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Continuous dependence for double diffusive convection in a Brinkman model with variable viscosity

Ghazi Abed Meften
Basrah Education Directorate,
Ministry of Education,
Basrah, Iraq.
email: ghazialbazony@gmail.com

Ali Hasan Ali
Department of Mathematics,
College of Education for Pure Sciences,
University of Basrah, Basrah, Iraq.
Doctoral School of Mathematical and
Computational Sciences,
University of Debrecen, H-4002
Debrecen, Pf. 400, Hungary.
email: ali.hasan@science.unideb.hu

Abstract. This current work is presented to deal with the model of double diffusive convection in porous material with variable viscosity, such that the equations for convective fluid motion in a Brinkman type are analysed when the viscosity varies with temperature quadratically. Hence, we carefully find a priori bounds when the coefficients depend only on the geometry of the problem, initial data, and boundary data, where this shows the continuous dependence of the solution on changes in the viscosity. A convergence result is also showen when the variable viscosity is allowed to tend to a constant viscosity.

1 Introduction

Studies in the exploration of double-diffusive convection topic in a fluid-saturated porous layer have been an active field for a long time making this topic closely

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related to many other research papers. The various ways of getting the heat and the mass combined to transfer, can be seen in a lot of real life problems.

In fact, the importance of the continuous dependence on changes in the boundary conditions, initial conditions, coefficients, or even in the system of the equations, has been increasingly recognized. This aspect of continuous dependence, or stability, is what we refer to as structural stability, cf. [1], and in many ways is more important than the classical idea of stability, or continuous dependence on the initial data. Continuous dependence on modeling in elasticity has been shown to be of considerable importance in a seminal paper, see [2].

The system of equations which explains the double diffusive convective flow in a porous medium using a Brinkman model has been proposed in [3].In addition, [4, 5, 6, 7] have presented nonlinear stability analyses for a model which does not employ a Brinkman term but instead includes a Forchheimer term. Moreover, a recent study which includes both Brinkman and Forchheimer is suggested in [8]. Brinkman model with a viscosity which depends linearly on temperature is introduced in [9]. Early studies dealing with structural stability issues in porous flows (cf. [10], [11]), have recently developed for porous flow model which has a viscosity depends on concentration [12]. In this paper we continue the work of Payne et al. [12] who study the continuous dependence Brinkman and Forchheimer models when the viscosity is linear function for concentration. However, we study the double diffusive convection in a Brinkman model when the viscosity is linear function for temperature.

The layout of this paper is constructed as follows. In the next section, we will present mathematical formulas of the system. In Section 3, we develop a priori bounds. The goal of Sections 4 and 5 is to demonstrate continuous dependence on changes in the viscosity coefficients. Finally, the convergence to the constants viscosity solution will be establish in Sections 6 and 7.

2 Basic equations

The momentum equation for flow in a porous saturated material of Brinkman type may be taken as

$$-\Delta u_i + (1 + \alpha T + \beta T^2)u_i = -\frac{\partial p}{\partial x_i} + g_i T + \mathcal{I}_i C, \tag{1}$$

where, α and β are constants, and u_i, T, C and p are velocity, temperature, concentration and pressure, respectively. g_i and $\mathcal{I}_i$ are vectors incorporating the gravity field which take $|g_i| \leq 1$ and $|\mathcal{I}_i| \leq 1$. The balance of mass equation is

$$\frac{\partial u_i}{\partial x_i} = 0. \tag{2}$$

Furthermore, the temperature and concentration equations, respectively, have the following forms

$$\begin{aligned} \frac{\partial T}{\partial t} + u_i \frac{\partial T}{\partial x_i} &= \Delta T, \\ \frac{\partial C}{\partial t} + u_i \frac{\partial C}{\partial x_i} &= \Delta C. \end{aligned} \tag{3}$$

Let Ω be a bounded domain in $\mathbb{R}^3$ with boundary $\partial\Omega$. Thus, Equ. (1)-(3) are defined on $\Omega \times (0, \mathcal{T})$, for $\mathcal{T} < \infty$, and the boundary conditions

$$u_i = f_i(x,t), \quad \text{on} \quad \partial\Omega \times (0, \mathcal{T}), \tag{4}$$

and

$$T(x,t) = h(x,t), \;\; C(x,t) = k(x,t), \;\; x \text{ on } \;\partial\Omega, \;\; t \in (0, \mathcal{T}), \tag{5}$$

where h and k are prescribed functions and n is the unit outward normal to $\partial\Omega$, and the initial data for the temperature and concentration is given as

$$T(x,0) = T_0(x), \quad C(x,0) = C_0(x), \quad x \in \Omega, \tag{6}$$

where T_0 and C_0 are prescribed functions.

3 A priori bounds

In this section, we derive bounds for various norms of u_i, T and C, in terms of data. These bounds will be used in the next sections in the continuous dependence and converges proof. To develop a priori bounds, we introduce the functions $G(x,t)$, $K(x,t)$, $F(x,t)$ and $H(x,t)$ as solutions to the boundary value problems

$$\begin{aligned} \Delta G(x,t) &= 0, && \text{in} \;\; \Omega, \\ G(x,t) &= h(x,t), && \text{on} \;\; \partial\Omega, \end{aligned} \tag{7}$$

$$\begin{aligned} \Delta K(x,t) &= 0, && \text{in} \;\; \Omega, \\ K(x,t) &= k(x,t), && \text{on} \;\; \partial\Omega, \end{aligned} \tag{8}$$

$$\begin{aligned} \Delta F(x,t) &= 0, && \text{in} \;\; \Omega, \\ F(x,t) &= h^{2r-1}(x,t), && \text{on} \;\; \partial\Omega, \end{aligned} \tag{9}$$

and

$$\begin{aligned} \Delta H(x,t) &= 0, && \text{in } \ \Omega, \\ H(x,t) &= k^{2r-1}(x,t), && \text{on } \ \partial\Omega, \end{aligned} \tag{10}$$

where r is a positive integer. We commence with deriving a bound for $\|\mathbf{u}\|$, and we let b_i solve the Stokes flow problem in Ω, namely

$$\begin{aligned} \Delta b_i &= \rho_{,i}, \ \frac{\partial b_i}{\partial x_i} && \text{in } \ \Omega, \\ b_i &= f_i, && \text{on } \ \partial\Omega, \end{aligned} \tag{11}$$

where ρ is a pressure term. By the triangle inequality,

$$\|\mathbf{u}\| \leq \|\mathbf{u}-\mathbf{b}\| + \|\mathbf{b}\|. \tag{12}$$

Next, we employ (1) and (11) to derive

$$\begin{aligned} &\|\nabla(\mathbf{u}-\mathbf{b})\|^2 + \int_\Omega (1+\alpha T+\beta T^2)(u_i-b_i)(u_i-b_i)\,d\mathbf{x} \\ &= -\int_\Omega (1+\alpha T+\beta T^2)(u_i-b_i)a\,d\mathbf{x} \\ &\quad + \int_\Omega g_i T(u_i-b_i)\,d\mathbf{x} + \int_\Omega \mathcal{I}_i C(u_i-b_i)\,d\mathbf{x}. \end{aligned} \tag{13}$$

The Cauchy-Schwarz inequality together with the arithmetic-geometric mean and Sobolev inequalities are used on the right-hand side to find

$$\begin{aligned} &\|\nabla(\mathbf{u}-\mathbf{b})\|^2 + \frac{1}{2}\int_\Omega (1+\alpha T+\beta T^2)(u_i-b_i)(u_i-b_i)\,d\mathbf{x} \\ &\leq \frac{3}{2}\int_\Omega (1+\alpha T+\beta T^2)b_i b_i\,d\mathbf{x} + \frac{3}{2}(\|T\|^2+\|C\|^2) \\ &\leq \frac{3}{2}(\|T\|^2+\|C\|^2) + \frac{3}{2}\|\mathbf{b}\|^2 + \frac{3}{2}\alpha\|T\|\|\mathbf{b}\|_4^2 + \frac{3}{2}\beta\|T\|^2\|\mathbf{b}\|^2 \\ &\leq \frac{3}{2}(\|T\|^2+\|C\|^2) + \frac{3}{2}(1+\beta\|T\|^2)\|\mathbf{b}\|^2 + \frac{3}{2}\mathcal{C}\alpha\|T\|(\|\mathbf{b}\|^2+\|\nabla\mathbf{b}\|^2), \end{aligned} \tag{14}$$

here $\mathcal{C}$ is a constant in the Sobolev inequality. As proposed in [14], we can see that

$$\|\mathbf{b}\|^2 \leq 6d\oint_{\partial\Omega} f_i f_i\,dA + 4d^2\int_\Omega (b_{i,j}-b_{j,i})(b_{i,j}-b_{j,i})\,d\mathbf{x}$$

$$\leq (6d + 4d^2\bar{k}_1) \oint_{\partial\Omega} f_i f_i dA + 4d^2\bar{k}_2 \oint_{\partial\Omega} |\nabla_s \mathbf{f}|^2 dA, \tag{15}$$

where d is the radius of the smallest circumscribed ball for $D, \bar{k}_1$ and $\bar{k}_2$ are a priori constants given in [14] and ∇_s denotes the tangential derivative. Furthermore, we have that

$$\begin{aligned} \|\nabla \mathbf{b}\|^2 &= \frac{1}{2}\int_\Omega (b_{i,j} - b_{j,i})(b_{i,j} - b_{j,i}) d\mathbf{x} + \int_\Omega b_{i,j} b_{j,i} d\mathbf{x} \\ &= \frac{1}{2}\int_\Omega (b_{i,j} - b_{j,i})(b_{i,j} - b_{j,i}) d\mathbf{x} + \oint_{\partial\Omega} (b_{i,j} b_j - b_{j,i} b_i) dA \\ &= \frac{1}{2}\int_\Omega (b_{i,j} - b_{j,i})(b_{i,j} - b_{j,i}) d\mathbf{x} + \oint_{\partial\Omega} n^i s^j (b_j \nabla_s b_i - b_i \nabla_s b_j) dA \\ &\leq \frac{1}{2}\bar{k}_1 \oint_{\partial\Omega} f_i f_i dA + \frac{1}{2}\bar{k}_2 \oint_{\partial\Omega} |\nabla_s \mathbf{f}|^2 dA + \oint_{\partial\Omega} n^i s^j (f_j \nabla_s f_i - f_i \nabla_s f_j) dA, \end{aligned} \tag{16}$$

where s^j denotes a tangential vector.
Let us consider the right-hand sides of (3) and (16) by $D_1^2(t)$ and $2D_2^2(t)$, respectively. Observe that D_1 and D_2 are data terms. Then from (14)-(16), yields

$$\begin{aligned} &\|\nabla(\mathbf{u} - \mathbf{b})\|^2 + \frac{1}{2}\int_\Omega (1 + \alpha T + \beta T^2)(u_i - b_i)(u_i - b_i) d\mathbf{x} \\ &\leq \frac{3}{2}(\|T\|^2 + \|C\|^2) + \frac{3}{2}(1 + \beta\|T\|^2) D_1^2 + 3\mathcal{C}\alpha\|T\| D_2^2 \\ &\leq \frac{3}{2}(\|T\| + \|C\|)^2 + 3(\|T\| + \|C\|) D_3 + \frac{3}{2} D_3^2 + \frac{3}{2}\beta\|T\|^2 D_3^2 \\ &\leq \frac{3}{2}\Big((\|T\| + \|C\|) D_3\Big)^2 + \frac{3}{2}\beta\Big((\|T\| + \|C\|) D_3\Big)^2, \end{aligned} \tag{17}$$

where $D_3 = D_1$ if $D_1 \geq \mathcal{C}\alpha D_2^2$, otherwise $D_3 = \mathcal{C}\alpha D_2^4$. Then,

$$\|\mathbf{u} - \mathbf{b}\| \leq \sqrt{3(1 + \beta)}(\|T\| + \|C\|) D_3. \tag{18}$$

Subsequently, from (12)

$$\|\mathbf{u}\| \leq \sqrt{3(1 + \beta)}(\|T\| + \|C\|) D_3 + D_1, \tag{19}$$

and from (19), we conclude

$$\|\mathbf{u}\|^2 \leq 12(1 + \beta)(\|T\|^2 + \|C\|^2) D_3^2 + 2D_1^2. \tag{20}$$

Now, adopting the following expressions

$$\int_0^t \int_\Omega (T-G)\left(\frac{\partial T}{\partial s} + u_i \frac{\partial T}{\partial x_i} - \Delta T\right) \mathbf{dx}\mathbf{ds} = 0, \tag{21}$$

and

$$\int_0^t \int_\Omega (C-K)\left(\frac{\partial C}{\partial s} + u_i \frac{\partial C}{\partial x_i} - \Delta C\right) \mathbf{dx}\mathbf{ds} = 0, \tag{22}$$

where t is some number such that $0 \le t \le \mathcal{T}$. Next, integrating by part in (21) and employing the boundary condition $(7)_2$, to see that

$$\|T\|^2 + 2\int_0^t \|\nabla T\|^2 ds \le \|T_0\|^2 + 2(G,T) + 2\Big|(G_0,T_0)\Big| + 2\Big|\int_0^t (G_{,s},T)ds\Big|$$

$$+2\int_0^t \int_\Omega G u_i T_{,i} \mathbf{dx}\mathbf{ds} + 2\int_0^t \oint_{\partial\Omega} h\left(\frac{\partial G}{\partial n}\right) dAds + \int_0^t \oint_{\partial\Omega} |f| h^2 dAds,$$

by using Cauchy-Schwarz and arithmetic-geometric mean inequalities in above inequality, we have

$$\begin{aligned}\frac{1}{2}\|T\|^2 + \int_0^t \|\nabla T\|^2 ds &\le 2\|T_0\|^2 + 2\|G\|^2 + \|G_0\|^2 + \int_0^t \|G_{,s}\|^2 ds \\ &+ G_m^2 \int_0^t \|\mathbf{u}\|^2 ds + \int_0^t \|T\|^2 ds + \int_0^t \oint_{\partial\Omega} h^2 dAds \\ &+ \int_0^t \oint_{\partial\Omega} \left(\frac{\partial G}{\partial n}\right)^2 dAds + \int_0^t \oint_{\partial\Omega} |f| h^2 dAds,\end{aligned} \tag{23}$$

with inserting (20) in (23), yields

$$\begin{aligned}\|T\|^2 + 2\int_0^t \|\nabla T\|^2 ds &\le 4\|T_0\|^2 + 4\|G\|^2 + 2\|G_0\|^2 + 2\int_0^t \|G_{,s}\|^2 ds \\ &+ 2\left(1 + 6G_m^2[1+\beta]D_3^2\right)\int_0^t \|T\|^2 ds \\ &+ 2G_m^2 \int_0^t \left(6[1+\beta]\|C\|^2 D_3^2 + D_1^2\right) ds \\ &+ 2\int_0^t \oint_{\partial\Omega} \left(\frac{\partial G}{\partial n}\right)^2 dAds + 2\int_0^t \oint_{\partial\Omega} (1+|f|) h^2 dAds.\end{aligned} \tag{24}$$

Next, we return to the Equ. (22) and realize integral by parts with the aid of the Cauchy-Schwarz and arithmetic-geometric mean inequalities, to find

$$\begin{aligned}\frac{1}{4}\|C\|^2+\frac{3}{4}\int_0^t\|\nabla C\|^2ds\le\|C_0\|^2+\frac{1}{2}\|K_0\|^2+\|K\|^2+\frac{1}{2}\int_0^t\|C\|^2ds\\+K_m^2\int_0^t\|\mathbf{u}\|^2ds+\frac{1}{2}\int_0^t\oint_{\partial\Omega}(1+|f|)k^2dAds\\+\frac{1}{2}\int_0^t\|K_{,s}\|^2ds+\frac{1}{2}\int_0^t\oint_{\partial\Omega}\left(\frac{\partial K}{\partial n}\right)^2dAds,\end{aligned}\tag{25}$$

from (20), we derive (25) into

$$\begin{aligned}\|C\|^2+3\int_0^t\|\nabla C\|^2ds\le 4\|C_0\|^2+2\|K_0\|^2+4\|K\|^2\\+2\left(1+12K_m^2[1+\beta]D_3^2\right)\int_0^t\|C\|^2ds\\+4K_m^2\int_0^t\left(6[1+\beta]\|T\|^2D_3^2+D_1^2\right)ds+2\int_0^t\oint_{\partial\Omega}(1+|f|)k^2dAds\\+2\int_0^t\|K_{,s}\|^2ds+2\int_0^t\oint_{\partial\Omega}\left(\frac{\partial K}{\partial n}\right)^2dAds,\end{aligned}\tag{26}$$

where G_m and K_m are the maximum value of G and K, respectively, on $\partial\Omega\times(0,\mathcal{T})$. Next, combining (24) and (26), yields

$$\begin{aligned}&\|T\|^2+\|C\|^2+2\int_0^t\|\nabla T\|^2ds+3\int_0^t\|\nabla C\|^2ds\\&\le 2\left(1+6G_m^2[1+\beta]D_3^2+12K_m^2[1+\beta]D_3^2\right)\int_0^t\|T\|^2ds\\&\quad+2\left(1+6G_m^2[1+\beta]D_3^2+12K_m^2[1+\beta]D_3^2\right)\int_0^t\|C\|^2ds+E(t)\\&=\lambda\int_0^t\left(\|T\|^2+\|C\|^2\right)ds+E(t),\end{aligned}\tag{27}$$

where

$$\lambda=2\left(1+6G_m^2[1+\beta]D_3^2+12K_m^2[1+\beta]D_3^2\right),$$

and $E(t)$ is given by

$$\begin{aligned}E(t) &= 4\|T_0\|^2 + 2\|G_0\|^2 + 4\|C_0\|^2 + 2\|K_0\|^2 + 4\|G\|^2 + 4\|K\|^2 \\ &+ 2\left(G_m^2 + 2K_m^2\right)\int_0^t D_1^2 ds + 2\int_0^t \|G_{,s}\|^2 ds + 2\int_0^t \|K_{,s}\|^2 ds \\ &+ 2\int_0^t \oint_{\partial\Omega} (1+|f|)h^2 dA ds + 2\int_0^t \oint_{\partial\Omega} (1+|f|)k^2 dA ds \\ &+ 2\int_0^t \oint_{\partial\Omega} \left(\frac{\partial G}{\partial n}\right)^2 dA ds + 2\int_0^t \oint_{\partial\Omega} \left(\frac{\partial K}{\partial n}\right)^2 dA ds.\end{aligned} \tag{28}$$

For a function ϕ, which satisfies [15]

$$\begin{aligned}\Delta\phi &= 0, \quad \text{in } \Omega, \\ \phi &= M, \quad \text{on } \partial\Omega,\end{aligned} \tag{29}$$

then one may use a Rellich identity, [13], to denote c_1 and c_2 such that

$$\|\nabla\phi\|^2 + c_1 \oint_{\partial\Omega} \left(\frac{\partial\phi}{\partial n}\right)^2 dA \le c_2 \oint_{\partial\Omega} |\nabla_s M|^2 dA, \tag{30}$$

where ∇_s refers to the surface gradient over $\partial\Omega$. Also observe that

$$2(\psi\nabla\phi, \nabla\phi) + \|\phi\|^2 \le \psi_1 \oint_{\partial\Omega} M^2 dA, \tag{31}$$

where

$$\psi_1 = \max_{\partial\Omega} \left|\frac{\partial\psi}{\partial n}\right|,$$

with solving the boundary value problem,

$$\begin{aligned}\Delta\psi &= -1, \quad \text{in } \Omega, \\ \psi &= 0, \quad \text{on } \partial\Omega.\end{aligned} \tag{32}$$

Thus, (31) and (32) lead to bounds for $E(t)$ in terms of data. In fact, one may show

$$E(t) \le \tilde{D}(t), \tag{33}$$

so that

$$
\begin{aligned}
D(t) = {}& 4\|T_0\|^2 + 4\|C_0\|^2 + 2\left(G_m^2 + 2K_m^2\right)\int_0^t D_1^2 ds + 2\psi_1 \oint_{\partial\Omega} h_0^2 dA \\
&+ 2\psi_1 \oint_{\partial\Omega} k_0^2 dA + 4\psi_1 \oint_{\partial\Omega} h^2 dA + 4\psi_1 \oint_{\partial\Omega} k^2 dA \\
&+ 2\psi_1 \int_0^t \oint_{\partial\Omega} h_{,\eta}^2 dA d\eta + 2\psi_1 \int_0^t \oint_{\partial\Omega} k_{,\eta}^2 dA d\eta \\
&+ 2\int_0^t \oint_{\partial\Omega} (1+|f|) h^2 dA d\eta + 2\int_0^t \oint_{\partial\Omega} (1+|f|) k^2 dA d\eta \\
&+ \frac{2c_2}{c_1}\int_0^t \oint_{\partial\Omega} |\nabla_s h|^2 dA d\eta + \frac{2c_2}{c_1}\int_0^t \oint_{\partial\Omega} |\nabla_s k|^2 dA d\eta.
\end{aligned}
\tag{34}
$$

From (28) which leads us to

$$
\mathcal{F}' - \lambda\mathcal{F} \le \tilde{D}(t), \tag{35}
$$

where we have introduced the function $\mathcal{F}$, which is defined by

$$
\mathcal{F}(t) = \int_0^t \left(\|T\|^2 + \|C\|^2\right) ds.
$$

Upon assuming

$$
\tilde{D}_1(t) = \int_0^t \tilde{D}(s) e^{\lambda(t-s)} ds, \tag{36}
$$

one integrates (35) to show

$$
\mathcal{F}(t) \le \tilde{D}_1(t). \tag{37}
$$

Furthermore, setting $\tilde{D}_2 = \lambda\tilde{D}_1 + \tilde{D}$, one uses (3) to find

$$
\|T\|^2 + \|T\|_4^4 + \|C\|^2 \le \tilde{D}_2(t). \tag{38}
$$

Then, (27), (37) and (38) give

$$
\begin{gathered}
\int_0^t \|T\|^2 ds \le \tilde{D}_1, \quad \int_0^t \|C\|^2 ds \le \tilde{D}_1, \\
\|T\|^2 \le \tilde{D}_2, \quad \|C\|^2 \le \tilde{D}_2, \\
\int_0^t \|\nabla T\|^2 ds \le \frac{1}{2}\tilde{D}_2, \quad \int_0^t \|\nabla C\|^2 ds \le \frac{1}{3}\tilde{D}_2.
\end{gathered}
\tag{39}
$$

The next step is to derive a bound for $\sup_{\Omega\times[0,T]}|T|$, from

$$\int_0^t\int_\Omega (T^{2r-1}-F)\left(\frac{\partial T}{\partial s}+u_i\frac{\partial T}{\partial x_i}-\Delta T\right)dxds=0. \tag{40}$$

Integrating by parts, we see that

$$\begin{aligned}\int_\Omega T^{2r}dx+\frac{2(2r-1)}{r}\int_0^t\int_\Omega \nabla T^r\nabla T^r dxds&=\int_\Omega T_0^{2r}dx+2r(T,F)-2r(T_0,F_0)\\ &-2r\int_0^t\int_F TF_{,s}dxds+2r\int_0^t\int_\Omega T_{,i}Fu_i dxds\\ &+2r\int_0^t\oint_{\partial\Omega} h\frac{\partial F}{\partial n}dAds-\int_0^t\oint_{\partial\Omega} fT^{2r}dAds\\ \le\int_\Omega T_0^{2r}dx+2r\left(\int_0^t\|F_{,s}\|^2ds\int_0^t\|T\|^2ds\right)^{1/2}&+2r(\|T\|\|F\|+\|T_0\|\|F_0\|)\\ +4rh_m^{2r-1}\left(\int_0^t\left[6[1+\beta](\|T\|^2+\|C\|^2)D_3^2+D_1^2\right]ds\int_0^t\|\nabla T\|^2ds\right)^{1/2}&\\ +2r\left(\int_0^t\oint_{\partial\Omega}h^2dAds\int_0^t\oint_{\partial\Omega}\left[\frac{\partial F}{\partial n}\right]^2dAds\right)^{1/2}&+\int_0^t\oint_{\partial\Omega}|f|h^{2r}dAds.\end{aligned} \tag{41}$$

Using arithmetic-geometric mean inequality and (30), (31) with (39), yield

$$\begin{aligned}\int_\Omega T^{2r}dx\le\int_\Omega T_0^{2r}dx+2r(\sqrt{\tilde D_2}+\|T_0\|)\left(\psi_1\oint_{\partial\Omega}h^{4r-2}dA\right)^{1/2}&\\ +2r\left(\tilde D_1\psi_1\int_0^t\oint_{\partial\Omega}\left[h_{,\eta}^{2r-1}\right]^2dAd\eta\right)^{1/2}+rh_m^{2r-1}\left(2\int_0^t D_1^2d\eta+24[1+\beta]\tilde D_1+2\tilde D_2\right)&\\ +2r\left(\frac{c_2}{c_1}\int_0^t\oint_{\partial\Omega}\left[\nabla_\eta h\right]^2dAd\eta\int_0^t\oint_{\partial\Omega}h^2dAd\eta\right)^{1/2}+\int_0^t\oint_{\partial\Omega}|f|h^{2r}dAd\eta.&\end{aligned} \tag{42}$$

Then, from further application for Cauchy-Schwarz, we get

$$\left(\oint_{\partial\Omega}h^{4r-2}dA\right)^{1/2}\le h_m^{2r-1}\left(\oint_{\partial\Omega}dA\right)^{1/2}=\frac{h_m^{2r}}{h_m}\sqrt{[m(\partial\Omega)]}, \tag{43}$$

$$\left(\int_0^t\oint_{\partial\Omega}h^{4r-4}h_{,\eta}^2dAd\eta\right)^{1/2}\le\frac{h_m^{2r}}{h_m^2}\left(\int_0^t\oint_{\partial\Omega}h_{,\eta}^2dAd\eta\right)^{1/2}, \tag{44}$$

and

$$\left(\int_0^t \oint_{\partial\Omega} h^{4r-4}\left[\nabla_\eta h\right]^2 dA d\eta\right)^{1/2} \leq \frac{h_m^{2r}}{h_m^2}\left(\int_0^t \oint_{\partial\Omega} \left[\nabla_\eta h\right]^2 dA d\eta\right)^{1/2}, \tag{45}$$

where $m(\partial\Omega)$ is the surface measure of $\partial\Omega$. Employing (43) − (45) in (3), lead to

$$\begin{aligned} \int_\Omega T^{2r} dx \leq & \int_\Omega T_0^{2r} dx + \frac{2rh_m^{2r}}{h_m}(\sqrt{\tilde{D}_2} + \|T_0\|)\sqrt{\psi_1[m(\partial\Omega)]} \\ & + \frac{2r(2r-1)h_m^{2r}}{h_m}\left(\tilde{D}_1\psi_1 \int_0^t \oint_{\partial\Omega} h_{,\eta}^2 dA d\eta\right)^{1/2} \\ & + \frac{2rh_m^{2r}}{h_m}\left(\int_0^t D_1^2 d\eta + 12[1+\beta]\tilde{D}_1 + \tilde{D}_2\right) + h_m^{2r}\int_0^t \oint_{\partial\Omega} |f| dA d\eta \\ & + \frac{2r(2r-1)h_m^{2r}}{h_m^2}\left(\frac{c_2}{c_1}\int_0^t \oint_{\partial\Omega} \left[\nabla_\eta h\right]^2 dA d\eta \int_0^t \oint_{\partial\Omega} h^2 dA d\eta\right)^{1/2}. \end{aligned} \tag{46}$$

After taking the power $1/2r$ of (46), we obtain

$$\|T\|_{2r} \leq \left(\|T_0\|_{2r}^{2r} + h_m^{2r}\sum_{i=1}^{5}\gamma_i\right)^{1/2r}, \tag{47}$$

where γ_i may be obtained from (46), here

$$h_m = \max_{\partial\Omega\times[0,\mathcal{T}]} |h|.$$

Taking the limit $r \to \infty$, yields a priori bound

$$\sup_{\Omega\times[0,\mathcal{T}]} |T| \leq \max\{|T_0|_m, \sup_{[0,\mathcal{T}]} h_m\}, \tag{48}$$

where

$$|T_0|_m = \max_\Omega |T_0|.$$

Finally, we have to find a bound for $\sup_{\Omega\times[0,\mathcal{T}]} |C|$. Now, form the expression

$$\int_0^t \int_\Omega (C^{2r-1} - H)\left(\frac{\partial C}{\partial s} + u_i\frac{\partial C}{\partial x_i} - \Delta C\right) dx ds = 0. \tag{49}$$

Following the same manner in (40)-(48), we have that

$$\sup_{\Omega\times[0,\mathcal{T}]} |C| \leq \max\{|C_0|_m, \sup_{[0,\mathcal{T}]} k_m\}, \tag{50}$$

where

$$|C_0|_m = \max_\Omega |C_0|.$$

4 Continuous dependence on α

To investigate continuous dependence on the viscosity coefficient α in (1), we let (u_i, T, C_1, p) and (v_i, S, C_2, q) be solutions to (1) − (6) for the same data functions f, h and T_0, but for different viscosity coefficients, α_1 and α_2, respectively. Define the difference solution (w_i, θ, ϕ, π) by

$$w_i = u_i - v_i, \quad \theta = T - S, \quad \phi = C_1 - C_2, \quad \pi = p - q, \quad \alpha = \alpha_1 - \alpha_2. \tag{51}$$

Then from (1) − (6), this solution satisfies the boundary-initial-value problem

$$\begin{gathered}
-\triangle w_i + (1 + \alpha_2 S + \beta S^2) w_i = -\alpha_2 \theta u_i - \alpha T u_i - \beta (T + S)\theta u_i - \pi_{,i} + g_i \theta + \mathcal{I}_i \phi, \\
w_{i,i} = 0, \\
\frac{\partial \theta}{\partial t} + w_i \frac{\partial S}{\partial x_i} + u_i \frac{\partial \theta}{\partial x_i} = \Delta T, \\
\frac{\partial \phi}{\partial t} + w_i \frac{\partial C_2}{\partial x_i} + u_i \frac{\partial \phi}{\partial x_i} = \Delta C_2, \\
w_i = \theta = \phi = 0 \ \text{ on } \ \partial\Omega \times [0, \mathcal{T}], \\
\theta(x, 0) = \phi(x, 0) = 0, \quad x \in \Omega.
\end{gathered} \tag{52}$$

Next, we multiply Equ. $(52)_1$ by w_i and integrate over Ω.

$$\begin{aligned}
\|\nabla \mathbf{w}\|^2 &+ \int_\Omega (1 + \alpha_2 S + \beta S^2) w_i w_i \, d\mathbf{x} = -\alpha \int_\Omega T u_i w_i \, d\mathbf{x} - \alpha_2 \int_\Omega \theta u_i w_i \, d\mathbf{x} \\
&- \beta \int_\Omega (T + S)\theta u_i w_i \, d\mathbf{x} + g_i(\theta, w_i) + \mathcal{I}_i(\phi, w_i) \\
&\leq \alpha T_m \|\mathbf{u}\| \|\mathbf{w}\| + \|\theta\| \|\mathbf{w}\| + \|\phi\| \|\mathbf{w}\| \\
&+ \Big(\alpha_2 + \beta [T_m + S_m]\Big) \|\theta\| \Big(\int_\Omega u_i w_i u_j w_j \, d\mathbf{x} \Big)^{1/2},
\end{aligned} \tag{53}$$

where, T_m and S_m are the maximum value of T and S, respectively. The last term in (53) is bounded as the following, [14], i.e.

$$\begin{aligned}
\int_\Omega u_i w_i u_j w_j \, d\mathbf{x} &\leq \frac{2}{\pi} \Big(\|\nabla \mathbf{w}\|^2 \oint_{\partial\Omega} f_i f_i \, dA + \|\nabla \mathbf{u}\|^2 \|\mathbf{w}\| \|\nabla \mathbf{w}\| \Big) \\
&\leq \frac{2}{\pi} \|\nabla \mathbf{w}\|^2 \Big(\oint_{\partial\Omega} f_i f_i \, dA + \kappa^{-1/2} \|\nabla \mathbf{u}\|^2 \Big),
\end{aligned} \tag{54}$$

where κ is the Poincaré constant for Ω.
To employ (53) and (54) we need data bounds for $\|\mathbf{u}\|$ and $\|\nabla \mathbf{u}\|$, thus, from

the triangle inequality

$$\|\nabla \mathbf{u}\| \leq \|\nabla(\mathbf{u}-\mathbf{b})\| + \|\nabla \mathbf{b}\|,$$

and then from the inequality before (14) and (16) we find

$$\|\nabla \mathbf{u}\|^2 \leq \left(\sqrt{6(1+\beta)\tilde{D}_2}D_3 + \sqrt{2}D_2\right)^2. \tag{55}$$

Hence, return to (53) we conclude that

$$\begin{aligned} &\|\nabla \mathbf{w}\|^2 + \int_\Omega (1+\alpha_2 S + \beta S^2) w_i w_i \, d\mathbf{x} \\ &\leq \alpha T_m D_4 \|\mathbf{w}\| + \|\theta\|\|\mathbf{w}\| + \|\phi\|\|\mathbf{w}\| + D_6\|\theta\|\|\nabla \mathbf{w}\|, \end{aligned} \tag{56}$$

where

$$D_4 = \sqrt{6(1+\beta)\tilde{D}_2}D_3 + D_1, \quad D_5 = \sqrt{6(1+\beta)\tilde{D}_2}D_3 + \sqrt{2}D_2,$$

and

$$D_6 = \sqrt{\frac{2}{\pi}}\left(\alpha_2 + \beta[T_m + S_m]\right)\left(\oint_{\partial\Omega} f_i f_i \, dA + \kappa^{-1/2} D_5\right)^{1/2}.$$

Thus, from (56), we may derive

$$\begin{aligned} \|\nabla \mathbf{w}\|^2 + \mu^* \|\mathbf{w}\|^2 &\leq \frac{3\alpha^2 T_m^2 D_4^2}{\mu^*} + \frac{3}{\mu^*}(\|\theta\|^2 + \|\phi\|^2) + D_6^2\|\theta\|^2 \\ &\leq \frac{3\alpha^2 T_m^2 D_4^2}{\mu^*} + \left(\frac{3}{\mu^*} + D_6^2)(\|\theta\|^2 + \|\phi\|^2\right), \end{aligned} \tag{57}$$

where

$$0 < \mu^* \leq 1 + \alpha_2 S + \beta S^2.$$

Moreover, multiplying $(52)_3$ by θ and $(52)_4$ by ϕ, with integrating over Ω, we can see that

$$\frac{d}{dt}\|\theta\|^2 \leq \frac{S_m^2}{2}\|\mathbf{w}\|^2, \tag{58}$$

and

$$\frac{d}{dt}\|\phi\|^2 \leq \frac{C_{2m}^2}{2}\|\mathbf{w}\|^2. \tag{59}$$

Employing (58) and (59) with integrating the result, yields

$$\begin{aligned}\|\theta\|^2 + \|\phi\|^2 &\leq \frac{1}{2}(S_m^2 + C_{2m}^2)\int_0^t \|\mathbf{w}\|^2 ds \\ &\leq \frac{1}{2\mu^*}(S_m^2 + C_{2m}^2)\int_0^t (\|\nabla \mathbf{w}\|^2 + \mu^*\|\mathbf{w}\|^2) ds,\end{aligned} \tag{60}$$

where C_{2m} is the maximum value of C_2. Next, substituting (60) in (57), we have

$$\|\nabla \mathbf{w}\|^2 + \mu^*\|\mathbf{w}\|^2 \leq \frac{3\alpha^2 T_m^2 D_4^2}{\mu^*} + \mathcal{J}\int_0^t (\|\nabla \mathbf{w}\|^2 + \mu^*\|\mathbf{w}\|^2) ds, \tag{61}$$

here

$$\mathcal{J} = \frac{1}{2\mu^*}(S_m^2 + C_{2m}^2)(\frac{3}{\mu^*} + D_6^2).$$

By integration (61), we find

$$\int_0^t (\|\nabla \mathbf{w}\|^2 + \mu^*\|\mathbf{w}\|^2) ds \leq \frac{3\alpha^2 T_m^2 D_4^2}{\mu^*} t + \mathcal{J}\int_0^t (t-s)(\|\nabla \mathbf{w}\|^2 + \mu^*\|\mathbf{w}\|^2) ds, \tag{62}$$

thus, from (62) we obtain

$$\int_0^t (t-s)(\|\nabla \mathbf{w}\|^2 + \mu^*\|\mathbf{w}\|^2) ds \leq \alpha^2 \mathcal{J}_2(t), \tag{63}$$

and

$$\int_0^t (\|\nabla \mathbf{w}\|^2 + \mu^*\|\mathbf{w}\|^2) ds \leq \alpha^2 \mathcal{J}_3(t), \tag{64}$$

where

$$\mathcal{J}_2(t) = \int_0^t \mathcal{J}_1(s) e^{\mathcal{J}(t-s)} ds, \quad \mathcal{J}_1(t) = \frac{3T_m^2 D_4^2}{\mu^*} t \quad \text{and} \quad \mathcal{J}_3(t) = \mathcal{J}_1 + \mathcal{J}\mathcal{J}_2.$$

Finally, inserting (64) in (60) we also find

$$\|\theta\|^2 + \|\phi\|^2 \leq \frac{1}{2\mu^*}\alpha^2 \mathcal{J}_3 (S_m^2 + C_{2m}^2). \tag{65}$$

Inequality (65) yields continuous dependence on the viscosity coefficient α and it is truly a priori such that the coefficients of α^2 depend only on boundary and initial conditions.

5 Continuous dependence on β

In this section, we study the continuous dependence on the coefficient β. We first, let (u_i, T, C_1, p) and (v_i, S, C_2, q) be solutions to (1) − (6) for the same data functions f, h and T_0, but for different viscosity coefficients, β_1 and β_2, respectively. Hence, define the difference solution (w_i, θ, ϕ, π) as

$$w_i = u_i - v_i, \quad \theta = T - S, \quad \phi = C_1 - C_2, \quad \pi = p - q, \quad \beta = \beta_1 - \beta_2. \tag{66}$$

Thus, from (1) − (6) this solution satisfies the boundary-initial-value problem

$$\begin{gathered}
-\triangle w_i + (1 + \alpha S + \beta_2 S^2) w_i = -\alpha\theta u_i - \beta S^2 u_i - \beta_1 (T+S)\theta u_i - \pi_{,i} + g_i\theta + \mathcal{I}_i\phi,\\
w_{i,i} = 0,\\
\frac{\partial\theta}{\partial t} + w_i\frac{\partial S}{\partial x_i} + u_i\frac{\partial\theta}{\partial x_i} = \Delta T,\\
\frac{\partial\phi}{\partial t} + w_i\frac{\partial C_2}{\partial x_i} + u_i\frac{\partial\phi}{\partial x_i} = \Delta C_2,\\
w_i = \theta = \phi = 0 \quad \text{on} \quad \partial\Omega \times [0, \mathcal{T}],\\
\theta(x,0) = \phi(x,0) = 0, \quad x \in \Omega.
\end{gathered} \tag{67}$$

Now, multiply Equ. $(67)_1$ by w_i and integrate over Ω, we have

$$\begin{aligned}
\|\nabla \mathbf{w}\|^2 + \int_\Omega (1 + \alpha S + \beta_2 S^2) w_i w_i \, d\mathbf{x} &= -\beta \int_\Omega S^2 u_i w_i \, d\mathbf{x} - \beta_1 \int_\Omega (T+S)\theta u_i w_i \, d\mathbf{x}\\
&\quad -\alpha \int_\Omega \theta u_i w_i \, d\mathbf{x} + g_i(\theta, w_i) + \mathcal{I}_i(\phi, w_i)\\
&\le \beta S_m^2 \|\mathbf{u}\|\|\mathbf{w}\| + \|\theta\|\|\mathbf{w}\| + \|\phi\|\|\mathbf{w}\|\\
&\quad + \Big(\alpha + \beta_1[T_m + S_m]\Big)\|\theta\|\Big(\int_\Omega u_i w_i u_j w_j \, d\mathbf{x}\Big)^{1/2}.
\end{aligned} \tag{68}$$

We are now performing a similar manner starting from (54), to obtain

$$\|\nabla \mathbf{w}\|^2 + \nu^*\|\mathbf{w}\|^2 \le \frac{3\beta^2 S_m^4 D_4^2}{\nu^*} + \frac{1}{2\nu^*}(S_m^2 + C_{2\,m}^2)(\frac{3}{\nu^*} + D_7^2)\int_0^t (\|\nabla \mathbf{w}\|^2 + \nu^*\|\mathbf{w}\|^2)\, ds, \tag{69}$$

where

$$0 < \nu^* \le 1 + \alpha S + \beta_2 S^2$$

and

$$D_7 = \sqrt{\frac{2}{\pi}}\Big(\alpha + \beta_1[T_m + S_m]\Big)\Big(\oint_{\partial\Omega} f_i f_i dA + \kappa^{-1/2} D_5\Big)^{1/2}.$$

By integrating (69), we can find

$$\int_0^t (\|\nabla \mathbf{w}\|^2 + \nu^* \|\mathbf{w}\|^2) ds \le \beta^2 \mathcal{K}_1 t + \mathcal{K}_2 \int_0^t (t-s)(\|\nabla \mathbf{w}\|^2 + \nu^* \|\mathbf{w}\|^2) ds, \quad (70)$$

where

$$\mathcal{K}_1 = \frac{3S_m^4 D_4^2}{\nu^*} \quad \text{and} \quad \mathcal{K}_2 = \frac{1}{2\nu^*}(S_m^2 + C_{2\,m}^2)(\frac{3}{\nu^*} + D_7^2).$$

Thus, from (70) we have

$$\int_0^t (t-s)(\|\nabla \mathbf{w}\|^2 + \nu^* \|\mathbf{w}\|^2) ds \le \beta^2 \int_0^t \mathcal{K}_1 t e^{\mathcal{K}_2 (t-s)} ds, \quad (71)$$

and

$$\int_0^t (\|\nabla \mathbf{w}\|^2 + \nu^* \|\mathbf{w}\|^2) ds \le \beta^2 \Big(\mathcal{K}_1 t + \mathcal{K}_2 \int_0^t \mathcal{K}_1 t e^{\mathcal{K}_2 (t-s)} ds\Big). \quad (72)$$

Further, (71) leads to

$$\|\theta\|^2 + \|\phi\|^2 \le \frac{1}{2\nu^*}\beta^2 (S_m^2 + C_{2\,m}^2)\Big(\mathcal{K}_1 t + \mathcal{K}_2 \int_0^t \mathcal{K}_1 t e^{\mathcal{K}_2 (t-s)} ds\Big). \quad (73)$$

Obviously, (73) demonstrates continuous dependence on the viscosity coefficient β and it is actually a priori such that the coefficients of β^2 depend only on boundary and initial data

6 Convergence to the constant viscosity α

Let now (u_i, T, C_1, p) and (v_i, S, C_2, q) be the solutions that satisfy the following boundary-initial-value problems:

$$\begin{aligned} -\Delta u_i + (1 + \alpha T + \beta T^2) u_i &= -\frac{\partial p}{\partial x_i} + g_i T + \mathcal{I}_i C_1, \\ u_{i,i} &= 0, \\ \frac{\partial T}{\partial t} + u_i \frac{\partial T}{\partial x_i} &= \Delta T, \\ \frac{\partial C_1}{\partial t} + u_i \frac{\partial C_1}{\partial x_i} &= \Delta C_1, \end{aligned} \quad (74)$$

in $\Omega \times (0, \mathcal{T})$, with

$$
\begin{aligned}
u_i &= f(x,t), \quad \text{on} \quad \partial\Omega \times (0, \mathcal{T}), \\
T(x,t) = h(x,t), \quad C_1(x,t) &= k(x,t), \quad x \quad \text{on} \quad \partial\Omega, \quad t \in (0, \mathcal{T}),
\end{aligned} \tag{75}
$$

$$
\begin{aligned}
-\Delta v_i + (1 + \beta S^2) v_i &= -\frac{\partial q}{\partial x_i} + g_i S + \mathcal{I}_i C_2, \\
v_{i,i} &= 0, \\
\frac{\partial S}{\partial t} + v_i \frac{\partial S}{\partial x_i} &= \Delta S, \\
\frac{\partial C_2}{\partial t} + v_i \frac{\partial C_2}{\partial x_i} &= \Delta C_2,
\end{aligned} \tag{76}
$$

in $\Omega \times (0, \mathcal{T})$, and

$$
\begin{aligned}
v_i &= f(x,t), \quad \text{on} \quad \partial\Omega \times (0, \mathcal{T}), \\
S(x,t) = h(x,t), \quad C_2(x,t) &= k(x,t), \quad x \quad \text{on} \quad \partial\Omega, \quad t \in (0, \mathcal{T}].
\end{aligned} \tag{77}
$$

The variables w_i, θ, ϕ and π were introduced in (51) which satisfy the boundary-initial-value problem

$$
\begin{aligned}
-\triangle w_i + (1 + \beta S^2) w_i &= -\pi_{,i} + g_i \theta + \mathcal{I}_i \phi - \alpha T u_i - \beta (T + S) \theta u_i, \\
w_{i,i} &= 0, \\
\frac{\partial \theta}{\partial t} + w_i \frac{\partial S}{\partial x_i} + u_i \frac{\partial \theta}{\partial x_i} &= \Delta T, \\
\frac{\partial \phi}{\partial t} + w_i \frac{\partial C_2}{\partial x_i} + u_i \frac{\partial \phi}{\partial x_i} &= \Delta C_2, \\
w_i = \theta = \phi &= 0 \quad \text{on} \quad \partial\Omega \times (0, \mathcal{T}), \\
\theta(x, 0) = \phi(x, 0) &= 0, \quad x \in \Omega.
\end{aligned} \tag{78}
$$

Next, we start with multiplying $(78)_1$ by w_i and integrating over Ω, with employing the Cauchy-Schwarz and arithmetic-geometric-mean inequalities, to derive

$$
\begin{aligned}
\|\nabla \mathbf{w}\|^2 + \int_\Omega (1 + \beta S^2) w_i w_i \, \mathbf{dx} &\le \alpha T_m \|\mathbf{w}\| \|\mathbf{u}\| + \|\theta\| \|\mathbf{w}\| + \|\phi\| \|\mathbf{w}\| \\
&+ \beta (T_m + S_m) \|\theta\| \left(\int_\Omega u_i w_i u_j w_j \mathbf{x} \right)^{1/2},
\end{aligned} \tag{79}
$$

Upon using (54) with data bounds for $\|\mathbf{u}\|$ and $\|\nabla \mathbf{u}\|$, yields

$$
\|\nabla \mathbf{w}\|^2 + \gamma^* \|\mathbf{w}\|^2 \le \alpha T_m D_4 \|\mathbf{w}\| + \|\theta\| \|\mathbf{w}\| + \|\phi\| \|\mathbf{w}\| + D_8 \|\theta\| \|\nabla \mathbf{w}\|, \tag{80}
$$

where

$$0<\gamma^{*}\leq 1+\beta S^{2}\quad\text{and}\quad D_{8}=\sqrt{\frac{2}{\pi}}\beta(T_{m}+S_{m})\left(\oint_{\partial\Omega}f_{i}f_{i}dA+\kappa^{-1/2}D_{5}\right)^{1/2}.$$

Thus, from (80), after employing arithmetic-geometric mean inequality, we conclude

$$\|\nabla\mathbf{w}\|^{2}+\gamma^{*}\|\mathbf{w}\|^{2}\leq\frac{3\gamma^{2}T_{m}D_{4}^{2}}{\gamma^{*}}+(\frac{3}{\gamma^{*}}+D_{8}^{2})(\|\theta\|+\|\phi\|).\tag{81}$$

Multiply $(78)_3$ and $(78)_4$ by θ and ϕ, respectively with integrating over Ω, to derive

$$\|\theta\|^{2}+\|\phi\|^{2}\leq\frac{1}{2\alpha^{*}}(S_{m}^{2}+C_{2\,m}^{2})\int_{0}^{t}(\|\nabla\mathbf{w}\|^{2}+\gamma^{*}\|\mathbf{w}\|^{2})ds.\tag{82}$$

Inserting (82) in (81), gives

$$\|\nabla\mathbf{w}\|^{2}+\|\mathbf{w}\|^{2}\leq\frac{3\alpha^{2}T_{m}D_{4}^{2}}{\gamma^{*}}+\frac{1}{2\gamma^{*}}(\frac{3}{\gamma^{*}}+D_{8}^{2})(S_{m}^{2}+C_{2\,m}^{2})\int_{0}^{t}(\|\nabla\mathbf{w}\|^{2}+\gamma^{*}\|\mathbf{w}\|^{2})ds.\tag{83}$$

Finally, integration (83), yields

$$\int_{0}^{t}(\|\nabla\mathbf{w}\|^{2}+\gamma^{*}\|\mathbf{w}\|^{2})ds\leq\frac{6\gamma^{*}\alpha^{2}T_{m}^{2}D_{4}^{2}\exp\left(\frac{1}{2\gamma^{*}}(\frac{3}{\gamma^{*}}+D_{8}^{2})(S_{m}^{2}+C_{2\,m}^{2})t\right)}{(\frac{3}{\gamma^{*}}+D_{8}^{2})(S_{m}^{2}+C_{2\,m}^{2})}.\tag{84}$$

Evidently, (84) demonstrates a convergence of u_i to v_i as $\alpha\to 0$, in the indicated measure. By combining (84) and (83), we also obtain a convergence of w_i in $L^2(\Omega)$ and $H^1(\Omega)$ norms , and from (82) we may obtain a convergence of θ and ϕ in the $L^2(\Omega)$ norm.

7 Convergence to the constant viscosity β

Let (u_i, T, C_1, p) and (v_i, S, C_2, q) be solutions that satisfy the following boundary-initial-value problems:

$$\begin{aligned}-\Delta u_{i}+(1+\alpha T+\beta T^{2})u_{i}&=-\frac{\partial p}{\partial x_{i}}+g_{i}T+\mathcal{I}_{i}C_{1},\\ u_{i,i}&=0,\\ \frac{\partial T}{\partial t}+u_{i}\frac{\partial T}{\partial x_{i}}&=\Delta T,\\ \frac{\partial C_{1}}{\partial t}+u_{i}\frac{\partial C_{1}}{\partial x_{i}}&=\Delta C_{1},\end{aligned}\tag{85}$$

in $\Omega \times (0, \mathcal{T})$, with

$$\begin{aligned} u_i = f(x,t), \quad \text{on} \quad \partial\Omega \times (0,\mathcal{T}), \\ T(x,t) = h(x,t), \quad C_1(x,t) = k(x,t), \quad x \quad \text{on} \quad \partial\Omega, \quad t \in (0,\mathcal{T}), \end{aligned} \tag{86}$$

$$\begin{aligned} -\Delta v_i + (1+\alpha S)v_i &= -\frac{\partial q}{\partial x_i} + g_i S + \mathcal{I}_i C_2, \\ v_{i,i} &= 0, \\ \frac{\partial S}{\partial t} + v_i \frac{\partial S}{\partial x_i} &= \Delta S, \\ \frac{\partial C_2}{\partial t} + v_i \frac{\partial C_2}{\partial x_i} &= \Delta C_2, \end{aligned} \tag{87}$$

in $\Omega \times (0, \mathcal{T})$, with

$$\begin{aligned} v_i = f(x,t), \quad \text{on} \quad \partial\Omega \times (0,\mathcal{T}), \\ S(x,t) = h(x,t), \quad C_2(x,t) = k(x,t), \quad x \quad \text{on} \quad \partial\Omega, \quad t \in (0,\mathcal{T}). \end{aligned} \tag{88}$$

The variables w_i, θ, ϕ and π were introduced in (51) which satisfy the boundary-initial-value problem

$$\begin{aligned} -\triangle w_i + (1+\alpha S)w_i &= -\pi_{,i} + g_i\theta + \mathcal{I}_i\phi - \beta T^2 u_i - \alpha\theta u_i, \\ w_{i,i} &= 0, \\ \frac{\partial \theta}{\partial t} + w_i \frac{\partial S}{\partial x_i} + u_i \frac{\partial \theta}{\partial x_i} &= \Delta T, \\ \frac{\partial \phi}{\partial t} + w_i \frac{\partial C_2}{\partial x_i} + u_i \frac{\partial \phi}{\partial x_i} &= \Delta C_2. \\ w_i = \theta = \phi = 0 \quad &\text{on} \quad \partial\Omega \times (0,\mathcal{T}), \\ \theta(x,0) = \phi(x,0) = 0, \quad &x \in \Omega. \end{aligned} \tag{89}$$

Now, we multiply $(89)_1$ by w_i and integrate over Ω, with the aid of the Cauchy-Schwarz inequality to obtain

$$\begin{aligned} \|\nabla \mathbf{w}\|^2 + \int_\Omega (1+\alpha S)w_i w_i \mathbf{dx} \leq \beta T_m^2 \|\mathbf{w}\|\|\mathbf{u}\| + \|\theta\|\|\mathbf{w}\| \\ + \|\phi\|\|\mathbf{w}\| + \alpha\|\theta\| \left(\int_\Omega u_i w_i u_j w_j \mathbf{x} \right)^{1/2}, \end{aligned} \tag{90}$$

Then, with further employing for (54) and data bounds for $\|\mathbf{u}\|$ and $\|\nabla \mathbf{u}\|$ with application of the arithmetic-geometric mean inequality to see that

$$\|\nabla \mathbf{w}\|^2 + \delta^* \|\mathbf{w}\|^2 \leq \frac{3\beta^2 T_m^4 D_4^2}{\delta^*} + (\frac{3}{\delta^*} + D_9^2)(\|\theta\|^2 + \|\phi\|^2), \tag{91}$$

here

$$0 < \delta^* \leq 1 + \alpha S \quad \text{and} \quad D_9 = \alpha\sqrt{\frac{2}{\pi}}\left(\oint_{\partial\Omega} f_i f_i dA + \kappa^{-1/2} D_5\right)^{1/2}.$$

Since

$$\|\theta\|^2 + \|\phi\|^2 \leq \frac{1}{2\delta^*}(S_m^2 + C_{2\,m}^2)\int_0^t (\|\nabla \mathbf{w}\|^2 + \delta^*\|\mathbf{w}\|^2)ds. \tag{92}$$

Substituting (92) in (91) we observe that

$$\|\nabla \mathbf{w}\|^2 + \delta^*\|\mathbf{w}\|^2 \leq \frac{3\beta^2 T_m^4 D_4^2}{\delta^*} + \frac{1}{2\delta^*}(3 + D_9^2)(S_m^2 + C_{2\,m}^2)\int_0^t (\|\nabla \mathbf{w}\|^2 + \delta^*\|\mathbf{w}\|^2)ds. \tag{93}$$

Finally, integrating (93), yields

$$\int_0^t (\|\nabla \mathbf{w}\|^2 + \delta^*\|\mathbf{w}\|^2)ds \leq \frac{6\delta^*\beta^2 T_m^4 D_4^2 \exp\left(\frac{1}{2\delta^*}(\frac{3}{\delta^*} + D_9^2)(S_m^2 + C_{2\,m}^2)t\right)}{(\frac{3}{\delta^*} + D_9^2)(S_m^2 + C_{2\,m}^2)}. \tag{94}$$

We can see that in (94), the convergence is demonstrated with u_i to v_i as $\beta \to 0$, in the indicated measure. By combining (93) and (94), we also obtain a convergence of w_i in $L^2(\Omega)$ and $H^1(\Omega)$ norm, and from (91) we may obtain a convergence of θ and ϕ in the $L^2(\Omega)$ norm.

8 Conclusions

In this current paper, the problem of double diffusive convection in a Brinkman model has been considered when the viscosity varies with temperature. Specifically, in this work we presented a priori bounds with coefficients that depend only on boundary data, initial data and we demonstrated that the solution depends continuously on changes in the viscosity coefficients α and β, respectively. Moreover, the convergence result is established on Brinkman model when the variable viscosity is allowed to approach to a constant viscosity.

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