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# Existence results of infinitely many weak solutions of a singular subelliptic system on the Heisenberg group

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**Abstract.** This article shows the existence and multiplicity of weak solutions for the singular subelliptic system on the Heisenberg group

$$\begin{cases} -\Delta_{\mathbb{H}^n} u + a(\xi)\frac{u}{(|z|^4+t^2)^{\frac{1}{2}}} = \lambda F_u(\xi, u, v) & \text{in } \Omega, \\ -\Delta_{\mathbb{H}^n} v + b(\xi)\frac{v}{(|z|^4+t^2)^{\frac{1}{2}}} = \lambda F_v(\xi, u, v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases}$$

The approach is based on variational methods.

## 1 Introduction

The aim of this article is to establish the existence and multiplicity of weak solutions for the singular subelliptic system

$$\begin{cases} -\Delta_{\mathbb{H}^n} u + a(\xi)\frac{u}{(|z|^4+t^2)^{\frac{1}{2}}} = \lambda F_u(\xi, u, v) & \text{in } \Omega, \\ -\Delta_{\mathbb{H}^n} v + b(\xi)\frac{v}{(|z|^4+t^2)^{\frac{1}{2}}} = \lambda F_v(\xi, u, v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

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where $\Omega \subset \mathbb{H}^n, n \geq 1$, is an open, bounded subset containing the origin with smooth boundary. $\lambda > 0$, $a, b \in L^\infty(\Omega)$ such that $\mathrm{ess}_\Omega \inf a > 0$ and $\mathrm{ess}_\Omega \inf b > 0$, $F : \bar{\Omega} \times \mathbb{R}^2 \to \mathbb{R}$ is a function such that $F(\cdot, s_1, s_2)$ is continuous in $\bar{\Omega}$, for all $(s_1, s_2) \in \mathbb{R}^2$ and $F(\xi, \cdot, \cdot)$ is $C^1$ in $\mathbb{R}^2$ for every $\xi \in \Omega$, and $F_u, F_v$ denote the partial derivatives of $F$, with respect to $u, v$ respectively.

Singular elliptic problems have been intensively studied in the last decades, in [6, 16] the authors investigated infinitely many solutions for singular elliptic problems.In [1] and [5] some problems which depend on continuous component of time like coherent states in quantum optics are probed. These problems are studied in a space which have a component of time and are known as Heisenberg group. Important topics where the Heisenberg group reveals itself as an essential factor are quantum mechanics, ergodic theory, representation theory of nilpotent Lie group, foundation of abelian harmonic analysis, and the theory of partial differential equations. We are now interested in the last one.

Recently the existence of radial solutions of Neumann problem on Heisenberg group is studied (see [10, 11, 12, 13]).

Here we recall some definitions and results on Heisenberg group (see [2, 8, 9, 15]). The Heisenberg group $\mathbb{H}^n = (\mathbb{R}^{2n+1}, \circ\ )$ is the space $\mathbb{R}^{2n+1}$ with the noncommutative law of product

$$(x, y, t) \circ (x', y', t') = (x + x', y + y', t + t' + 2(\langle y, x'\rangle - \langle x, y'\rangle)),$$

where $x, x', y, y' \in \mathbb{R}^n,\ t, t' \in \mathbb{R}$ and $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^n$. This operation endows $\mathbb{H}^n$ with the structure of a Lie group. The Lie algebra of $\mathbb{H}^n$ is generated by the left-invariant vector fields

$$T = \frac{\partial}{\partial t},\quad X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t},\quad Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t},\quad i = 1, 2, 3, \cdots, n.$$

These generators satisfy the noncommutative formula

$$[X_i, Y_j] = -4\delta_{ij} T,\quad [X_i, X_j] = [Y_i, Y_j] = [X_i, T] = [Y_i, T] = 0.$$

Let $z = (x, y) \in \mathbb{R}^{2n}$ and $\xi = (z, t) \in \mathbb{H}^n$. The parabolic dilation $\delta_\lambda \xi = (\lambda x, \lambda y, \lambda^2 t)$, satisfies $\delta_\lambda(\xi_0 \circ \xi) = \delta_\lambda \xi \circ \delta_\lambda \xi_0$, and $|\xi|_{\mathbb{H}^n} = (|z|^4 + t^2)^{\frac{1}{4}} = ((x^2 + y^2)^2 + t^2)^{\frac{1}{4}}$, is a norm with respect to the parabolic dilation which is known as Korányi gauge norm $N(z, t)$. The Heisenberg distance between two points $(z, t)$ and $(z', t')$ is given by $\rho(z, t; z', t') = |(z', t')^{-1} \circ (z, t)|_{\mathbb{H}^n}$. Clearly, the vector fields $X_i, Y_i, i = 1, 2, \cdots, n$, are homogeneous of degree 1 under the norm $|.|_{\mathbb{H}^n}$ and T is homogenous of degree 2. The Korányi ball of center $\xi_0$ and radius $r$ is defined by

$$B_{\mathbb{H}^n}(\xi_0, r) = \{\xi : |\xi^{-1} \circ \xi_0|_{\mathbb{H}^n} \leq r\},$$

and it satisfies $|B_{\mathbb{H}^n}(\xi_0, r)| = |B_{\mathbb{H}^n}(0, r)| = r^{2n+2}|B_{\mathbb{H}^n}(0,1)|$. The Heisenberg gradient and the Kohn-Laplacian (the Heisenberg Laplacian) operator on $\mathbb{H}^n$ are given by

$$\nabla_{\mathbb{H}^n} = (X_1, X_2, \cdots, X_n, Y_1, Y_2, \cdots, Y_n) \text{ and } \Delta_{\mathbb{H}^n} = \sum_{i=1}^{n} X_i^2 + Y_i^2,$$

respectively. We define the associated Sobolev space as following:

$$H^1(\Omega, \mathbb{H}^n) := \{u \in L^2(\Omega) : \ X_i u, Y_i u \in L^2(\Omega), i = 1, 2, \cdots, n\},$$

and $H_0^1(\Omega, \mathbb{H}^n)$ is the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega, \mathbb{H}^n)$ with respect to the norm

$$\|u\|_{H^1(\Omega,\mathbb{H}^n)} = \left(\int_\Omega (|\nabla_{\mathbb{H}^n} u|^2 + |u|^2) d\xi\right)^{\frac{1}{2}},$$

where $u : \Omega \subset \mathbb{H}^n \to \mathbb{R}$. A norm on $H_0^1(\Omega, \mathbb{H}^n)$ is

$$\|u\|_{H_0^1(\Omega,\mathbb{H}^n)} = \left(\int_\Omega |\nabla_{\mathbb{H}^n} u|^2 d\xi\right)^{\frac{1}{2}},$$

which is equivalent to the standard one. The dual space of $H_0^1(\Omega, \mathbb{H})$ is denoted by $H^{-1}(\Omega, \mathbb{H})$. Here we recall Hardy's inequality and some results on the Heisenberg group.

**Lemma 1** [7] *For* $n \geq 1$ *and for any* $u \in H_0^1(\Omega, \mathbb{H}^n)$, *we have*

$$\int_\Omega \frac{|u|^2}{(|z|^4 + |t|^2)^{\frac{1}{2}}} \, d\xi \leq \left(\frac{n+1}{n^2}\right)^2 \int_\Omega |\nabla_{\mathbb{H}^n} u|^2 d\xi.$$

For convenience, we write the above inequality by

$$\int_\Omega \frac{|u|^2}{(|z|^4 + |t|^2)^{\frac{1}{2}}} \, d\xi \leq \frac{1}{C_n} \int_\Omega |\nabla_{\mathbb{H}^n} u|^2 d\xi,$$

where $C_n = (\frac{n^2}{n+1})^2$.

**Lemma 2** [14] *Let $\Omega \subset \mathbb{H}^n$ be a bounded open set. Then the compact embedding $H_0^1(\Omega, \mathbb{H}^n) \subset\subset L^p(\Omega)$ for $1 \leq p < Q^*$ is satisfied, where $Q^* = \frac{2Q}{Q-2}$ is critical exponent of $Q = 2n + 2$, which is homogeneous dimension of $\mathbb{H}^n$.*

We denote the Sobolev embedding constant of the above compact embedding by $S_p > 0$, i.e.

$$\|u\|_{L^p(\Omega)} \leq S_p \|u\|_{H_0^1(\Omega,\mathbb{H}^n)} \quad \text{for all } u \in H_0^1(\Omega, \mathbb{H}^n),\ 1 \leq p < Q^*. \tag{2}$$

In the sequel, $X$ will denote the space $H_0^1(\Omega, \mathbb{H}^n) \times H_0^1(\Omega, \mathbb{H}^n)$, which is a reflexive Banach space endowed with the norm

$$\|(u, v)\| = \|u\|_{H_0^1(\Omega,\mathbb{H}^n)} + \|v\|_{H_0^1(\Omega,\mathbb{H}^n)}.$$

We use the following multiple critical points theorem due to G. Bonanno [3].

**Theorem 1** *Let $X$ be a reflexive real Banach space, let $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that $\Phi$ is strong continuous, sequentially weakly lower semi-continuous, coercive and $\Psi$ is sequentially weakly upper-semi-continuous. For every $r > \inf_X \Phi$, let*

$$\varphi(r) := \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{(\sup_{v\in\Phi^{-1}(-\infty,r)} \Psi(v)) - \Psi(u)}{r - \Phi(u)},$$

$$\gamma := \liminf_{r\to+\infty} \varphi(r), \quad \delta := \liminf_{r\to(\inf_X \Phi)^+} \varphi(r).$$

*Then*

(a) *If $\gamma < +\infty$ then, for each $\lambda \in \left(0, \frac{1}{\gamma}\right)$, the following alternative holds: either*

(a1) $I_\lambda := \Phi - \lambda\Psi$ *possesses a global minimum, or*

(a2) *there is a sequence $\{u_n\}$ of critical points (local minima) of $I_\lambda$ such that $\lim_{n\to\infty} \Phi(u_n) = +\infty$.*

(b) *If $\delta < +\infty$ then, for each $\lambda \in \left(0, \frac{1}{\delta}\right)$, the following alternative holds: either*

(b1) *there is a global minimum of $\Phi$ that is a local minimum of $I_\lambda$, or*

(b2) *there is a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of $I_\lambda$ that weakly converges to a global minimum of $\Phi$ .*

## 2 Weak solutions

In this section we prove the existence of weak solutions for the system (1). Du to do this, we the definition of the weak solution.

**Definition 1** *One says that* $(u, v) \in X$ *is a weak solution to the system* (1) *if*

$$\int_\Omega \nabla_{\mathbb{H}^n} u \cdot \nabla_{\mathbb{H}^n} \varphi d\xi + \int_\Omega \frac{u\varphi}{(|z|^4 + t^2)^{\frac{1}{2}}} a(\xi) d\xi - \lambda \int_\Omega F_u(\xi, u, v)\varphi d\xi$$
$$+ \int_\Omega \nabla_{\mathbb{H}^n} v \cdot \nabla_{\mathbb{H}^n} \psi d\xi + \int_\Omega \frac{v\psi}{(|z|^4 + t^2)^{\frac{1}{2}}} b(\xi) d\xi - \lambda \int_\Omega F_v(\xi, u, v)\psi d\xi = 0,$$

*for every* $(\varphi, \psi) \in X$.

Define the functional $I_\lambda : X \to \mathbb{R}$ by

$$I_\lambda(u, v) = \Phi(u, v) - \lambda\Psi(u, v),$$

for all $(u, v) \in X$, where

$$\Phi(u, v) = \frac{1}{2}\|u\|^2_{H^1_0(\Omega,\mathbb{H}^n)} + \frac{1}{2}\int_\Omega \frac{|u|^2}{(|z|^4 + t^2)^{\frac{1}{2}}} a(\xi) d\xi$$
$$+ \frac{1}{2}\|v\|^2_{H^1_0(\Omega,\mathbb{H}^n)} + \frac{1}{2}\int_\Omega \frac{|v|^2}{(|z|^4 + t^2)^{\frac{1}{2}}} b(\xi) d\xi,$$

and $\Psi(u, v) = \int_\Omega F(\xi, u, v) d\xi$.
Lemma 1 implies

$$\tfrac{1}{2}\|u\|^2_{H^1_0(\Omega,\mathbb{H}^n)} + \tfrac{1}{2}\|v\|^2_{H^1_0(\Omega,\mathbb{H}^n)} < \Phi(u, v)$$
$$< (\tfrac{C_n + \|a\|_\infty}{2C_n})\|u\|^2_{H^1_0(\Omega,\mathbb{H}^n)} + (\tfrac{C_n + \|b\|_\infty}{2C_n})\|v\|^2_{H^1_0(\Omega,\mathbb{H}^n)},$$

which implies that $\Phi$ is coercive. Moreover, from the weakly lower semicontinuity of norm, we known that $\Phi$ is sequentially weakly lower semicontinuous. Notice that functionals $\Phi, \Psi$ are well defined and continuously Gâteaux differentiable functionals whose derivatives at the point $(u, v) \in X$ are the functionals $\Phi'(u, v)$ and $\Psi'(u, v)$ given by

$$\langle \Phi'(u, v), (\varphi, \psi)\rangle = \int_\Omega \nabla_{\mathbb{H}^n} u \cdot \nabla_{\mathbb{H}^n} \varphi d\xi + \int_\Omega \frac{u\varphi}{(|z|^4 + t^2)^{\frac{1}{2}}} a(\xi) d\xi$$
$$+ \int_\Omega \nabla_{\mathbb{H}^n} v \cdot \nabla_{\mathbb{H}^n} \psi d\xi + \int_\Omega \frac{v\psi}{(|z|^4 + t^2)^{\frac{1}{2}}} b(\xi) d\xi,$$

$$\langle \Psi'(u, v), (\varphi, \psi)\rangle = \int_\Omega F_u(\xi, u, v)\varphi d\xi + \int_\Omega F_v(\xi, u, v)\psi d\xi,$$

for all $(\varphi, \psi) \in X$.
$\Psi' : X \to X^*$ is a compact operator, indeed, it is enough to show that $\Psi'$ is strongly continuous on $X$. For this end, for fixed $(u, v) \in X$, let $(u_k, v_k) \to (u, v)$ weakly in $X$ as $k \to \infty$, by the Lemma 2, we deduce that $(u_k, v_k) \to (u, v)$ in $L^p(\Omega) \times L^p(\Omega)$, therefore, $(u_k(\xi), v_k(\xi)) \to (u(\xi), v(\xi))$ for a.e $\xi \in \Omega$. Since $F(\xi, \cdot, \cdot)$ is $C^1$ in $\mathbb{R}^n$ for every $\xi \in \Omega$, so $\Psi'(\xi, u_k, v_k) \to \Psi'(\xi, u, v)$ strongly as $k \to +\infty$. Thus we have that $\Psi'$ is strongly continuous on $X$, which implies that $\Psi'$ is a compact operator by Proposition 26.2, [17], it follows that $\Psi$ is sequentially weakly continuous.

Here is an example to show a function $F$ with the conditions defined in (1) can exists.

**Example 1** *It could be possible to consider the same example given in* [4]*. Let $\Omega$ be a bounded domain in $\mathbb{R}^2$ containing the origin and with smooth boundary $\partial\Omega$. Consider the increasing sequence of positive real numbers given by*

$$a_1 := 2, \qquad a_{n+1} := n!a_n^2 + 2,$$

*for every $n \geq 1$. Define the $C^1$-function $F : \mathbb{R}^2 \to \mathbb{R}$ as follows*

$$F(s_1, s_2) = \begin{cases} (a_{n+1})^4 e^{1-\frac{1}{1-[\|s_1-a_{n+1}\|p+\|s_2-a_{n+1}\|p]}} & (s_1, s_2) \in \cup_{n\geq 1} B\big((a_{n+1}, a_{n+1}), 1\big), \\ 0 & \textit{otherwise}, \end{cases}$$

*where $B\big((a_{n+1}, a_{n+1}), 1\big)$ is an open unit ball of center $(a_{n+1}, a_{n+1})$.*

Due to study the existence of infinitely many weak solutions, suppose there exist $R_0 > 0$ such that $R_0 < \operatorname{dist}(0, \partial\Omega)$ and $\zeta \in (0, 1)$. Set

$$\begin{aligned} L_a &:= \frac{R_0^2(1-\zeta)^2}{8S_p(\frac{C_n+\|a\|_\infty}{2C_n})(1-\frac{t^2}{r^2})(1+4t^2)\omega_{2n+1}R_0^{2n+1}(1-\zeta^{2n+1})} \\ L_b &:= \frac{R_0^2(1-\zeta)^2}{8S_p(\frac{C_n+\|b\|_\infty}{2C_n})(1-\frac{t^2}{r^2})(1+4t^2)\omega_{2n+1}R_0^{2n+1}(1-\zeta^{2n+1})}, \end{aligned} \tag{3}$$

where $w_n$ denotes the volume of the $n$-dimensional unit ball in $\mathbb{R}^n$ and $S_p$ is given by (2).

**Theorem 2** *Assume that*
(i$_1$) $F(\xi, s_1, s_2) \geq 0$ *for every* $(\xi, s_1, s_2) \in \Omega \times \mathbb{R}^2_+$,

(i$_2$) *There exist* $R_0 > 0$ *such that* $R_0 < \mathtt{dist}(0, \partial\Omega)$ *and* $\zeta \in (0,1)$, *given by* (3). *Assume* $A < LB$, *where*

$$
\begin{aligned}
A &:= \liminf_{\sigma \to +\infty} \frac{\int_\Omega \sup_{\|s_1\|_{L^p(\Omega)} + \|s_2\|_{L^p(\Omega)} \le \sigma} F(\xi, s_1, s_2) d\xi}{\sigma^2}, \\
B &:= \limsup_{s_1, s_2 \to +\infty} \frac{\int_{B_{2n+1}(0, \zeta R_0)} F(\xi, s_1, s_2) d\xi}{s_1^2 + s_2^2}, \\
L &:= \min\{L_a, L_b\}.
\end{aligned}
$$

*Then for each* $\lambda \in \Lambda := \frac{1}{8S_p}\left(\frac{1}{LB}, \frac{1}{A}\right)$, *problem* (1) *has an unbounded sequence of weak solutions in* $X$.

**Proof.** We apply the part (a) of Theorem 1. Certainly, the weak solutions of the problem (1) are exactly the solutions of the equation $I'_\lambda(u,v) = 0$. The functional $\Phi$ and $\Psi$ satisfy the assumptions of Theorem 1. Now we show that $\gamma < +\infty$. Since $X$ compactly embedded in $L^p(\Omega) \times L^p(\Omega)$, and from (2) one has

$$
\|u\|_{L^p(\Omega)} \le S_p \|u\|_{H_0^1(\Omega, \mathbb{H}^n)} \quad \text{and} \quad \|v\|_{L^p(\Omega)} \le S_p \|v\|_{H_0^1(\Omega, \mathbb{H}^n)},
$$

for all $(u,v) \in X$. Thus

$$
\frac{1}{2}\|u\|^2_{L^p(\Omega)} + \frac{1}{2}\|v\|^2_{L^p(\Omega)} < S_p\left(\frac{1}{2}\|u\|^2_{H_0^1(\Omega,\mathbb{H}^n)} + \frac{1}{2}\|v\|^2_{H_0^1(\Omega,\mathbb{H}^n)}\right).
$$

So, for each $r > 0$

$$
\begin{aligned}
\Phi^{-1}(]-\infty, r[) :&= \{(u,v) \in X : \Phi(u,v) < r\} \\
&= \{(u,v) \in X : \frac{1}{2}\|u\|^2_{H_0^1(\Omega,\mathbb{H}^n)} + \frac{1}{2}\|v\|^2_{H_0^1(\Omega,\mathbb{H}^n)} < r\} \\
&\subseteq \{(u,v) \in X : \frac{1}{2}\|u\|^2_{L^p(\Omega)} + \frac{1}{2}\|v\|^2_{L^p(\Omega)} < S_p r\},
\end{aligned} \tag{4}
$$

and it follows that

$$
\sup_{(u,v)\in\Phi^{-1}(]-\infty,r[)} \Psi(u,v) < \sup_{\{(u,v)\in X: \frac{1}{2}\|u\|_{L^p(\Omega)} + \frac{1}{2}\|v\|_{L^p(\Omega)} < S_p r\}} \int_\Omega F(\xi, u, v) d\xi.
$$

Note that $\Phi(0,0)=0$ and $\Psi(0,0)\geq 0$. Therefore, for every $r>0$,

$$\begin{aligned}\varphi(r):&=\inf_{\Phi(u,v)<r}\frac{\left(\sup_{(u',v')\in\Phi^{-1}(]-\infty,r[)}\Psi(u',v')\right)-\Psi(u,v)}{r-\Phi(u,v)}\\&\leq\frac{\sup_{\Phi^{-1}(]-\infty,r[)}\Psi}{r}\\&\leq\frac{1}{r}\sup_{\frac{1}{2}\|s_1\|_{L^p(\Omega)}+\frac{1}{2}\|s_2\|_{L^p(\Omega)}<S_p r}\int_\Omega F(\xi,s_1,s_2)d\xi.\end{aligned}$$

Let $\{\sigma_k\}$ be a real sequence of positive numbers such that $\lim_{k\to+\infty}\sigma_k=+\infty$ and

$$\lim_{k\to+\infty}\frac{\int_\Omega\sup_{\|s_1\|_{L^p(\Omega)}+\|s_2\|_{L^p(\Omega)}\leq\sigma_k}F(\xi,s_1,s_2)d\xi}{\sigma_k^2}=A<+\infty. \tag{5}$$

Set $r_k=\frac{1}{2S_p}(\frac{\sigma_k}{2})^2$, from (4), one has

$$\frac{1}{2}\|u\|^2_{L^p(\Omega)}+\frac{1}{2}\|v\|^2_{L^p(\Omega)}<S_p r_k,\ \text{ for all }\xi\in\Omega.$$

So,

$$\|u\|_{L^p(\Omega)}\leq\sqrt{2S_p r_k}\quad\text{and}\quad\|v\|_{L^p(\Omega)}\leq\sqrt{2S_p r_k},$$

thus, for each $k\in\mathbb{N}$ large enough

$$\|u\|_{L^p(\Omega)}+\|v\|_{L^p(\Omega)}\leq\sqrt{2S_p r_k}+\sqrt{2S_p r_k}=2\sqrt{2S_p r_k}=\sigma_k.$$

Hence,

$$\begin{aligned}\varphi(r_k)&\leq\frac{\sup_{\{(u,v)\in X:\|u\|_{L^p(\Omega)}+\|v\|_{L^p(\Omega)}<\sigma_k\}}\int_\Omega F(\xi,u,v)d\xi}{\frac{1}{2S_p}(\frac{\sigma_k}{2})^2}\\&\leq 8S_p\frac{\int_\Omega\sup_{\|s_1\|_{L^p(\Omega)}+\|s_2\|_{L^p(\Omega)}<\sigma_k}F(\xi,s_1,s_2)d\xi}{\sigma_k^2}.\end{aligned}\tag{6}$$

Hence, from (5) and (6), one has

$$\begin{aligned}\gamma&\leq\liminf_{k\to+\infty}\varphi(r_k)\leq 8S_p\lim_{k\to+\infty}\frac{\int_\Omega\sup_{\|s_1\|_{L^p(\Omega)}+\|s_2\|_{L^p(\Omega)}<\sigma_k}F(\xi,s_1,s_2)d\xi}{\sigma_k^2}\\&=8S_pA<+\infty.\end{aligned}$$

This implies

$$\gamma\leq 8S_pA<\frac{1}{\lambda}.$$

We conclude that $\Lambda \subseteq ]0, \frac{1}{\gamma}[$. For $\lambda \in \Lambda$, we show that the functional $I_\lambda = \Phi - \lambda\Psi$ is unbounded from below. Indeed, since $\frac{1}{\lambda} < 8S_pLB$, we can consider two positive real sequences $\{\eta_{i,k}\}_{i=1}^2$ and $\theta > 0$ such that $\sqrt{\sum_{i=1}^2 \eta_{i,k}} \to +\infty$ as $k \to \infty$ and

$$\frac{1}{\lambda} < \theta < 8S_pL\frac{\int_{B_{2n+1}(0,\zeta R_0)} F(\xi, \eta_{1,k}, \eta_{2,k})d\xi}{\eta_{1,k}^2 + \eta_{2,k}^2}, \tag{7}$$

for $k$ large enough. Suppose $u_k(\xi) = (u_{1k}(\xi), u_{2k}(\xi))$ be a sequence in $X$ defined by

$$u_{ik}(\xi) = \begin{cases} 0 & \text{if } \xi \in \mathbb{H}^n \setminus B_{2n+1}(0, R_0) \\ \eta_{i,k} & \text{if } \xi \in B_{2n+1}(0, \zeta R_0) \\ \dfrac{\eta_{i,k}}{R_0(1-\zeta)}(R_0 - |\xi|) & \text{if } \xi \in B_{2n+1}(0, R_0) \setminus B_{2n+1}(0, \zeta R_0), \end{cases} \tag{8}$$

for $i = 1, 2$, where $B_n(0, r)$ denotes the $n$-dimensional open ball with center $0$ and radius $r > 0$.
Bearing (3) in mind, we have

$$\begin{aligned}
\Phi(u_{1k}, u_{2k}) &= \frac{1}{2}\int_\Omega |\nabla_{\mathbb{H}^n} u_{1k}(\xi)|^2 d\xi + \frac{\|a\|_\infty}{2}\int_\Omega \frac{|u_{1k}(\xi)|^2}{(|z|^4+t^2)^{\frac{1}{2}}}d\xi \\
&+ \frac{1}{2}\int_\Omega |\nabla_{\mathbb{H}^n} u_{2k}(\xi)|^2 d\xi + \frac{\|b\|_\infty}{2}\int_\Omega \frac{|u_{2k}(\xi)|^2}{(|z|^4+t^2)^{\frac{1}{2}}}d\xi \\
&\leq \left(\frac{C_n + \|a\|_\infty}{2C_n}\right)\int_{B_{2n+1}(0,R_0)\setminus B_{2n+1}(0,\zeta R_0)} |\nabla_{\mathbb{H}^n} u_{1k}(\xi)|^2 d\xi \\
&+ \left(\frac{C_n + \|b\|_\infty}{2C_n}\right)\int_{B_{2n+1}(0,R_0)\setminus B_{2n+1}(0,\zeta R_0)} |\nabla_{\mathbb{H}^n} u_{2k}(\xi)|^2 d\xi \\
&= \left(\frac{C_n + \|a\|_\infty}{2C_n}\right)\frac{\eta_{1,k}^2}{R_0^2(1-\zeta)^2}\left(1 - \frac{t^2}{r^2}\right)(1+4t^2)\omega_{2n+1}R_0^{2n+1}(1-\zeta^{2n+1}) \\
&+ \left(\frac{C_n + \|b\|_\infty}{2C_n}\right)\frac{\eta_{2,k}^2}{R_0^2(1-\zeta)^2}\left(1 - \frac{t^2}{r^2}\right)(1+4t^2)\omega_{2n+1}R_0^{2n+1}(1-\zeta^{2n+1}) \\
&= \frac{1}{8S_p}\left(\frac{\eta_{1,k}^2}{L_a} + \frac{\eta_{2,k}^2}{L_b}\right).
\end{aligned} \tag{9}$$

On the other hand, by assumption that $F(\xi, s_1, s_2) \geq 0$, we have

$$\Psi(u_{1k}, u_{2k}) = \int_{\Omega} F(\xi, u_{1k}, u_{2k}) d\xi \geq \int_{B_{2n+1}(0,\zeta R_0)} F(\xi, \eta_{1,k}, \eta_{2,k}) d\xi. \tag{10}$$

So, it follows from (7), (9) and (10) that

$$\begin{aligned}
I_\lambda(u_{1k}, u_{2k}) &= \Phi(u_{1k}, u_{2k}) - \lambda\Psi(u_{1k}, u_{2k}) \\
&\leq \frac{1}{8S_p L}(\eta_{1,k}^2 + \eta_{2,k}^2) - \lambda \int_{B_{2n+1}(0,\zeta R_0)} F(\xi, \eta_{1,k}, \eta_{2,k}) d\xi \\
&\leq \frac{(1 - \lambda\theta)}{8S_p L}(\eta_{1,k}^2 + \eta_{2,k}^2),
\end{aligned}$$

for $k$ large enough, so $\lim_{k\to+\infty} I_\lambda(u_{1k}, u_{2k}) = -\infty$, and hence, the claim follows.

The alternative of Theorem 1 case (a) assures the existence of unbounded sequence $\{u_k = (u_{1k}, u_{2k})\} \subset X$ of critical points of the functional $I_\lambda$ and the proof of Theorem 2 is complete. $\square$

**Theorem 3** *Assume that* $(i_1)$ *holds and*

$(i_3)$ $F(\xi, 0, 0) = 0$ *for every* $\xi \in \Omega$.

$(i_4)$ *There exist* $R_0 > 0$ *such that* $R_0 < \operatorname{dist}(0, \partial\Omega)$ *and* $\zeta \in (0, 1)$, *given by* (3). *Assume* $A' < LB'$ *where*

$$\begin{aligned}
A' &:= \liminf_{\sigma\to 0^+} \frac{\int_\Omega \sup_{\|s_1\|_{L^p(\Omega)}+\|s_2\|_{L^p(\Omega)}\leq\sigma} F(\xi, s_1, s_2) d\xi}{\sigma^2}, \\
B' &:= \limsup_{s_1,s_2\to 0^+} \frac{\int_{B_{2n+1}(0,\zeta R_0)} F(\xi, s_1, s_2) d\xi}{s_1^2 + s_2^2}, \\
L &:= \min\{L_a, L_b\}.
\end{aligned}$$

*Then for each* $\lambda \in \Lambda := \frac{1}{8S_p}\left(\frac{1}{LB'}, \frac{1}{A'}\right)$, *problem* (1) *admits a sequence of weak solutions which converges to* $0$.

**Proof.** Note that $\Phi(0, 0) = 0$ and $\Psi(0, 0) = 0$. Therefore, for every $r > 0$, Let $\{\sigma_k\}$ be a real sequence of positive numbers such that $\sigma_k \to 0^+$ as $k \to +\infty$ and

$$\lim_{k\to+\infty} \frac{\int_\Omega \sup_{\|s_1\|_{L^p(\Omega)}+\|s_2\|_{L^p(\Omega)}\leq\sigma_k} F(\xi, s_1, s_2) d\xi}{\sigma_k^2} = A' < +\infty. \tag{11}$$

Put $r_k = \frac{1}{2S_p}(\frac{\sigma_k}{2})^2$ for all $k \in \mathbb{N}$, and $\delta := \liminf_{r\to 0^+} \varphi(r)$. Hence, from (6) and (11), one has

$$\begin{aligned}\delta &\leq \liminf_{k\to+\infty} \varphi(r_k)\\ &\leq 8S_p \lim_{k\to+\infty} \frac{\int_\Omega \sup_{\|s_1\|_{L^p(\Omega)}+\|s_2\|_{L^p(\Omega)}<\sigma_k} F(\xi, s_1, s_2)d\xi}{\sigma_k^2}\\ &= 8S_p A' < +\infty.\end{aligned}$$

We conclude that $\Lambda \subseteq ]0, \frac{1}{\delta}[$. For $\lambda \in \Lambda$, we show that the functional $I_\lambda$ is unbounded from below. Indeed, since

$$\frac{1}{\lambda} < 8S_p LB',$$

we can consider two positive real sequences $\{\eta_{i,k}\}_{i=1}^2$ and $\theta > 0$ such that $\sqrt{\sum_{i=1}^2 \eta_{i,k}} \to 0$ as $k \to \infty$ and

$$\frac{1}{\lambda} < \theta < 8S_p L \frac{\int_{B_{2n+1}(0,\zeta R_0)} F(\xi, \eta_{1,k}, \eta_{2,k})d\xi}{\eta_{1,k}^2 + \eta_{2,k}^2}, \tag{12}$$

for $k$ large enough. Let $(u_k)$ be the sequence defined in (8). By combining (9), (10) and (12), we obtain

$$\begin{aligned}I_\lambda(u_{1k}, u_{2k}) &= \Phi(u_{1k}, u_{2k}) - \lambda\Psi(u_{1k}, u_{2k})\\ &\leq \frac{1}{8S_p L}(\eta_{1,k}^2 + \eta_{2,k}^2) - \lambda \int_{B_{2n+1}(0,\zeta R_0)} F(\xi, \eta_{1,k}, \eta_{2,k})d\xi\\ &\leq \frac{(1-\lambda\theta)}{8S_p L}(\eta_{1,k}^2 + \eta_{2,k}^2),\end{aligned}$$

for $k$ large enough, so $\lim_{k\to+\infty} I_\lambda(u_{1k}, u_{2k}) = -\infty$, and hence, the claim follows.

The alternative of Theorem 1 case (b) assures the existence of sequence $(u_k)$ of pairwise distinct critical points of $I_\lambda$ which weakly converges to $0$. This completes the proof of Theorem 3. □

**Remark 1** *We observe that, if* $F_{t_i}(\xi, 0, 0) \neq 0$*, then, by Theorem 2 and 3 we obtain the existence of infinitely many non-trivial weak solutions.*

**Corollary 1** *Let* $\Omega$ *be a bounded open subset of* $\mathbb{H}^n$*,* $f : \Omega \times \mathbb{R} \to \mathbb{R}$ *be a Carathéodory function. Suppopse* $F : \Omega \times \mathbb{R} \to \mathbb{R}$ *is defined by* $F(\xi, s) := \int_0^s f(\xi, \iota) d\iota$*, for all* $\xi \in \Omega$ *and* $s \in \mathbb{R}$*, and satisfying*

i) $F(\xi, s) > M|s|^2$,

ii) $|F(\xi, s)| < C\|s\|^2_{L^p(\Omega)}$,

*for some positive constants* $M$ *and* $C$*. Let* $2s_p C|\Omega| < L'M\zeta^{2n+1}$*, where*

$$L' := \frac{2C_n}{(C_n + \|a\|_\infty)} \frac{R_0^2(1-\zeta)^2}{\left(1 - \frac{t^2}{r^2}\right)(1+4t^2)},$$

*then for each* $\lambda \in (\frac{1}{L'M\zeta^{2n+1}}, \frac{1}{2s_p C|\Omega|})$*, the problem*

$$\begin{cases} -\Delta_{\mathbb{H}^n} u + a(\xi) \frac{u}{(|z|^4+t^2)^{\frac{1}{2}}} = \lambda f(\xi, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{13}$$

*has an unbounded sequence of weak solutions in* $H_0^1(\Omega, \mathbb{H}^n)$*.*

**Proof.** We apply Theorem 1 part of $(a)$. The strategy of the proof is similar to Theorem 2, hence, we omit the details. Let $\{\sigma_k\}$ be a real sequence of positive numbers such that $\lim_{k\to+\infty} \sigma_k = +\infty$. Set $r_k = \frac{1}{2S_p}{\sigma_k}^2$. By $(ii)$ we have

$$\begin{aligned} \gamma &\le \liminf_{k\to+\infty} \varphi(r_k) \\ &\le 2S_p \lim_{k\to+\infty} \frac{\int_\Omega \sup_{\|s\|_{L^p(\Omega)}<\sigma_k} F(\xi, s) d\xi}{\sigma_k^2} \\ &= 2S_p C|\Omega| < +\infty. \end{aligned}$$

This implies $\gamma \le 2S_p C|\Omega| < \dfrac{1}{\lambda}$. Let $(u_k)$ be the sequence defined in (8). Then,

$$\begin{aligned} \Phi(u_k) &< \left(\frac{C_n + \|a\|_\infty}{2C_n}\right) \int_{B_{2n+1}(0,R_0)\setminus B_{2n+1}(0,\zeta R_0)} |\nabla_{\mathbb{H}^n} u_k(\xi)|^2 d\xi \\ &\le \frac{(C_n + \|a\|_\infty)}{2C_n} \frac{\eta_k^2}{R_0^2(1-\zeta)^2}\left(1 - \frac{t^2}{r^2}\right)(1+4t^2)\omega_{2n+1} R_0^{2n+1}(1-\zeta^{2n+1}) \\ &= \frac{\eta_k^2 \omega_{2n+1} R_0^{2n+1}}{L'}. \end{aligned} \tag{14}$$

On other hand, by $(\mathrm{i})$, we have

$$\Psi(u_k) = \int_\Omega F(\xi, u_k)d\xi \geq \int_{B_{2n+1}(0,\zeta R_0)} F(\xi, \eta_k)d\xi > M\eta_k^2\zeta^{2n+1}\omega_{2n+1}R_0^{2n+1}. \tag{15}$$

So, it follows from (14) and (15) that

$$\begin{aligned} I_\lambda(u_k) &= \Phi(u_k) - \lambda\Psi(u_k) \\ &\leq \eta_k^2\omega_{2n+1}R_0^{2n+1}\left(\frac{1}{L'} - \lambda M\zeta^{2n+1}\right), \end{aligned}$$

for $k$ large enough, so $\lim_{k\to+\infty} I_\lambda(u_k) = -\infty$, and hence, the claim follows. The alternative of Theorem 1 case (a) assures the existence of unbounded sequence $\{u_k\}$ of critical points of the functional $I_\lambda$, and the proof is complete. □

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