the following properties*

$$m_2 \leq G_2(t,s) \leq M_2$$

where

$$m_2 = \frac{T}{\left(e^L - 1\right)^2} \text{ and } M_2 = \frac{T \exp\left(\int_0^T a_1(v)\,dv\right)}{\left(e^l - 1\right)^2}.$$

Lemma 5 **([8])** *For any* $t_1, t_2 \in \mathbb{R}$

$$\int_{t_1}^{t_1+T} \left|G_2(t_2,s) - G_2(t_1,s)\right| ds \leq \mu \left|t_2 - t_1\right|,$$

where

$$\mu = Te^{2L}\eta\left[T\lambda_2\gamma\left(2e^{2L}+1\right) + e^L + 1\right],$$

and

$$\begin{aligned} \eta &= \frac{1}{\left[\exp\left(\int_0^T a(v)\,dv\right) - 1\right]\left[\exp\left(\int_0^T b(v)\,dv\right) - 1\right]}, \\ \lambda_2 &= \max_{t\in[0,T]} |b(t)|, \ \gamma = \exp\left(-\int_0^T b(v)\,dv\right). \end{aligned}$$

Lemma 6 **([11])** *Suppose the conditions of Lemma 2 hold and* $e \in P_T$*. Then the equation*

$$x'''(t) + p(t)x''(t) + q(t)x'(t) + r(t)x(t) = e(t),$$

has a T*-periodic solution. Moreover, the periodic solution can be expressed by*

$$x(t) = \int_t^{t+T} G(t,s)\,e(s)\,ds, \tag{9}$$

where

$$G(t,s) = \int_t^{t+T} G_2(t,\sigma)\,G_1(\sigma,s)\,d\sigma. \tag{10}$$

Corollary 3 ([14]) *Green's function* G *satisfies the following property*

$$m \leq G(t,s) \leq M,$$

where

$$m = \frac{T^2}{\left(e^L - 1\right)^2 \left(\exp(\rho T) - 1\right)} \text{ and } M = \frac{T^2\left(\rho T + \exp\left(\int_0^T a(v)\,dv\right)\right)}{\left(e^l - 1\right)^2\left(\exp(\rho T) - 1\right)}.$$

Lemma 7 ([20]) *For any* $\varphi, \psi \in P_T(L, K)$,

$$\left\|\varphi^{[i]} - \psi^{[i]}\right\| \leq \sum_{j=0}^{i-1} K^j \left\|\varphi - \psi\right\|, \ i = 1, 2, ...$$

Lemma 8 ([19]) *It holds*

$$P_T(N, K) = \{x \in P_T, \ \|x\| \leq N, \ |x(t_2) - x(t_1)| \leq K|t_2 - t_1|, \ \forall t_1, t_2 \in [0, T]\}.$$

Lemma 9 *Suppose (H1), (H2) and (5) hold. The function* $x \in P_T(N, K)$ *is a solution of (1) if and only if*

$$x(t) = \int_t^{t+T} r(s) H(x(s)) G(t, s) ds + \sum_{i=1}^{n} \int_t^{t+T} c_i(s) x(s) x^{[i]}(s) G(t, s) ds, \tag{11}$$

where

$$H(x) = x - h(x). \tag{12}$$

Proof. Let $x \in P_T(N, K)$ be a solution of (1). Rewrite (1) as

$$\begin{aligned} & x'''(t) + p(t) x''(t) + q(t) x'(t) + r(t) x(t) \\ & = r(t) H(x(t)) + x(t) \sum_{k=1}^{n} c_k(t) x^{[k]}(t). \end{aligned}$$

From Lemma 6, we have

$$x(t) = \int_t^{t+T} G(t, s) \left[r(s) H(x(s)) + x(s) \sum_{k=1}^{n} c_k(s) x^{[k]}(s) \right] ds.$$

The proof is completed. □

Definition 1 **(Large contraction** [10]**)** *Let* $(\mathbb{M}, d)$ *be a metric space and consider* $\mathcal{B} : \mathbb{M} \to \mathbb{M}$. *Then* $\mathcal{B}$ *is said to be a large contraction if given* $\phi, \varphi \in \mathbb{M}$ *with* $\phi \neq \varphi$ *then* $d(\mathcal{B}\phi, \mathcal{B}\varphi) \leq d(\phi, \varphi)$ *and if for all* $\varepsilon > 0$, *there exists a* $\delta \in (0, 1)$ *such that*

$$[\phi, \varphi \in \mathbb{M}, \ d(\phi, \varphi) \geq \varepsilon] \Rightarrow d(\mathcal{B}\phi, \mathcal{B}\varphi) \leq \delta d(\phi, \varphi).$$

Theorem 1 (Krasnoselskii-Burton [10]) *Let $\mathbb{M}$ be a closed bounded convex nonempty subset of a Banach space $(\mathbb{B}, \|.\|)$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ map $\mathbb{M}$ into $\mathbb{M}$ such that*

(i) $x, y \in \mathbb{M}$, implies $\mathcal{A}x + \mathcal{B}y \in \mathbb{M}$,

(ii) $\mathcal{A}$ is compact and continuous,

(iii) $\mathcal{B}$ is a large contraction mapping.

Then there exists $z \in \mathbb{M}$ with $z = \mathcal{A}z + \mathcal{B}z$.

We will use this theorem to show the existence of periodic solutions for (1).

Theorem 2 ([1]) *Let $\|.\|$ be the supremum norm, $\mathbb{M} = \{\varphi \in P_T : \|\varphi\| \leq N\}$ where N is a positive constant. Suppose that h is satisfying the following conditions*

(I) $h : \mathbb{R} \to \mathbb{R}$ *is continuous on* $[-N, N]$ *and differentiable on* $(-N, N)$,

(II) *the function* h *is strictly increasing on* $[-N, N]$,

(III) $\sup_{t \in (-N,N)} h'(t) \leq 1$.

Then the mapping H define by (12) is a large contraction on the set $\mathbb{M}$.

3 Existence of periodic solutions

To apply the Theorem 1 we need to define a Banach space $\mathbb{B}$, a closed bounded convex subset $\mathbb{M}$ of $\mathbb{B}$ and construct two mappings; one is a compact and the other is a large contraction. So, we let $(\mathbb{B}, \|.\|) = (P_T, \|.\|)$ and

$$\begin{aligned}\mathbb{M} &= P_T(N, K) \\ &= \{\varphi \in P_T,\ \|\varphi\| \leq N,\ |\varphi(t_2) - \varphi(t_1)| \leq K|t_2 - t_1|,\ \forall t_1, t_2 \in [0, T]\},\end{aligned} \tag{13}$$

with $N, K > 0$. Define a mapping $\mathcal{S} : \mathbb{M} \to P_T$ by

$$\begin{aligned}(\mathcal{S}\varphi)(t) &= \int_t^{t+T} r(s) H(\varphi(s)) G(t, s)\, ds \\ &\quad + \sum_{i=1}^{n} \int_t^{t+T} c_i(s) \varphi(s) \varphi^{[i]}(s) G(t, s)\, ds.\end{aligned}$$

Therefore, we express the above mapping as

$$\mathcal{S}\varphi = \mathcal{A}\varphi + \mathcal{B}\varphi,$$

where $\mathcal{A}, \mathcal{B} : \mathbb{M} \to P_T$ are given by

$$(\mathcal{A}\varphi)(t) = \sum_{i=1}^{n} \int_t^{t+T} c_i(s)\,\varphi(s)\,\varphi^{[i]}(s)\,G(t,s)\,ds, \tag{14}$$

and

$$(\mathcal{B}\varphi)(t) = \int_t^{t+T} r(s)\,H(\varphi(s))\,G(t,s)\,ds, \tag{15}$$

where c_i in P_T, $i = \overline{1,n}$.

Remark 1 *The compactness of* $P_T(N,K)$ *results immediately from the application of Lemma 8 and Ascoli-Arzela theorem.*

We need the next lemma in our next results. This lemma and its proof can be found in [9].

Lemma 10 *For any* $\varphi, \psi \in \mathbb{M}$*,*

$$\left\|\varphi\varphi^{[i]} - \psi\psi^{[i]}\right\| \leq N\left(1 + \sum_{j=0}^{i-1} K^j\right)\|\varphi - \psi\|,\ i = 1, 2, \ldots.$$

We will show set of preparatory lemmas to use them in the proof of the main existence results.

Lemma 11 *Suppose that (H1), (H2), (5) hold and* $c_i \in P_T(N_{c_i}, K_{c_i})$*,* $i = \overline{1,n}$*. If*

$$J\left(TMN^2 \sum_{i=1}^{n} N_{c_i}\right) \leq N, \tag{16}$$

and

$$J\left(\mu M_1 T^2 + 2M_1 M_2 T + 2M\right) N^2 \sum_{i=1}^{n} N_{c_i} \leq K, \tag{17}$$

hold, where J *is a positive constant with* $J \geq 3$*. Then the operator* $\mathcal{A}$ *defined by (14) is continuous and compact on* $\mathbb{M}$*.*

Proof. Let $\varphi \in \mathbb{M}$. For having $\mathcal{A}\varphi \in \mathbb{M}$ we show that $\mathcal{A}\varphi \in P_T$, $\|\mathcal{A}\varphi\| \leq N$ and $|(\mathcal{A}\varphi)(t_2) - (\mathcal{A}\varphi)(t_1)| \leq K|t_2 - t_1|$, $\forall t_1, t_2 \in [0, T]$. First it is easy to show that $(\mathcal{A}\varphi)(t+T) = (\mathcal{A}\varphi)(t)$. That is, if $\varphi \in P_T$ then $\mathcal{A}\varphi$ is periodic with period T. By using Corollary 2 and (16), we obtain

$$\begin{aligned}
|(\mathcal{A}\varphi)(t)| &\le \sum_{i=1}^{n} \int_{t}^{t+T} |c_i(s)| \, |\varphi(s)| \left|\varphi^{[i]}(s)\right| |G(t,s)| \, ds \\
&\le TMN^2 \sum_{i=1}^{n} N_{c_i} \le \frac{N}{J} \le N.
\end{aligned}$$

So, we get

$$\|\mathcal{A}\varphi\| \le N.$$

Second we prove that, for any $\varphi \in \mathbb{M}$ the function $\mathcal{A}\varphi$ is K-Lipschitzian. Let $\varphi \in \mathbb{M}$ and $t_1, t_2 \in [0,T]$ with $t_1 < t_2$, we have

$$\begin{aligned}
&|(\mathcal{A}\varphi)(t_2) - (\mathcal{A}\varphi)(t_1)| \\
&\le \left| \sum_{i=1}^{n} \int_{t_2}^{t_2+T} c_i(s)\, \varphi(s)\, \varphi^{[i]}(s)\, G(t_2,s)\, ds \right. \\
&\left. - \sum_{i=1}^{n} \int_{t_1}^{t_1+T} c_i(s)\, \varphi(s)\, \varphi^{[i]}(s)\, G(t_1,s)\, ds \right| \\
&\le \sum_{i=1}^{n} \int_{t_2}^{t_1} |c_i(s)| \, |\varphi(s)| \left|\varphi^{[i]}(s)\right| |G(t_2,s)| \, ds \\
&+ \sum_{i=1}^{n} \int_{t_1+T}^{t_2+T} |c_i(s)| \, |\varphi(s)| \left|\varphi^{[i]}(s)\right| |G(t_2,s)| \, ds \\
&+ \sum_{i=1}^{n} \int_{t_1}^{t_1+T} |c_i(s)| \, |\varphi(s)| \left|\varphi^{[i]}(s)\right| |G(t_2,s) - G(t_1,s)| \, ds.
\end{aligned}$$

It follows from Corollaries 2, 2 and Lemma 5 that

$$\begin{aligned}
&|G(t_2,s) - G(t_1,s)| \\
&= \left| \int_{t_2}^{t_2+T} G_2(t_2,\sigma)\, G_1(\sigma,s)\, d\sigma - \int_{t_1}^{t_1+T} G_2(t_1,\sigma)\, G_1(\sigma,s)\, d\sigma \right| \\
&\le \int_{t_2}^{t_1} |G_2(t_2,\sigma)| \, |G_1(\sigma,s)| \, d\sigma + \int_{t_1+T}^{t_2+T} |G_2(t_2,\sigma)| \, |G_1(\sigma,s)| \, d\sigma \\
&+ \int_{t_1}^{t_1+T} |G_1(\sigma,s)| \, |G_2(t_2,\sigma) - G_2(t_1,\sigma)| \, d\sigma.
\end{aligned}$$

So
$$|G(t_2,s)-G(t_1,s)| \le (2M_1M_2+\mu TM_1)|t_2-t_1|. \tag{18}$$
Using Corollary 3 and (18), we get
$$\begin{aligned}
&|(\mathcal{A}\varphi)(t_2)-(\mathcal{A}\varphi)(t_1)| \\
&\le 2MN^2\left(\sum_{i=1}^{n} N_{c_i}\right)|t_2-t_1| + TN^2(2M_1M_2+\mu TM_1)\left(\sum_{i=1}^{n} N_{c_i}\right)|t_2-t_1| \\
&= N^2\left(2M+2TM_1M_2+\mu T^2M_1\right)\left(\sum_{i=1}^{n} N_{c_i}\right)|t_2-t_1| \\
&\le \frac{K}{J}|t_2-t_1| \le K|t_2-t_1|.
\end{aligned}$$
So, we have
$$|(\mathcal{A}\varphi)(t_2)-(\mathcal{A}\varphi)(t_1)| \le K|t_2-t_1|.$$
which shows $\mathcal{A} : \mathbb{M} \to \mathbb{M}$.

Now, For $\varphi, \psi \in \mathbb{M}$, $c_i \in P_T(N_{c_i}, K_{c_i})$, $i = \overline{1,n}$, and from Corollary 3, we obtain
$$\begin{aligned}
&|(\mathcal{A}\varphi)(t)-(\mathcal{A}\psi)(t)| \\
&\le \sum_{i=1}^{n}\int_t^{t+T} |c_i(s)|\,|G(t,s)|\left|\varphi(s)\varphi^{[i]}(s)-\psi(s)\psi^{[i]}(s)\right| ds \\
&\le M\sum_{i=1}^{n}\int_t^{t+T} |c_i(s)|\left|\varphi(s)\varphi^{[i]}(s)-\psi(s)\psi^{[i]}(s)\right| ds.
\end{aligned}$$
Using Lemma 10, we get
$$|(\mathcal{A}\varphi)(t)-(\mathcal{A}\psi)(t)| \le NMT\sum_{i=1}^{n} N_{c_i}\left(1+\sum_{j=0}^{i-1}K^j\right)\|\varphi-\psi\|.$$
This implies the continuity of $\mathcal{A}$. We use Remark 1 and the fact that continuous operators maps compact sets into compact sets we deduce that $\mathcal{A}$ is a compact operator. □

The next result proves the relationship between the mappings H and $\mathcal{B}$ in the sense of large contractions. Assume that
$$\theta MT \le 1, \tag{19}$$

$$\max\left(\left|H\left(-N\right)\right|,\left|H\left(N\right)\right|\right)\leq\frac{\left(J-1\right)}{J}N,\tag{20}$$

and

$$\left(2M+2TM_1M_2+\mu T^2M_1\right)\theta N\leq K,\tag{21}$$

where $\theta=\max\limits_{t\in[0,T]}\left|r\left(t\right)\right|$.

Lemma 12 *Let $\mathcal{B}$ be defined by (15). Suppose (H1), (H2), ((5)), (19)–(21) and all conditions of Theorem 2 hold. Then $\mathcal{B}$ is a large contraction on $\mathbb{M}$.*

Proof. Let $\mathcal{B}$ be defined by (15). Obviously, $\mathcal{B}\varphi$ is continuous and it is easy to show that $(\mathcal{B}\varphi)(t+T)=(\mathcal{B}\varphi)(t)$. For having $\mathcal{B}\varphi\in\mathbb{M}$ we will show that $\|\mathcal{B}\varphi\|\leq N$ and $\left|\left(\mathcal{B}\varphi\right)\left(t_2\right)-\left(\mathcal{B}\varphi\right)\left(t_1\right)\right|\leq K\left|t_2-t_1\right|,\ \forall t_1,t_2\in[0,T]$. Let $\varphi\in\mathbb{M}$, by (19) and (20) we get

$$\begin{aligned}\left|\left(\mathcal{B}\varphi\right)\left(t\right)\right|&\leq\int_t^{t+T}\left|G\left(t,s\right)\right|\left|r\left(s\right)\right|\left|H\left(\varphi\left(s\right)\right)\right|ds\\&\leq\theta MT\max\left\{\left|H\left(-N\right)\right|,\left|H\left(N\right)\right|\right\}\leq\frac{\left(J-1\right)N}{J}\leq N.\end{aligned}$$

Then, for any $\varphi\in\mathbb{M}$, we have

$$\|\mathcal{B}\varphi\|\leq N.$$

Let $t_1,t_2\in[0,T]$ with $t_1<t_2$, for any $\varphi\in\mathbb{M}$, we have

$$\begin{aligned}&\left|\left(\mathcal{B}\varphi\right)\left(t_2\right)-\left(\mathcal{B}\varphi\right)\left(t_1\right)\right|\\&\leq\left|\int_{t_2}^{t_2+T}G\left(t_2,s\right)r\left(s\right)H\left(\varphi\left(s\right)\right)ds-\int_{t_1}^{t_1+T}G\left(t_1,s\right)r\left(s\right)H\left(\varphi\left(s\right)\right)ds\right|\\&\leq\int_{t_2}^{t_1}\left|G\left(t_2,s\right)\right|\left|r\left(s\right)\right|\left|H\left(\varphi\left(s\right)\right)\right|ds\\&+\int_{t_1+T}^{t_2+T}\left|G\left(t_2,s\right)\right|\left|r\left(s\right)\right|\left|H\left(\varphi\left(s\right)\right)\right|ds\\&+\int_{t_1}^{t_1+T}\left|G\left(t_2,s\right)-G\left(t_1,s\right)\right|\left|r\left(s\right)\right|\left|H\left(\varphi\left(s\right)\right)\right|ds.\end{aligned}$$

Using Corollary 3 and (18), we have

$$\left|\left(\mathcal{B}\varphi\right)\left(t_2\right)-\left(\mathcal{B}\varphi\right)\left(t_1\right)\right|$$

$$\leq 2M\theta\frac{(J-1)N}{J}|t_2-t_1|+\theta\frac{(J-1)N}{J}T(2M_1M_2+\mu TM_1)|t_2-t_1|$$
$$=\left(2M+2TM_1M_2+\mu T^2M_1\right)\theta\frac{(J-1)N}{J}|t_2-t_1|.$$

From (21), we obtain

$$\begin{aligned}|(\mathcal{B}\varphi)(t_2)-(\mathcal{B}\varphi)(t_1)| &\leq \frac{(J-1)K}{J}|t_2-t_1| \\ &\leq K|t_2-t_1|.\end{aligned}$$

Consequently, we have $\mathcal{B}:\mathbb{M}\to\mathbb{M}$.

It remains to show that $\mathcal{B}$ is a large contraction. By Theorem 2 H is large contraction on $\mathbb{M}$, then for any $\varphi,\psi\in\mathbb{M}$, with $\varphi\neq\psi$ we have

$$\begin{aligned}&|(\mathcal{B}\varphi)(t)-(\mathcal{B}\psi)(t)| \\ &\leq \left|\int_t^{t+T}G(t,s)r(s)[H(\varphi(s))-H(\psi(s))]ds\right| \\ &\leq \theta MT\|\varphi-\psi\|\leq\|\varphi-\psi\|.\end{aligned}$$

Then $\|\mathcal{B}\varphi-\mathcal{B}\psi\|\leq\|\varphi-\psi\|$. Now, let $\varepsilon\in(0,1)$ be given and let $\varphi,\psi\in\mathbb{M}$, with $\|\varphi-\psi\|\geq\varepsilon$, from the proof of Theorem 2, we have found a $\delta\in(0,1)$, such that

$$|(H\varphi)(t)-(H\psi)(t)|\leq\delta\|\varphi-\psi\|.$$

Thus,

$$\begin{aligned}&|(\mathcal{B}\varphi)(t)-(\mathcal{B}\psi)(t)| \\ &\leq \left|\int_t^{t+T}G(t,s)r(s)[H(\varphi(s))-H(\psi(s))]ds\right| \\ &\leq \theta MT\delta\|\varphi-\psi\|\leq\delta\|\varphi-\psi\|.\end{aligned}$$

The proof is complete. □

Theorem 3 *Suppose the hypotheses of Lemmas 11, 12 hold. Let $\mathbb{M}$ defined by (13). Then (1) has a T-periodic solution in $\mathbb{M}$.*

Proof. By Lemmas 11, $\mathcal{A}:\mathbb{M}\to\mathbb{M}$ is continuous and $\mathcal{A}(\mathbb{M})$ is contained in a compact set. Also, from Lemma 12, the mapping $\mathcal{B}:\mathbb{M}\to\mathbb{M}$ is a large contraction. Next, we show that if $\varphi,\psi\in\mathbb{M}$, we have $\|\mathcal{A}\varphi+\mathcal{B}\psi\|\leq N$ and

$|(\mathcal{A}\varphi+\mathcal{B}\psi)(t_2)-(\mathcal{A}\varphi+\mathcal{B}\psi)(t_1)| \leq K|t_2-t_1|,\ \forall t_1,t_2 \in [0,T]$. Let $\varphi,\psi \in \mathbb{M}$. By (16), (19) and (20), we obtain

$$\|\mathcal{A}\varphi+\mathcal{B}\psi\| \leq TMN^2\sum_{i=1}^{n} N_{c_i} + \frac{(J-1)N}{J} \leq \frac{N}{J} + \frac{(J-1)N}{J} = N.$$

Now, let $\varphi,\psi \in \mathbb{M}$ and $t_1,t_2 \in [0,T]$. By (17), (21), we get

$$\begin{aligned}
&|(\mathcal{A}\varphi+\mathcal{B}\psi)(t_2)-(\mathcal{A}\varphi+\mathcal{B}\psi)(t_1)| \\
&\leq |(\mathcal{A}\varphi)(t_2)-(\mathcal{A}\varphi)(t_1)| + |(\mathcal{B}\psi)(t_2)-(\mathcal{B}\psi)(t_1)| \\
&\leq \frac{K}{J}|t_2-t_1| + \frac{(J-1)K}{J}|t_2-t_1| \\
&\leq K|t_2-t_1|.
\end{aligned}$$

Clearly, all the hypotheses of Krasnoselskii-Burton's theorem are satisfied. Thus there exists a fixed point $z \in \mathbb{M}$ such that $z = \mathcal{A}z + \mathcal{B}z$. By Lemma 9 this fixed point is a solution of (1). Hence (1) has a T-periodic solution. $\square$

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