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Norm attaining bilinear forms on the plane with the l_1-norm

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Abstract. For given unit vectors $x_1, \cdots, x_n$ of a real Banach space E, we define

$$NA(\mathcal{L}(^n E))(x_1, \cdots, x_n) = \{T \in \mathcal{L}(^n E) : |T(x_1, \cdots, x_n)| = \|T\| = 1\},$$

where $\mathcal{L}(^n E)$ denotes the Banach space of all continuous n-linear forms on E endowed with the norm $\|T\| = \sup_{\|x_k\|=1, 1\le k\le n} |T(x_1, \ldots, x_n)|$.

In this paper, we classify $NA(\mathcal{L}(^2 l_1^2))((x_1, x_2), (y_1, y_2))$ for unit vectors $(x_1, x_2), (y_1, y_2) \in l_1^2$, where $l_1^2 = \mathbb{R}^2$ with the l_1-norm.

1 Introduction

Let $n \in \mathbb{N}, n \ge 2$. We write S_E for the unit sphere of a real Banach space E. We denote by $\mathcal{L}(^n E)$ the Banach space of all continuous n-linear forms on E endowed with the norm $\|T\| = \sup_{\|x_k\|=1, 1\le k\le n} |T(x_1, \ldots, x_n)|$. The subspace of all continuous symmetric n-linear forms on E is denoted by $\mathcal{L}_s(^n E)$. A mapping $P : E \to \mathbb{R}$ is a continuous n-homogeneous polynomial if there exists $T \in \mathcal{L}(^n E)$ such that $P(x) = T(x, \ldots, x)$ for every $x \in E$. We denote by $\mathcal{P}(^n E)$ the Banach space of all continuous n-homogeneous polynomials from E into $\mathbb{R}$ endowed

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with the norm $\|P\| = \sup_{\|x\|=1} |P(x)|$. For more details about the theory of multilinear mappings and polynomials on a Banach space, we refer to [4].

Elements $x_1, \ldots, x_n \in E$ is called *norming points* of $T \in \mathcal{L}({}^nE)$ if $\|x_1\| = \cdots = \|x_n\| = 1$ and $|T(x_1, \ldots, x_n)| = \|T\|$. In this case, T is called a *norm attaining* n-linear form at $x_1, \ldots, x_n$. Similarly, an element $x \in E$ is called a *norming point* of $P \in \mathcal{P}({}^nE)$ if $\|x\| = 1$ and $|P(x)| = \|P\|$. In this case, P is called a *norm attaining* n-homogeneous polynomial at x. Let $X = \mathcal{L}({}^nE)$ or $\mathcal{L}_s({}^nE)$. For $x, x_1, \cdots, x_n \in S_E$, we define

$$NA(X)(x_1, \cdots, x_n) = \{T \in X : |T(x_1, \cdots, x_n)| = \|T\| = 1\}$$

and

$$NA(\mathcal{P}({}^nE))(x) = \{P \in \mathcal{P}({}^nE) : |P(x)| = \|P\| = 1\}.$$

Notice that

$$NA(\mathcal{L}({}^nE))(x_1, \cdots, x_n) = NA(\mathcal{L}({}^nE))(\pm x_1, \cdots, \pm x_n),$$

$$NA(\mathcal{L}_s({}^nE))(x_1, \cdots, x_n) = NA(\mathcal{L}_s({}^nE))(\pm x_{\sigma(1)}, \cdots, \pm x_{\sigma(n)})$$

and

$$NA(\mathcal{P}({}^nE))(x) = NA(\mathcal{P})({}^nE))(-x)$$

for all $x, x_1, \cdots, x_n \in S_E$ and for all permutation σ on $\{1, \ldots, n\}$.

Let us introduce a brief history of norm attaining multilinear forms and polynomials on Banach spaces. In 1961 Bishop and Phelps [2] initiated and showed that the set of norm attaining functionals on a Banach space is dense in the dual space. Shortly after, attention was paid to possible extensions of this result to more general settings, specially bounded linear operators between Banach spaces. The problem of denseness of norm attaining functions has moved to other types of mappings like multilinear forms or polynomials. The first result about norm attaining multilinear forms appeared in a joint work of Aron, Finet and Werner [1], where they showed that the Radon-Nikodym property is sufficient for the denseness of norm attaining multilinear forms. Choi and Kim [3] showed that the Radon-Nikodym property is also sufficient for the denseness of norm attaining polynomials. Jimenez-Sevilla and Paya [5] studied the denseness of norm attaining multilinear forms and polynomials on preduals of Lorentz sequence spaces.

It seems to be natural and interesting to study about

$$NA(\mathcal{L}({}^nE))(x_1, \cdots, x_n), NA(\mathcal{L}_s({}^nE))(x_1, \cdots, x_n) \text{ and } NA(\mathcal{P}({}^nE))(x)$$

for $x, x_1, \cdots, x_n \in S_E$. Kim [6] classified $NA(\mathcal{P}(^2 l_p^2))((x_1, x_2))$ for $(x_1, x_2) \in S_{l_p^2}$ and $p = 1, 2, \infty$, where $l_p^2 = \mathbb{R}^2$ with the l_p-norm.

In this paper, we classify $NA(\mathcal{L}(^2 l_1^2))((x_1, x_2), (y_1, y_2))$ for $(x_1, x_2), (y_1, y_2) \in S_{l_1^2}$.

2 Results

Let $T((x_1, y_1), (x_2, y_2)) = ax_1x_2 + by_1y_2 + cx_1y_2 + dx_2y_1 \in \mathcal{L}(^2 l_1^2)$ for some $a, b, c, d \in \mathbb{R}$. For simplicity, we denote $T = (a, b, c, d)$.

Theorem 1 *Let* $T = (a, b, c, d) \in \mathcal{L}(^2 l_1^2)$ *for some* $a, b, c, d \in \mathbb{R}$. *Then,*

$$\|T\| = \max\{|a|, |b|, |c|, |d|\}.$$

Proof. Let $M := \max\{|a|, |b|, |c|, |d|\}$. Let $(x_j, y_j) \in S_{l_1^2}$ for $j = 1, 2$. It follows that

$$\begin{aligned} |T((x_1, y_1), (x_2, y_2))| &\leq |a|\ |x_1x_2| + |b|\ |y_1y_2| + |c|\ |x_1y_2| + |d|\ |x_2y_1| \\ &\leq M\ (|x_1x_2| + |y_1y_2| + |x_1y_2| + |x_2y_1|) \\ &= M(|x_1| + |y_1|)(|x_2| + |y_2|) = M \\ &= \max\{|T(1,0),(1,0)|, |T(0,1),(0,1)|, |T(1,0),(0,1)|, \\ &\qquad |T(0,1),(1,0)|\} \leq \|T\|. \end{aligned}$$

Therefore, $\|T\| = M$. □

Notice that if $\|T\| = 1$, then $|a| \leq 1, |b| \leq 1, |c| \leq 1$ and $|d| \leq 1$.

Lemma 1 *Let* $T = (a, b, c, d) \in \mathcal{L}(^2 l_1^2)$ *for some* $a, b, c, d \in \mathbb{R}$. *The following are equivalent: let* $(x_1, y_1), (x_2, y_2) \in S_{l_1^2}$.

(a) $T \in NA(\mathcal{L}(^2 l_1^2))((x_1, y_1), (x_2, y_2))$;
(b) $T_1 := (b, a, d, c) \in NA(\mathcal{L}(^2 l_1^2))((y_1, x_1), (y_2, x_2))$;
(c) $T_2 := (a, b, -c, -d) \in NA(\mathcal{L}(^2 l_1^2))((x_1, -y_2), (x_2, -y_2))$;
(d) $T_3 := (-a, -b, -c, -d) \in NA(\mathcal{L}(^2 l_1^2))((-x_1, -y_1), (x_2, y_2))$;
(e) $T_4 := (a, b, d, c) \in NA(\mathcal{L}(^2 l_1^2))((x_2, y_2), (x_1, y_1))$;
(f) $T_5 := (a, -b, -c, d) \in NA(\mathcal{L}(^2 l_1^2))((x_1, y_1), (x_2, -y_2))$.

The following theorem classifies $NA(\mathcal{L}(^2 l_1^2))((x_1, x_2), (y_1, y_2))$ for unit vectors $(x_1, x_2), (y_1, y_2) \in l_1^2$.

Theorem 2 *Let* $(x_1, y_1), (x_2, y_2) \in S_{l_1^2}$. *Then the following statements holds:*

Case 1. $x_j y_j \neq 0$ *for all* $j = 1, 2$.

If $x_j y_j > 0$ *for all* $j = 1, 2$, *then*

$$NA(\mathcal{L}(^2 l_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(1, 1, 1, 1)\}.$$

If $x_j y_j < 0$ *for all* $j = 1, 2$, *then*

$$NA(\mathcal{L}(^2 l_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(1, 1, -1, -1)\}.$$

If $x_1 y_1 > 0$ *and* $x_2 y_2 < 0$, *then*

$$NA(\mathcal{L}(^2 l_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(1, -1, -1, 1)\}.$$

If $x_1 y_1 < 0$ *and* $x_2 y_2 > 0$, *then*

$$NA(\mathcal{L}(^2 l_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(1, -1, 1, -1)\}.$$

Case 2. $x_1 y_1 = 0$ *and* $x_2 y_2 \neq 0$

If $x_1 = 0$ *and* $x_2 y_2 > 0$, *then*

$$NA(\mathcal{L}(^2 l_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(a, 1, c, 1) : |a| \leq 1, |c| \leq 1\}.$$

If $x_1 = 0$ *and* $x_2 y_2 < 0$, *then*

$$NA(\mathcal{L}(^2 l_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(a, 1, c, -1) : |a| \leq 1, |c| \leq 1\}.$$

If $y_1 = 0$ *and* $x_2 y_2 > 0$, *then*

$$NA(\mathcal{L}(^2 l_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(1, b, 1, d) : |b| \leq 1, |d| \leq 1\}.$$

If $y_1 = 0$ *and* $x_2 y_2 < 0$, *then*

$$NA(\mathcal{L}(^2 l_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(1, b, -1, d) : |b| \leq 1, |d| \leq 1\}.$$

Case 3. $x_2 y_2 = 0$ *and* $x_1 y_1 \neq 0$

If $x_2 = 0$ *and* $x_1 y_1 > 0$, *then*

$$NA(\mathcal{L}(^2 l_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(a, 1, 1, d) : |a| \leq 1, |d| \leq 1\}.$$

If $x_2 = 0$ *and* $x_1 y_1 < 0$, *then*

$$NA(\mathcal{L}(^2 l_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(a, 1, -1, d) : |a| \le 1, |d| \le 1\}.$$

If $y_2 = 0$ *and* $x_1 y_1 > 0$, *then*

$$NA(\mathcal{L}(^2 l_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(1, b, c, 1) : |b| \le 1, |c| \le 1\}.$$

If $y_2 = 0$ *and* $x_1 y_1 < 0$, *then*

$$NA(\mathcal{L}(^2 l_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(1, b, c, -1) : |b| \le 1, |c| \le 1\}.$$

Case 4. $x_1 y_1 = x_2 y_2 = 0$

If $x_1 = x_2 = 0$, *then*

$$NA(\mathcal{L}(^2 l_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(a, 1, c, d) : |a| \le 1, |c| \le 1, |d| \le 1\}.$$

If $x_1 = y_2 = 0$, *then*

$$NA(\mathcal{L}(^2 l_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(a, b, c, 1) : |a| \le 1, |b| \le 1, |c| \le 1\}.$$

If $x_2 = y_1 = 0$, *then*

$$NA(\mathcal{L}(^2 l_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(a, b, 1, d) : |a| \le 1, |b| \le 1, |d| \le 1\}.$$

If $y_1 = y_2 = 0$, *then*

$$NA(\mathcal{L}(^2 l_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(1, b, c, d) : |b| \le 1, |c| \le 1, |d| \le 1\}.$$

Proof. Let $(x_1, y_1), (x_2, y_2) \in S_{l_1^2}$. Let $T = (a, b, c, d) \in NA(\mathcal{L}(^2 l_\infty^2))((x_1, y_1), (x_2, y_2))$ for some $a, b, c, d \in \mathbb{R}$. By Theorem 1, $|a| \le 1, |b| \le 1, |c| \le 1$ and $|d| \le 1$. By Lemma 1, we may assume that $a \ge 0$. We consider four cases.

Case 1. $x_j y_j \neq 0$ for all $j = 1, 2$.

Suppose that $x_j y_j > 0$ for all $j = 1, 2$.

Claim. $NA(\mathcal{L}(^2 l_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(1, 1, 1, 1)\}$.

It is obvious that $\{\pm(1, 1, 1, 1)\} \subseteq NA(\mathcal{L}(^2 l_1^2))((x_1, y_1), (x_2, y_2))$. It follows that

$$\begin{aligned} 1 &= |T((x_1, y_1), (x_2, y_2))| = |a x_1 x_2 + b y_1 y_2 + c x_1 y_2 + d x_2 y_1| \\ &\le a\, |x_1 x_2| + |b|\, |y_1 y_2| + |c|\, |x_1 y_2| + |d|\, |x_2 y_1| \end{aligned}$$

$$\begin{aligned} &\le |x_1x_2| + |y_1y_2| + |x_1y_2| + |x_2y_1| \\ &= (|x_1| + |y_1|)(|x_2| + |y_2|) = 1, \end{aligned}$$

which shows that $a = |b| = |c| = |d| = 1$. We will show that $b = 1$. Assume that $b = -1$. Then

$$\begin{aligned} 1 &= |T((x_1, y_1), (x_2, y_2))| = |x_1x_2 - y_1y_2 + cx_1y_2 + dx_2y_1| \\ &= |x_1(x_2 + cy_2) + y_1(dx_2 - y_2)| \\ &\le |x_1|\ |x_2 + cy_2| + |y_1|\ |dx_2 - y_2| \\ &\le |x_1| + |y_1| = 1, \end{aligned}$$

which shows that

$$|x_2 + cy_2| = |dx_2 - y_2| = 1$$

because $|x_1| > 0$ and $|y_1| > 0$. Since $x_2y_2 > 0$, $c = 1, d = -1$. Hence,

$$\begin{aligned} 1 &= |T((x_1, y_1), (x_2, y_2))| = |x_1x_2 - y_1y_2 + x_1y_2 - x_2y_1| \\ &= |x_1 - y_1|\ |x_2 + y_2| = |x_1 - y_1| < 1, \end{aligned}$$

which is a contradiction. Therefore, $b = 1$. It follows that

$$\begin{aligned} 1 &= |T((x_1, y_1), (x_2, y_2))| = |x_1x_2 + y_1y_2 + cx_1y_2 + dx_2y_1| \\ &\le |x_1|\ |x_2 + cy_2| + |y_1|\ |dx_2 + y_2| \\ &= |x_1| + |y_1| = 1, \end{aligned}$$

which shows that

$$|x_2 + cy_2| = |dx_2 + y_2| = 1$$

because $|x_1| > 0$ and $|y_1| > 0$. Hence, $c = d = 1$. Therefore, $T = (1, 1, 1, 1)$, which concludes $NA(\mathcal{L}(^2l_1^2))((x_1, y_1), (x_2, y_2)) \subseteq \{\pm(1, 1, 1, 1)\}$. Therefore, we have shown the claim.

Suppose that $x_jy_j < 0$ for all $j = 1, 2$. Lemma 1 implies that

$$T \in NA(\mathcal{L}(^2l_1^2))((x_1, y_1), (x_2, y_2))$$

if and only if

$$T_2 := (a, b, -c, -d) \in NA(\mathcal{L}(^2l_1^2))((x_1, -y_2), (x_2, -y_2)).$$

Since $x_j(-y_j) > 0$ for all $j = 1, 2$, by the above claim,

$$NA(\mathcal{L}(^2l_1^2))((x_1, -y_2), (x_2, -y_2)) = \{\pm(1, 1, 1, 1)\}.$$

Hence,

$$NA(\mathcal{L}(^2 l_1^2))((x_1,y_2),(x_2,y_2)) = \{\pm(1,1,-1,-1)\}.$$

Suppose that $x_1y_1 > 0$ and $x_2y_2 < 0$. Lemma 1 implies that

$$T \in NA(\mathcal{L}(^2 l_1^2))((x_1,y_1),(x_2,y_2))$$

if and only if

$$T_5 := (a,-b,-c,d) \in NA(\mathcal{L}(^2 l_1^2))((x_1,y_2),(x_2,-y_2)).$$

Since $x_1y_1 > 0$ and $x_2(-y_2) > 0$, by the above claim,

$$NA(\mathcal{L}(^2 l_1^2))((x_1,y_2),(x_2,-y_2)) = \{\pm(1,1,1,1)\}.$$

Hence,

$$NA(\mathcal{L}(^2 l_1^2))((x_1,y_2),(x_2,y_2)) = \{\pm(1,-1,-1,1)\}.$$

Suppose that $x_1y_1 < 0$ and $x_2y_2 > 0$. Lemma 1 implies that

$$T \in NA(\mathcal{L}(^2 l_1^2))((x_1,y_1),(x_2,y_2))$$

if and only if

$$T_2 := (a,b,-c,-d) \in NA(\mathcal{L}(^2 l_1^2))((x_1,-y_2),(x_2,-y_2)).$$

Since $x_1(-y_1) > 0$ and $x_2(-y_2) < 0$, by the above claim,

$$NA(\mathcal{L}(^2 l_1^2))((x_1,-y_2),(x_2,-y_2)) = \{\pm(1,-1,-1,1)\}.$$

Hence,

$$NA(\mathcal{L}(^2 l_1^2))((x_1,y_2),(x_2,y_2)) = \{\pm(1,-1,1,-1)\}.$$

Case 2. $x_1y_1 = 0$ and $x_2y_2 \neq 0$

Suppose that $x_1 = 0$ and $x_2y_2 > 0$.

Claim. $NA(\mathcal{L}(^2 l_1^2))((x_1,y_1),(x_2,y_2)) = \{\pm(a,1,c,1) : |a| \leq 1, |c| \leq 1\}$.

It is obvious that $\{\pm(a,1,c,1) : |a| \leq 1, |c| \leq 1\} \subseteq NA(\mathcal{L}(^2 l_1^2))((x_1,y_1),(x_2,y_2))$. Since $x_1 = 0$, $|y_2| = 1$ and

$$\begin{aligned} 1 &= |T((x_1,y_1),(x_2,y_2))| = |by_1y_2 + dx_2y_1| \\ &\leq |dx_2 + by_2| \leq |d|\,|x_2| + |b|\,|y_2| \\ &\leq |x_2| + |y_2| = 1, \end{aligned}$$

which shows that $|dx_2 + by_2| = 1 = |b| = |d|$. Since $x_2y_2 > 0$, $b = d = 1$ or $b = d = -1$. Hence, $T = \pm(a, 1, c, 1)$ for some $|a| \leq 1, |c| \leq 1$. Therefore, we have shown the claim.

Suppose that $x_1 = 0$ and $x_2y_2 < 0$. Lemma 1 implies that

$$T \in NA(\mathcal{L}(^2l_1^2))((x_1, y_1), (x_2, y_2))$$

if and only if

$$T_2 := (a, b, -c, -d) \in NA(\mathcal{L}(^2l_1^2))((x_1, -y_2), (x_2, -y_2)).$$

Since $x_1(-y_1) = 0$ and $x_2(-y_2) < 0$, by the above claim,

$$NA(\mathcal{L}(^2l_1^2))((x_1, -y_2), (x_2, -y_2)) = \{\pm(a, 1, c, 1) : |a| \leq 1, |c| \leq 1\}.$$

Hence,

$$NA(\mathcal{L}(^2l_1^2))((x_1, y_2), (x_2, y_2)) = \{\pm(a, 1, c, -1) : |a| \leq 1, |c| \leq 1\}.$$

Suppose that $y_1 = 0$ and $x_2y_2 > 0$.
Lemma 1 implies that

$$T \in NA(\mathcal{L}(^2l_1^2))((x_1, y_1), (x_2, y_2))$$

if and only if

$$T_1 := (b, a, d, c) \in NA(\mathcal{L}(^2l_1^2))((y_1, x_1), (y_2, x_2)).$$

By the above claim,

$$NA(\mathcal{L}(^2l_1^2))((y_1, x_1), (y_2, x_2)) = \{\pm(a, 1, c, 1) : |a| \leq 1, |c| \leq 1\}.$$

Hence, $NA(\mathcal{L}(^2l_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(1, b, 1, d) : |b| \leq 1, |d| \leq 1\}$.

Suppose that $y_1 = 0$ and $x_2y_2 < 0$. Lemma 1 implies that

$$T \in NA(\mathcal{L}(^2l_1^2))((x_1, y_1), (x_2, y_2))$$

if and only if

$$T_2 := (a, b, -c, -d) \in NA(\mathcal{L}(^2l_1^2))((x_1, -y_2), (x_2, -y_2)).$$

Since $-y_1 = 0$ and $x_2(-y_2) > 0$, by the above claim,

$$NA(\mathcal{L}(^2l_1^2))((x_1, -y_2), (x_2, -y_2)) = \{\pm(1, b, 1, d) : |b| \leq 1, |d| \leq 1\}.$$

Hence,

$$NA(\mathcal{L}(^2l_1^2))((x_1, y_2), (x_2, y_2)) = \{\pm(1, b, -1, d) : |b| \leq 1, |d| \leq 1\}.$$

Case 3. $x_2y_2 = 0$ and $x_1y_1 \neq 0$

Lemma 1 implies that $T \in NA(\mathcal{L}(^2l_1^2))((x_1, y_1), (x_2, y_2))$ if and only if $T_1 := (b, a, d, c) \in NA(\mathcal{L}(^2l_1^2))((y_1, x_1), (y_2, x_2))$. By Case 2, the assertions of Case 3 hold.

Case 4. $x_1y_1 = x_2y_2 = 0$

Suppose that $x_1 = x_2 = 0$.

Claim. $NA(\mathcal{L}(^2l_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(a, 1, c, d) : |a| \leq 1, |c| \leq 1, |d| \leq 1\}$.

It is obvious that

$$\{\pm(a, 1, c, d) : |a| \leq 1, |c| \leq 1, |d| \leq 1\} \subseteq NA(\mathcal{L}(^2l_1^2))((x_1, y_1), (x_2, y_2)).$$

Since $x_1 = x_2 = 0$, $|y_1| = |y_2| = 1$ and

$$1 = |T((x_1, y_1), (x_2, y_2))| = |by_1y_2| = |b|$$

which shows that $|b| = 1$. Hence, $T = \pm(a, 1, c, d)$ for some $|a| \leq 1, |c| \leq 1, |d| \leq 1$. Therefore, we have shown the claim.

If $x_1 = y_2 = 0$, then $|y_1| = |x_2| = 1$ and

$$1 = |T((x_1, y_1), (x_2, y_2))| = |dx_2y_1| = |d|$$

which shows that $|d| = 1$. Hence, $T = \pm(a, b, c, 1)$ for some $|a| \leq 1, |b| \leq 1, |c| \leq 1$. Hence, $NA(\mathcal{L}(^2l_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(a, b, c, 1) : |a| \leq 1, |b| \leq 1, |c| \leq 1\}$.

If $x_2 = y_1 = 0$, then $|x_1| = |y_2| = 1$ and

$$1 = |T((x_1, y_1), (x_2, y_2))| = |cx_1y_2| = |c|$$

which shows that $|c| = 1$. Hence, $T = \pm(a, b, 1, d)$ for some $|a| \leq 1, |b| \leq 1, |d| \leq 1$. Hence, $NA(\mathcal{L}(^2l_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(a, b, 1, d) : |a| \leq 1, |b| \leq 1, |d| \leq 1\}$.

If $y_1 = y_2 = 0$, then $|x_1| = |x_2| = 1$ and

$$1 = |T((x_1, y_1), (x_2, y_2))| = |ax_1x_2| = |a| = a$$

which shows that $a = 1$. Hence, $T = \pm(1, b, c, d)$ for some $|b| \leq 1, |c| \leq 1, |d| \leq 1$. Hence, $NA(\mathcal{L}(^2l_1^2))((x_1, y_1), (x_2, y_2)) = \{\pm(1, b, c, d) : |b| \leq 1, |c| \leq 1, |d| \leq 1\}$. Therefore, we complete the proof. □

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