Acta Univ. Sapientiae, Mathematica, **14**, 1 (2022) 104–114

DOI: 10.2478/ausm-2022-0007

A power of a meromorphic function sharing a set with its higher order derivative

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Abstract. In this paper, we deduce the form of a nonconstant meromorphic function f when some power of f shares certain set counting multiplicities in the weak sense with the k-th derivative of the power. The results of this paper generalize the results due to Lahiri and Zeng [Afr. Mat. 27 (2016), 941-947].

1 Introduction, definitions and results

Let f and g be two nonconstant meromorphic functions in the complex plane and $a \in \mathbb{C} \cup \{\infty\}$. If the zeros of $f - a$ and $g - a$ coincide both in locations and multiplicities then we say that f and g share the value a CM (counting multiplicities) and if they coincide only in locations (may or may not have the same multiplicities) then we say that f and g share the value a IM (ignoring multiplicities). For a meromorphic function f in the complex plane, we denote by $S(r, f)$ any quantity satisfying $S(r, f) = o\{T(r, f)\}$ for all r outside a possible exceptional set of finite logarithmic measure. Throughout this paper, we adopt the standard notations of Nevanlinna Theory as described in [1] and [8]. We now recall the following definitions.

2010 Mathematics Subject Classification: 30D35
Key words and phrases: meromorphic functions, WCM sharing, Nevanlinna theory

Definition 1 [3] *For* $a \in \mathbb{C} \cup \{\infty\}$ *we denote by* $N(r, a; f| = 1)$ *the counting function of simple* a*-points of* f*. For a positive integer* m *we denote by* $N(r, a; f| \leq m)$ *(*$N(r, a; f| \geq m)$*) the counting function of those* a*-points of* f *whose multiplicities are not greater (less) than* m *where each* a*-point is counted according to its multiplicity.*

$\overline{N}(r, a; f| \leq m)$ *(*$\overline{N}(r, a; f| \geq m)$*) are defined analogously, where in counting the* a*-points of* f *we ignore the multiplicities.*

Definition 2 [2] *Let* a *be any value in the extended complex plane, and let* p *be an arbitrary nonnegative integer. We denote by* $N_p(r, a; f)$ *the counting function of* a*-points of* f*, where an* a*-point of multiplicity* m *is counted* m *times if* $m \leq p$ *and* p *times if* $m > p$*. Then*

$$N_p(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f \,|\geq 2) + \ \ldots \ + \overline{N}(r, a; f \,|\geq p).$$

Clearly $N_1(r, a; f) = \overline{N}(r, a; f)$*.*

In 1983, Mues and Steinmetz [7] proved the following result.

Theorem A *Let* f *be a nonconstant meromorphic function and* a*,* b *be two distinct finite complex numbers. If* f *and* f' *share* a*,* b *CM, then* $f = ce^z$*, where* c *is a nonzero constant.*

In 2004, Lin and Huang [6] proved the following result considering certain power of a meromorphic function.

Theorem B *Let* f *be a nonconstant meromorphic function,* $n(\geq 8)$ *be an integer and* a *be a nonzero complex number. If* f^n *and* $(f^n)'$ *share the value* a *CM, then* $f = ce^{\frac{z}{n}}$*, where* c *is a nonzero constant.*

In 2008, Lei, Fang, Yang and Wang [5] improved Theorem B by relaxing the lower bound of n and proved the result for $n \geq 4$.

For $a \in \mathbb{C} \cup \{\infty\}$, let $E(a, f)$ denote the set of all a-points of f where an a-point is counted according to its multiplicity and $\overline{E}(a, f)$ denote the set of distinct a-points of f. If $S \subset \mathbb{C} \cup \{\infty\}$, then we define $E(S, f) = \cup_{a \in S} E(a, f)$. We say that f and g share the set S counting multiplicities (CM) if $E(S, f) = E(S, g)$. Similarly we define $\overline{E}(S, f) = \cup_{a \in S} \overline{E}(a, f)$.

Let $a \in \mathbb{C} \cup \{\infty\}$ and $B \subset \mathbb{C} \cup \{\infty\}$. We denote by $\overline{E}_B(a; f, g)$ the set of all those distinct a-points of f which are b-points of g with same multiplicities for some $b \in B$ and $\overline{E}_B(A; f, g) = \cup_{a \in A} \overline{E}_B(a; f, g)$ for $A \subset \mathbb{C} \cup \{\infty\}$.

For $S \subset \mathbb{C} \cup \{\infty\}$, we now put $Y = \{\overline{E}(S, f) \cup \overline{E}(S, g)\} \backslash \overline{E}_S(S; f, g)$. We say that f and g share the set S counting multiplicities in the weak sense or WCM if $\overline{N}_Y(r, a; f) = S(r, f)$ and $\overline{N}_Y(r, a; g) = S(r, g)$ for every $a \in S$, where $\overline{N}_Y(r, a; f)$ denotes the reduced counting function of those a-points of f which lie in the set Y (see [4]). Intuitively, sharing WCM is little less than sharing CM by an unimportant error term. We also see that f and g share the set S CM if and only if $Y = \phi$. Further, WCM value sharing is same as "CM" value sharing when $S = \{a\}$(p. 226, [8]).

In 2016, using the concept of WCM value sharing of a set, Lahiri and Zeng [4] proved the following theorems which improve Theorem B.

Theorem C Let f be a nonconstant meromorphic function, m, $n(\geq 4)$ be positive integers and $S = \{a_1, a_2, ..., a_m\} \subset \mathbb{C} \backslash \{0\}$ be a set of distinct numbers. If f^n and $(f^n)'$ share the set S WCM, then $f = ce^{\frac{\omega z}{n}}$, where $c(\neq 0)$, ω are constants and $\omega^m = 1$. Further $f = ce^{\frac{z}{n}}$ if either $\sum_{i=1}^{m} a_i \neq 0$ or m is prime and $S \neq \{az : z^m = 1\}$, where a is any nonzero number.

Remark 1 [4] *If* $\sum_{i=1}^{m} a_i = 0$*, then* ω *may not be equal to* 1*. For example, let* $S = \{1, -1, 2, -2\}$ *and* $f = ce^{\frac{-z}{4}}$*, where* c *is a nonzero constant.*

Remark 2 [4] *If* $S = \{az : z^m = 1\}$*, then* ω *may not be equal to* 1 *even if* m *is prime. For, let* $S = \{2, 2\omega, 2\omega^2\}$ *and* $f = ce^{\frac{\omega z}{4}}$*, where* c *is a nonzero constant and* ω *is an imaginary cube root of unity.*

Remark 3 [4] *If* m *is not a prime, then* ω *may not be equal to* 1 *even if* $S \neq \{az : z^m = 1\}$*, where* a *is any nonzero constant. The example in Remark 1 makes it evident.*

Theorem D *Let* f *be a nonconstant meromorphic function,* $m(\geq 2)$*,* $n(\geq 3)$ *be positive integers and* $S = \{a_1, a_2, ..., a_m\} \subset \mathbb{C} \backslash \{0\}$ *be a set of distinct numbers such that* $\sum_{i=1}^{m} a_i = 0$*. If* f^n *and* $(f^n)'$ *share the set* S *WCM, then* $f = ce^{\frac{\omega z}{n}}$*, where* $c(\neq 0)$*,* ω *are constants and* $\omega^m = 1$*.*

Regarding Theorems C and D, it is natural to ask the following question which is the motive of this paper.

Question 1 *What happens if the function* f^n *share the set* S *WCM with its* k*-th derivative in Theorems C and D?*

In this paper, we find possible answer to the above question and prove the following theorems.

Theorem 1 *Let* f *be a nonconstant meromorphic function,* m*,* n*,* $k(\geq 1)$ *be positive integers satisfying* $n \geq k+1+\sqrt{k+2}$ *and* $S=\{a_1, a_2, ..., a_m\} \subset \mathbb{C}\backslash\{0\}$ *be a set of distinct complex numbers. If* f^n *and* $(f^n)^{(k)}$ *share the set* S *WCM then either* $f = ce^{\frac{\omega^{\frac{1}{k}}}{n}\nu z}$*, where* $c(\neq 0)$*,* ω *and* ν *are constants with* $\omega^m = 1$ *and* $\nu^k = 1$ *or* f^n *is a linear combination of* $e^{\omega^{\frac{1}{k}}\nu_1 z}$*,* $e^{\omega^{\frac{1}{k}}\nu_2 z}$*, . . . ,* $e^{\omega^{\frac{1}{k}}\nu_k z}$*, where* ν_i*'s are the distinct* k*-th roots of unity. Further, if either* $\sum_{i=1}^{m} a_i \neq 0$ *or* m *is prime and* $S \neq \{az : z^m = 1\}$*, where* a *is any nonzero number, then* $\omega = 1$*.*

Theorem 2 *Let* f *be a nonconstant meromorphic function,* $m(\geq 2)$*,* n*,* k *be positive integers satisfying* $n > \frac{3(k+1)+\sqrt{k^2+10k+17}}{4}$ *and* $S = \{a_1, a_2, ..., a_m\} \subset \mathbb{C}\backslash\{0\}$ *be a set of distinct complex numbers such that* $\sum_{i=1}^{m} a_i = 0$*. If* f^n *and* $(f^n)^{(k)}$ *share the set* S *WCM, then either* $f = ce^{\frac{\omega^{\frac{1}{k}}}{n}\nu z}$*, where* $c(\neq 0)$*,* ω *and* ν *are constants with* $\omega^m = 1$ *and* $\nu^k = 1$ *or* f^n *is a linear combination of* $e^{\omega^{\frac{1}{k}}\nu_1 z}$*,* $e^{\omega^{\frac{1}{k}}\nu_2 z}$*, . . . ,* $e^{\omega^{\frac{1}{k}}\nu_k z}$*, where* ν_i*'s are the distinct* k*-th roots of unity.*

Remark 4 *Theorems C and D can be obtained by putting* $k = 1$ *in Theorems 1 and 2, as in this case, we obtain* $\nu = 1$*.*

2 Lemmas

Let a, a_1, a_2, . . . , a_m be distinct finite complex numbers. We put $z_i = a - a_i$ for $i = 1, 2, ..., m$ and $\sigma_0 = 1$, $\sigma_1 = \sum_{i=1}^{m} z_i$, $\sigma_2 = \sum_{1\leq i<j\leq m} z_i z_j$, . . ., $\sigma_m = z_1 z_2 \ . \ . \ .z_m$. We say that a complex number C satisfies the property (A) if $\sigma_i(C^i - 1) = 0$ and a complex number K satisfies the property (B), if $K^i \sigma_{m-i} = \sigma_i \sigma_m$, $i = 1, 2, 3, ..., m$ (see [8], p.482).

Now we state some lemmas which will be needed in the sequel.

Lemma 1 *Let* f *be a nonconstant meromorphic function and* $S = \{a_1, a_2, \ldots, a_m\} \subset \mathbb{C}$ *be a set of distinct complex numbers. Further suppose that* $N(r, a; f) + N(r, a; f^{(k)}) + N(r, \infty; f) = S(r, f)$ *for some* $a \in \mathbb{C}\backslash S$*. If* f *and* $f^{(k)}$ *share the set* S *WCM, then either* $f^{(k)} - a \equiv C(f - a)$ *or* $(f^{(k)} - a)(f - a) \equiv K$*, where* C *satisfies the property (A) and* K *satisfies the property (B).*

Proof. Clearly $N(r, a; f) = N(r, a; f^{(k)}) = N(r, \infty; f) = S(r, f)$.

If z_0 is a pole of f of order l then z_0 is a pole of $f^{(k)}$ of order $l + k$. Now, $l + k \leq (k + 1)l$, therefore $N(r, \infty; f^{(k)}) \leq (k + 1)N(r, \infty; f)$, which implies

$N(r,\infty;f^{(k)}) = S(r,f)$. Thus, using Lemma 3.8 of [8] (p.193) we deduce that

$$\delta(a,f) = \delta(\infty,f) = \delta(a,g) = \delta(\infty,g) = 1,$$

where $g = f^{(k)}$. The rest of the proof can be completed in the line of Theorem 10.26 of [8], (p. 482). □

Lemma 2 [8]*(Theorem 1.24, p.39) Let* f *be a nonconstant meromorphic function and* k *be a positive integer. Then*

$$N(r,0;f^{(k)}) \leq N(r,0;f) + k\overline{N}(r,\infty;f) + S(r,f).$$

Lemma 3 [9] *Let* f *be a nonconstant meromorphic function and* p, k *be two positive integers. Then*

$$N_p(r,0;f^{(k)}) \leq k\overline{N}(r,\infty;f) + N_{p+k}(r,0;f) + S(r,f).$$

Lemma 4 *Let* f *be a nonconstant meromorphic function,* m, k, $n(> k+1)$ *be positive integers and* $S = \{a_1, a_2, ..., a_m\}$ *be a set of distinct nonzero complex numbers. If* f^n *and* $(f^n)^{(k)}$ *share the set* S *WCM, then one of the following holds:*

(i) $N(r,0;f) \leq \frac{1}{n-k-1}\overline{N}(r,\infty;f) + S(r,f)$;

(ii) $(f^n)^{(k)} \equiv \omega f^n$, *where* $\omega^m = 1$.

Proof. Let $g = f^n$. Put

$$\phi = \sum_{i=1}^{m} \frac{g'}{g-a_i} - \sum_{i=1}^{m} \frac{g^{(k+1)}}{g^{(k)}-a_i}. \tag{1}$$

Now we consider the following cases.

Case 1. Let $\phi \not\equiv 0$. Then $m(r,\phi) = S(r,g) = S(r,f)$. If z_0 is a zero of f with multiplicity l, then z_0 is a zero of ϕ with multiplicity at least $l(n-k-1)$. Since g and $g^{(k)}$ share S WCM, from (1) we get $N(r,\infty;\phi) \leq \overline{N}(r,\infty;f) + S(r,f)$. Therefore

$$\begin{aligned} N(r,0;f) &\leq \frac{1}{n-k-1}N(r,0;\phi) \\ &\leq \frac{1}{n-k-1}T(r,\phi) + O(1) \end{aligned}$$

$$\begin{aligned} &= \frac{1}{n-k-1} N(r, \infty; \phi) + S(r, f) \\ &\le \frac{1}{n-k-1} \overline{N}(r, \infty; f) + S(r, f). \end{aligned}$$

Case 2. Let $\phi \equiv 0$. Then

$$\sum_{i=1}^{m} \frac{g'}{g - a_i} \equiv \sum_{i=1}^{m} \frac{g^{(k+1)}}{g^{(k)} - a_i}.$$

Integrating,

$$\prod_{i=1}^{m} (g - a_i) \equiv c \prod_{i=1}^{m} (g^{(k)} - a_i), \tag{2}$$

where c is a nonzero constant.

If $N(r, 0; f) = S(r, f)$, then (i) holds. So we assume that $N(r, 0; f) \neq S(r, f)$. If z_0 is a zero of f with multiplicity l, then z_0 is a zero of g and $g^{(k)}$ of multiplicities nl and $nl - k$ respectively. So from (2) we see that $c = 1$. Also we have $g^{(nl)}(z_0) \neq 0$. Thus from (2) we obtain

$$\begin{aligned} g^m &+ \sum_{i=1}^{m} (-a_i) g^{m-1} + \sum_{1 \le i < j \le m} (a_i a_j) g^{m-2} + \ldots + \sum_{i=1}^{m} (-1)^{m-1} \frac{a_1 a_2 \ldots a_m}{a_i} g \\ &\equiv (g^{(k)})^m + \sum_{i=1}^{m} (-a_i)(g^{(k)})^{m-1} + \sum_{1 \le i < j \le m} (a_i a_j)(g^{(k)})^{m-2} \\ &+ \ldots + \sum_{i=1}^{m} (-1)^{m-1} \frac{a_1 a_2 \ldots a_m}{a_i} g^{(k)}. \end{aligned} \tag{3}$$

If $m = 1$, then $(f^n)^{(k)} = f^n$. Let $m \ge 2$. We differentiate (3) $nl - k$ times and put $z = z_0$ to obtain

$$\sum_{i=1}^{m} (-1)^{m-1} \frac{a_1 a_2 \ldots a_m}{a_i} = 0.$$

Hence from (3) we get

$$g^m + \sum_{i=1}^{m}(-a_i)g^{m-1} + \sum_{1\le i<j\le m}(a_i a_j)g^{m-2} + \ldots + \sum_{i=1}^{m}(-1)^{m-2}\frac{a_1 a_2 \ldots a_m}{a_i a_j}g^2$$
$$\equiv (g^{(k)})^m + \sum_{i=1}^{m}(-a_i)(g^{(k)})^{m-1} + \sum_{1\le i<j\le m}(a_i a_j)(g^{(k)})^{m-2} \quad (4)$$
$$+ \ldots + \sum_{i=1}^{m}(-1)^{m-2}\frac{a_1 a_2 \ldots a_m}{a_i a_j}(g^{(k)})^2.$$

Differentiating both sides of (4) $2(nl-k)$ times and putting $z = z_0$, we get

$$\sum_{1\le i<j\le m}\frac{a_1 a_2 \ldots a_m}{a_i a_j} = 0.$$

Proceeding similarly, we get

$$\sum_{i=1}^{m} a_i = \sum_{1\le i<j\le m} a_i a_j = \ldots = 0.$$

Hence from (3) we get $g^m \equiv (g^{(k)})^m$ and so $(f^n)^{(k)} \equiv \omega f^n$, where $\omega^m = 1$. This proves the lemma. □

Lemma 5 *Let* f *be a nonconstant meromorphic function,* m, $n(\ge 2)$ *be positive integers and* $S = \{a_1, a_2, \ldots, a_m\}$ *be a set of distinct nonzero complex numbers. If* f^n *and* $(f^n)^{(k)}$ *share the set* S *WCM, then*

$$N(r,\infty;f) \le \frac{k+2}{n-1}\overline{N}(r,0;f) + \frac{k}{n-1}\overline{N}(r,\infty;f) + S(r,f).$$

Proof. Let $g = f^n$. We put

$$\phi = \frac{mg'}{g} - \sum_{i=1}^{m}\frac{g'}{g-a_i} - \frac{mg^{(k+1)}}{g^{(k)}} + \sum_{i=1}^{m}\frac{g^{(k+1)}}{g^{(k)}-a_i}. \quad (5)$$

Casa 1: Let $\phi \not\equiv 0$. Then $m(r,\phi) = S(r,g) = S(r,f)$. We can write (5) as

$$\phi = \frac{g'}{g\prod_{i=1}^{m}(g-a_i)}\left[\sum_{i=1}^{m}(-a_i)g^{m-1} + P_{m-2}(g)\right]$$

$$-\frac{g^{(k+1)}}{g^{(k)}\prod_{i=1}^{m}(g^{(k)}-a_i)}\left[\sum_{i=1}^{m}(-a_i)(g^{(k)})^{m-1}+P_{m-2}(g^{(k)})\right], \qquad (6)$$

where $P_{m-2}(z)$ is a polynomial of degree at most $m-2$ if $m \geq 2$ and $P_{-1}(z) \equiv 0$.

If z_0 is a pole of f with multiplicity l then z_0 is a zero of ϕ with multiplicity at least $(n-1)l$. Since g and $g^{(k)}$ share the set S WCM, using Lemma 3 we see that

$$\begin{aligned} N(r,\infty;\phi)=\overline{N}(r,\infty;\phi) &\leq \overline{N}(r,0;f)+\overline{N}(r,0;g^{(k)})+S(r,f) \\ &\leq \overline{N}(r,0;f)+k\overline{N}(r,\infty;f)+N_{k+1}(r,0;f^n)+S(r,f) \\ &\leq (k+2)\overline{N}(r,0;f)+k\overline{N}(r,\infty;f)+S(r,f). \end{aligned}$$

Hence we obtain

$$\begin{aligned} N(r,\infty;f) &\leq \frac{1}{n-1}N(r,0;\phi) \\ &\leq \frac{1}{n-1}T(r,\phi)+S(r,f) \\ &= \frac{1}{n-1}N(r,\infty;\phi) \\ &\leq \frac{k+2}{n-1}\overline{N}(r,0;f)+\frac{k}{n-1}\overline{N}(r,\infty;f)+S(r,f). \end{aligned}$$

Case 2: Let $\phi \equiv 0$. Then integrating (5) we have,

$$g^m\prod_{i=1}^{m}(g^{(k)}-a_i) \equiv c(g^{(k)})^m\prod_{i=1}^{m}(g-a_i), \qquad (7)$$

where $c(\neq 0)$ is a constant.

Now (7) can be rewritten in concise form as

$$\prod_{i=1}^{m}\left(1-\frac{a_i}{g^{(k)}}\right)=c\prod_{i=1}^{m}\left(1-\frac{a_i}{g}\right).$$

From the above we note that if f has a pole at $z=z_0$, say, then $c=1$. Hence from (7) we get

$$\begin{aligned} &\left(-\sum_{i=1}^{m}a_i\right)g^m\left(g^{(k)}\right)^{m-1}+g^mQ_{m-2}\left(g^{(k)}\right) \\ &=\left(-\sum_{i=1}^{m}a_i\right)\left(g^{(k)}\right)^m g^{m-1}+\left(g^{(k)}\right)^m Q_{m-2}(g), \end{aligned} \qquad (8)$$

where $Q_{m-2}(z)$ is a polynomial of degree at least $m-2$ if $m \geq 2$ and $Q_{-1}(z) \equiv 0$.

Let $\sum_{i=1}^{m} a_i \neq 0$. If z_0 is a pole of f with multiplicity l, then z_0 is a pole of multiplicity $2mnl + mk - nl - k$ of the left hand side of (8) and a pole of multiplicity $2mnl + mk - nl$ of the right hand side of the same, which can not happen. Now we assume $\sum_{i=1}^{m} a_i = 0$. If z_0 is a pole of f with multiplicity l, then z_0 is a pole of multiplicity $2mnl + mk - 2k - 2nl$ of the left hand side of (8) and a pole of multiplicity $2mnl + mk - 2nl$ of the right hand side of the same, which is impossible. Thus f has no pole in both the cases and hence the lemma. □

Lemma 6 *Let f be a nonconstant meromorphic function, $m(\geq 2)$, n be integers and $S = \{a_1, a_2, ..., a_m\}$ be a set of distinct nonzero complex numbers with $\sum_{i=0}^{m} a_i = 0$. If f^n and $(f^n)^{(k)}$ share the set S WCM, then*

$$N(r, \infty; f) \leq \frac{k+2}{2n-1}\overline{N}(r, 0; f) + \frac{k}{2n-1}\overline{N}(r, \infty; f) + S(r, f).$$

Proof. The lemma can be proved in a similar way as in Lemma 5 noting that if z_0 is a pole of f with multiplicity l, then it is a zero of ϕ with multiplicity at least $(2n-1)l$. □

3 Proof of theorems

Proof. [Proof of Theorem 1] First, we suppose that $(f^n)^{(k)} \not\equiv \omega f^n$ for any constant ω satisfying $\omega^m = 1$. Then using (i) of Lemma 4 and Lemma 5 we have

$$\begin{aligned} N(r, \infty; f) &\leq \frac{k+2}{n-1}\overline{N}(r, 0; f) + \frac{k}{n-1}\overline{N}(r, \infty; f) + S(r, f) \\ &\leq \frac{k+2}{n-1}N(r, 0; f) + \frac{k}{n-1}\overline{N}(r, \infty; f) + S(r, f) \\ &\leq \frac{1}{n-1}\left(\frac{k+2}{n-k-1} + k\right)\overline{N}(r, \infty; f) + S(r, f). \end{aligned} \tag{9}$$

Since $n > k + 1 + \sqrt{k+2}$, from (9) it is clear that $N(r, \infty; f) = S(r, f)$, $N(r, 0; f) = S(r, f)$ and hence $N(r, 0; f^{(k)}) = S(r, f)$, by Lemma 2.

Now,

$$\begin{aligned} T\big(r, (f^n)^{(k)}\big) &= m\big(r, (f^n)^{(k)}\big) + N\big(r, \infty; (f^n)^{(k)}\big) \\ &\leq m(r, f^n) + N(r, \infty; f^n) + k\overline{N}(r, \infty; f) + S(r) \\ &= T(r, f^n) + S(r), \end{aligned} \tag{10}$$

and

$$\begin{aligned} T(r,f^n) &\le T\big(r,(f^n)^{(k)}\big) + T\left(r,\frac{f^n}{(f^n)^{(k)}}\right) + S(r) \\ &\le T\big(r,(f^n)^{(k)}\big) + N\left(r,\infty;\frac{(f^n)^{(k)}}{f^n}\right) + S(r) \\ &\le T\big(r,(f^n)^{(k)}\big) + N\big(r,\infty;(f^n)^{(k)}\big) + N(r,0;f^n) + S(r) \\ &\le T\big(r,(f^n)^{(k)}\big) + n(k+1)N\big(r,\infty;f\big) + nN(r,0;f) + S(r) \\ &= T\big(r,(f^n)^{(k)}\big) + S(r), \end{aligned} \tag{11}$$

where $S(r) = \max\{S(r,f), S(r,f^n), S\big(r,(f^n)^{(k)}\big)\}$.

From (10) and (11) we obtain $T\big(r,(f^n)^{(k)}\big) = T(r,f^n)+S(r) = nT(r,f)+S(r)$ and therefore $S(r,f) = S(r,f^n) = S\big(r,(f^n)^{(k)}\big)$. Also by Lemma 2 we see that

$$\begin{aligned} N(r,0;f^n) + N\big(r,0;(f^n)^{(k)}\big) + N(r,\infty;f^n) \le 2nN(r;0;f) \\ +k\overline{N}(r,\infty;f) + nN(r,\infty;f) = S(r,f). \end{aligned}$$

So by Lemma 1, we obtain either $(f^n)^{(k)} \equiv Cf^n$ or $(f^n)^{(k)}f^n \equiv K$, where C and K satisfy properties (A) and (B) respectively as given earlier with $a = 0$.

As $\sigma_m(C^m - 1) = 0$ and $\sigma_m \ne 0$, we get $C = \omega$, where $\omega^m = 1$. Therefore, $(f^n)^{(k)} \equiv \omega f^n$ where ω is a constant satisfying $\omega^m = 1$, a contradiction with our assumption. Therefore $(f^n)^{(k)}f^n \equiv K$, where $K^m = (\sigma_m)^2 \ne 0$. From this it follows that f is an entire function having no zero. Thus we may put $f^n = e^{\alpha}$, where α is a nonconstant entire function. So from above we get $e^{2\alpha}P(\alpha', \ldots, \alpha^{(k)}) \equiv K$, where $P(\alpha', \ldots, \alpha^{(k)})$ is a differential polynomial in $\alpha', \alpha'', \ldots, \alpha^{(k)}$. Since α is an entire function, we have $T(r,\alpha^{(j)}) = S(r,f)$ for $j \in \{1,2,\ldots,k\}$, and hence $T(r,P) = S(r,f) = S(r,e^{\alpha})$. Thus, we obtain

$$2T(r,e^{\alpha}) = T(r,P) + O(1) = S(r,e^{\alpha}),$$

a contradiction.

Hence we must have $(f^n)^{(k)} \equiv \omega f^n$ for some constant ω satisfying $\omega^m = 1$. On solving this k-th order differential equation for f, we obtain either $f = ce^{\frac{\omega^{\frac{1}{k}}}{n}\nu z}$, where $c(\ne 0)$ and ν are constants with $\nu^k = 1$ or f^n is a linear combination of $e^{\omega^{\frac{1}{k}}\nu_1 z}, e^{\omega^{\frac{1}{k}}\nu_2 z}, \ldots, e^{\omega^{\frac{1}{k}}\nu_k z}$, where ν_i's are the distinct k-th roots of unity. The rest of the proof can be completed in a similar way as done in the last part of the proof of Theorem 1.1 in [4]. □

Proof. [Proof of Theorem 2] Using Lemma 6 instead of Lemma 5, this theorem can be proved in the line of Theorem 1. Here we omit the details. □

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Received: November 29, 2021