Acta Univ. Sapientiae, Mathematica, **14**, 1 (2022) 147–159

DOI: 10.2478/ausm-2022-0010

On a metric topology on the set of bivariate means

Mustapha Raïssouli
Department of Mathematics,
Faculty of Science, Moulay Ismail
University, Meknes, Morocco.
email: raissouli.mustapha@gmail.com

Mohamed Chergui
Department of Mathematics,
CRMEF Kénitra,
EREAM Team, LaREAMI-Lab,
Kénitra, Morocco.
email: chergui_m@yahoo.fr

Abstract. In this paper, we define a distance d on the set $\mathcal{M}$ of bivariate means. We show that $(\mathcal{M}, \mathrm{d})$ is a bounded complete metric space which is not compact. Other algebraic and topological properties of $(\mathcal{M}, \mathrm{d})$ are investigated as well.

1 Introduction

A bivariate mean m, as proposed by Cauchy in [5], is a map from $(0, \infty) \times (0, \infty)$ into $(0, \infty)$ satisfying the following condition

$$\forall x, y > 0 \qquad \min(x, y) \le m(x, y) \le \max(x, y). \tag{1}$$

The two maps $(x, y) \longmapsto \min(x, y)$ and $(x, y) \longmapsto \max(x, y)$, will be denoted by min and max, are trivial means called the lower mean and the upper mean, respectively. The property in (1) is called by Audenaert in [1], the in-betweenness property of the mean m. Some other standard examples of means are given in

2010 Mathematics Subject Classification: 26E60, 54E50
Key words and phrases: bivariate means, metric topology

the following:

$$\begin{cases} A := A(x,y) = & \dfrac{x+y}{2} \\ G := G(x,y) = & \sqrt{xy} \\ H := H(x,y) = & \dfrac{2xy}{x+y} \end{cases} \tag{2}$$

and are known, in the literature, as the arithmetic mean, geometric mean and harmonic mean, respectively. These three means are included in the following families of means, with $t \in [0,1]$,

$$\begin{cases} A_t := A_t(x,y) = (1-t)x + ty, \\ G_t := G_t(x,y) = x^{1-t}y^t, \\ H_t := H_t(x,y) = \Big((1-t)x^{-1} + ty^{-1}\Big)^{-1}. \end{cases} \tag{3}$$

These are called weighted arithmetic mean, weighted geometric mean and weighted harmonic mean, respectively. Another example of classical means is the Heron mean He_t defined by, [8], $He_t = tG+(1-t)A$ with $t \in [0,1]$. Clearly, $He_0 = A$ and $He_1 = G$. For further examples of means, we refer the reader to [4, 3, 9] for instance and the related references cited therein.

Otherwise, it is not hard to check that the following relationship

$$A_t(x,y) - H_t(x,y) = t(1-t)\frac{(x-y)^2}{tx+(1-t)y} \tag{4}$$

holds for any $x, y > 0$ and $t \in [0,1]$.

We define symmetric (resp. homogeneous, monotone) means in the habitual way. The weighted means (3) are homogeneous, monotone, not symmetric unless $t = 1/2$, case for which they coincide with A, G and H, respectively. In the literature, see [9] for instance, we can find a lot of symmetric, homogeneous monotone means. For example, the following

$$L(x,y) := \frac{x-y}{\log(x/y)}, \; x \neq y; \;\; \text{with } L(x,x) = x \tag{5}$$

is known as the logarithmic mean of $x > 0$ and $y > 0$.

As example of mean which is not monotone, we can mention the contra-harmonic mean defined for all $x, y > 0$ by, $C(x,y) = \dfrac{x^2+y^2}{x+y}$. It is well known that the following inequalities, [9]

$$H(x,y) \leq G(x,y) \leq L(x,y) \leq A(x,y) \leq C(x,y) \tag{6}$$

hold for any $x, y > 0$.

The mean C is a particular case of the so-called Lehmer power mean defined for $p \in \mathbb{R}$ by

$$L_p(x, y) = \frac{x^p + y^p}{x^{p-1} + y^{p-1}}.$$

It is easy to see that $L_1 = A$, $L_0 = H$ and $L_2 = C$. We can also check that the equality

$$L_p(x, y) - A(x, y) = \frac{(x-y)(x^{p-1} - y^{p-1})}{2(x^{p-1} + y^{p-1})} \tag{7}$$

holds for any $x, y > 0$ and $p \in \mathbb{R}$.

The set of all (bivariate) means will be denoted by $\mathcal{M}$. We also denote by $\mathcal{M}_s$, resp. $\mathcal{M}_h$, the set of all symmetric means, resp. homogeneous means. As pointed out in [2], the sets $\mathcal{M}$, $\mathcal{M}_s$ and $\mathcal{M}_h$ are convex.

In Section 2 below, we will define a metric d on the set $\mathcal{M}$ and we study its algebraic properties as well as some examples for computations of $d(m_1, m_2)$ when $m_1, m_2 \in \mathcal{M}$. Afterwards, Section 3 is devoted to investigate some topological properties of the metric space $(\mathcal{M}, d)$.

2 Metric topology on $\mathcal{M}$

Let $m_1, m_2 \in \mathcal{M}$. From (1), we immediately deduce that, for all $x, y > 0$, we have

$$|m_1(x, y) - m_2(x, y)| \leq \max(x, y) - \min(x, y) = |x - y|. \tag{8}$$

We can then put the following definition.

Definition 1 *Let $m_1, m_2 \in \mathcal{M}$. For all $x, y > 0$, we define*

$$\mathcal{T}(m_1, m_2)(x, y) = \begin{cases} \dfrac{m_1(x, y) - m_2(x, y)}{x - y} & \textit{if } x \neq y, \\ 0 & \textit{if } x = y \end{cases} \tag{9}$$

Following (8), $\mathcal{T}(m_1, m_2)$ is a map from $(0, \infty) \times (0, \infty)$ into $[-1, 1]$, i.e.

$$\forall x, y > 0 \qquad |\mathcal{T}(m_1, m_2)(x, y)| \leq 1. \tag{10}$$

Further, if m_1 and m_2 are both symmetric and homogenous then we have,

$$\mathcal{T}(m_1, m_2)(x, y) = \mathcal{T}(m_1, m_2)(y, x) \text{ and}$$

$$\mathcal{T}\big(m_1,m_2\big)(tx,ty)=\mathcal{T}\big(m_1,m_2\big)(x,y),$$

for all $x,y>0$ and $t>0$.

Now, we are in a position to state the following definition.

Definition 2 *For $m_1,m_2\in\mathcal{M}$, we set*

$$d(m_1,m_2)=\sup_{x,y>0}\Big|\mathcal{T}\big(m_1,m_2\big)(x,y)\Big|. \tag{11}$$

It is clear that if $m_1,m_2\in\mathcal{M}_h$ then we have

$$d(m_1,m_2)=\sup_{0<x}|\mathcal{T}\big(m_1,m_2\big)(x,1)|. \tag{12}$$

Proposition 1 $(\mathcal{M},d)$ *is a bounded metric space.*

Proof. It is easy to check that for all $m_1,m_2,m_3\in\mathcal{M}$ the next properties are satisfied:

- $d(m_1,m_2)=d(m_2,m_1)$.
- $d(m_1,m_2)=0\Longleftrightarrow m_1=m_2$.
- $d(m_1,m_3)\le d(m_1,m_2)+d(m_2,m_3)$.

These confirm that d establishes a distance on $\mathcal{M}$. Otherwise, from inequality (10), we immediately deduce that $(\mathcal{M},d)$ is bounded. □

Remark 1 i) *In the particular case of symmetric means defined on a symmetric domain in $\mathbb{R}$, Farhi gave in [7] the following formula*

$$d(m_1,m_2)=\sup_{0<x,y}\left(\frac{1}{e^{f_{m_1}(x,y)}+1}-\frac{1}{e^{f_{m_2}(x,y)}+1}\right) \tag{13}$$

where for a mean $m\in\mathcal{M}_s$, $f_m(x,y)$ is defined for all $0<x,\ y$ by

$$\begin{cases} f_m(x,y)=\log\left(-\dfrac{x-m(x,y)}{y-m(x,y)}\right), & \text{for } x\neq y\\ f_m(x,x)=0 \end{cases} \tag{14}$$

which can be useful in a computational point of view as well as the relation (12) as we will see later.

ii) *If the means m_1 and m_2 are symmetric then the distance $d(m_1,m_2)$ can be also defined by the next formula*

$$d(m_1,m_2)=\sup_{0<x<y}\Big|\mathcal{T}\big(m_1,m_2\big)(x,y)\Big|.$$

Now we will state some important examples where distances between some special means are determined with respect to the metric d.

Example 1 *Simple computations lead to the following:*

(i) $d(A_{t_1}, A_{t_2}) = |t_1 - t_2|, \forall t_1, t_2 \in [0,1]$. *In particular,* $d(A, A_t) = |t - 1/2|$, $\forall t \in [0,1]$.

(ii) $d(A, H) = d(A, G) = d(A, C) = 1/2$.

(iii) $d(H, C) = d(\min, C) = 1,\ d(G, C) = 1$.

(iv) $d(A, He_t) = t/2$ *with* $t \in [0,1]$. *In particular* $d\big(A, \frac{A+G}{2}\big) = 1/4$, *since* $He_{1/2} = \frac{A+G}{2}$.

Remark 2 *One can check that, if* m, m_1 *and* m_2 *are three means such that* $m \le m_1 \le m_2$, *(resp.* $m_1 \le m_2 \le m$*), then we have*

$$d(m, m_1) \le d(m, m_2), \quad (\textit{resp. } d(m, m_1) \ge d(m, m_2)).$$

This, with the arithmetic-geometric-harmonic mean inequality, namely $H \le G \le A$, *yields the following:*

- *If* $m \in \mathcal{M}$ *is such that* $m \le H$ *then* $d(m, H) \le d(m, G) \le d(m, A)$.
- *If* $m \in \mathcal{M}$ *is such that* $m \ge A$ *then* $d(m, A) \le d(m, G) \le d(m, H)$.

We now state the following proposition which contains more examples giving the computation of $d(m_1, m_2)$ when m_1 and m_2 are among the previous standard bivariate means.

Proposition 2 *The following equalities hold,*

(i) *For any* $m \in \mathcal{M}$ *we have* $d(m, A) \le 1/2$.

(ii) $d\big(A_t, H_t\big) = \max(t, 1-t)$ *for any* $t \in (0,1)$.

(iii) $d\big(A_t, G_t\big) = \max(t, 1-t)$ *for all* $t \in (0,1)$.

(iv) $d\big(L_p, A\big) = 1/2$ *for any real number* $p \ne 1$.

(v) $d(A, L) = 1/2$.

Proof. (i) Let $m \in \mathcal{M}$ and $x, y > 0$. We have

$$|m(x,y) - A(x,y)| \le \max\big(x - A(x,y), y - A(x,y)\big) = \max\Big(\frac{x-y}{2}, \frac{y-x}{2}\Big), \tag{15}$$

which yields $d(m, A) \le 1/2$.

Since the means A_t, G_t,H_t,L_p and L are homogenous, then it suffices to use the formulae (12) for establishing the equalities (ii)-(v).

(ii) Let $x > 0$. According to (4) with $y = 1$, we obtain

$$\big|\mathcal{T}\big(A_t, H_t\big)(x,1)\big| = t(1-t)\frac{|1-x|}{1-t+tx}.$$

Studying the variations of the function $\phi : x \longmapsto \dfrac{|1-x|}{1-t+tx}$, defined for $x \in (0,\infty)$, we conclude that it decreases on $(0,1]$ and increases on $[1,\infty)$. Then we have

$$\sup_{x\in(0,\infty)} \phi(x) = \max\big(\phi(0^+), \lim_{x\to\infty} \phi(x)\big) = \max\Big(\frac{1}{t}, \frac{1}{1-t}\Big).$$

It follows that

$$d\big(A_t, H_t\big) = \sup_{0<x} \big|\mathcal{T}\big(A_t, H_t\big)(x,1)\big| = \max(t, 1-t).$$

(iii) Now, we have for any $x \in (0,\infty)$ with $x \neq 1$

$$\big|\mathcal{T}\big(A_t, G_t\big)(x,1)\big| = \left|\frac{(1-t)x + t - x^{1-t}}{1-x}\right|.$$

If for $x \in (0,1)$ we set $u_t(x) = \dfrac{(1-t)x + t - x^{1-t}}{1-x}$ then simple computation leads to

$$u_t'(x) = \frac{1-(1-t)x^{-t} - tx^{1-t}}{(1-x)^2} = \frac{v_t(x)}{(1-x)^2},$$

where

$$v_t(x) = 1 - (1-t)x^{-t} - tx^{1-t} \ \text{ and } \ v_t'(x) = t(1-t)x^{-t}(x^{-1} - 1) \ge 0.$$

It follows that v_t is a strictly increasing function on $(0,1)$ and so $v_t(x) \le v_t(1^-) = 0$ for any $x \in (0,1)$. We then deduce that u_t is a strictly decreasing function on $(0,1)$. Hence,

$$\sup_{0<x<1} \big|\mathcal{T}\big(A_t, G_t\big)(x,1)\big| = u_t(0) = t.$$

With similar computations we can prove that,

$$\sup_{x>1} \left|\mathcal{T}\big(A_t, G_t\big)(x,1)\right| = 1 - t.$$

So we get $d\big(A_t, G_t\big) = \max(t, 1-t)$.
(iv) Let $x \in (0,\infty)$. By virtue of (7) we have

$$\mathcal{T}(L_p, A)(x,1) := \frac{|x^{p-1} - 1|}{2(x^{p-1} + 1)}.$$

The case $p = 1$ is trivial, since $L_1 = A$ and so $d(L_1, A) = 0$. Assume that $p \neq 1$. Setting $x^{p-1} = z$ and using elementary techniques of real analysis, we find $d\big(L_p, A\big) = 1/2$.

(v) As previously, with the help of (5) and (6), we have for any $x \in (0,1)$ (after a simple reduction)

$$\mathcal{T}(L, A)(x,1) := \frac{(x+1)\log(x) - 2(x-1)}{2|1-x|\,\log(x)}.$$

We first show that

$$\forall x \in (0,1) \qquad \frac{(x+1)\log(x) - 2(x-1)}{2(1-x)\log(x)} \leq 1/2. \tag{16}$$

After simple manipulations, (16) is reduced to the following one

$$\forall x \in (0,1) \qquad x\log(x) - x + 1 \geq 0,$$

which is so easy to show by similar way as previously.
For $x > 1$ the inequality,

$$\frac{(x+1)\log(x) - 2(x-1)}{2(x-1)\log(x)} \leq 1/2. \tag{17}$$

is equivalent to

$$\log(x) - x + 1 \leq 0,$$

which can be simply confirmed.
The results (16) and (21) enable us to write for all $x > 0$ that,

$$\frac{(x+1)\log(x) - 2(x-1)}{2|1-x|\,\log(x)} \leq \frac{1}{2}. \tag{18}$$

Now, by l'Hopital's rule we can check that

$$\lim_{x\to 1}\frac{(x+1)\log(x)-2(x-1)}{(1-x)\log(x)}=1.$$

This, when combined with (18), yields the desired result. □

Remark 3 *According to the preceding proposition, the relationship* $d(L_p,A)=1/2$ *shows that the map* $p\longmapsto d(L_p,A)$ *is discontinuous at* $p=1$*, since* $L_1=A$ *and so* $d(L_1,A)=0$*.*

Proposition 3 *The distance between two harmonic weighted means* H_{t_1} *and* H_{t_2} *is given by*

$$d\big(H_{t_1},H_{t_2}\big)=\frac{|t_1-t_2|\theta(t_1,t_2)}{2(1-t_1)(1-t_2)+(t_1+t_2-2t_1t_2)\theta(t_1,t_2)},$$

where we set

$$\theta(t_1,t_2):=\sqrt{\frac{(1-t_1)(1-t_2)}{t_1t_2}}.$$

Proof. We also use (12). For $x\in(0,\infty)$, simple computation leads to

$$\left|\mathcal{T}\big(H_{t_1},H_{t_2}\big)(x,1)\right|=\frac{|t_1-t_2|x}{(1-t_1+t_1x)(1-t_2+t_2x)}.$$

Let us set

$$\forall x\in(0,\infty)\qquad g(x):=\frac{x}{(1-t_1+t_1x)(1-t_2+t_2x)}.$$

By computing the derivative of g we easily obtain

$$\forall x\in(0,\infty)\qquad g^{'}(x)=\frac{-t_1t_2x^2+(1-t_1)(1-t_2)}{(1-t_1+t_1x)^2(1-t_2+t_2x)^2}.$$

We then deduce that g attains its maximum point at $\theta(t_1,t_2)$. Computing $g\big(\theta(t_1,t_2)\big)$ we find the desired result after simple reductions. The proof is completed. □

Another result of interest is recited in the following.

Proposition 4 *The map* $(m_1, m_2) \longmapsto d(m_1, m_2)$ *is jointly convex and separately convex. That is, the two following inequalities*

$$d\Big((1-t)m_1 + tm_3, (1-t)m_2 + tm_4\Big) \leq (1-t)d(m_1, m_2) + td(m_3, m_4) \quad (19)$$

$$d\big((1-t)m_1 + tm_3, m_2\big) \leq (1-t)d(m_1, m_2) + td(m_3, m_2) \quad (20)$$

hold for any $m_1, m_2, m_3, m_4 \in \mathcal{M}$ *and* $t \in [0, 1]$.

Proof. Let $m_1, m_2, m_3, m_4 \in \mathcal{M}$ and $t \in [0, 1]$. For the sake of simplicity, we set

$$\delta := d\Big((1-t)m_1 + tm_3, (1-t)m_2 + tm_4\Big).$$

Then we have

$$\begin{aligned}
\delta &= \sup_{0<x,y} \left| \frac{\big((1-t)m_1 + tm_3\big)(x,y) - \big((1-t)m_2 + tm_4\big)(x,y)}{x-y} \right| \\
&= \sup_{0<x,y} \left| \frac{(1-t)\big(m_1(x,y) - m_2(x,y)\big) + t\big(m_3(x,y) - m_4(x,y)\big)}{x-y} \right| \\
&\leq \sup_{0<x,y} \left| \frac{(1-t)\big(m_1(x,y) - m_2(x,y)\big)}{x-y} \right| + \sup_{0<x,y} \left| \frac{t\big(m_3(x,y) - m_4(x,y)\big)}{x-y} \right| \\
&= (1-t)d(m_1, m_2) + td(m_3, m_4)
\end{aligned}$$

Every jointly convex map is separately convex. That is, (20) follows from (19) when we take $m_4 = m_2$. The proof is finished. □

Remark 4 (i) *According to Example 1,* (iii) *the relation* $d\big(A, \frac{A+G}{2}\big) = 1/4$ *shows that the convexity of the map* $(m_1, m_2) \longmapsto d(m_1, m_2)$ *is not strict.*
(ii) *From the preceding proposition we immediately deduce that, every ball (closed or open) of* $(\mathcal{M}, d)$ *is convex.*

Corollary 1 *The next inequalities hold*
(i) $d(A, \lambda A_t + (1-\lambda)L_p) \leq \lambda|t - 1/2| + \dfrac{1-\lambda}{2}$, *for* $t, \lambda \in [0, 1]$ *and* $p \in \mathbb{R}$.
(ii) $d(A, \lambda A_t + (1-\lambda)He_\alpha) \leq \lambda|t - 1/2| + \dfrac{\alpha(1-\lambda)}{2}$, *for* $t, \lambda, \alpha \in [0, 1]$.
(iii) $d(A, \lambda L_p + (1-\lambda)He_\alpha) \leq \dfrac{\lambda}{2} + \dfrac{\alpha(1-\lambda)}{2}$, *for* $t, \lambda, \alpha \in [0, 1]$ *and* $p \in \mathbb{R}$.

Proof. According to (20), with the help of Example 1 and Proposition 2, we easily obtain the desired inequalities. The details are simple and therefore omitted here for the reader. □

3 Topological properties of $(\mathcal{M}, d)$

We preserve the same notations as in the previous sections.

Proposition 5 *$(\mathcal{M}, d)$ coincides with its closed ball of center A, the arithmetic mean, and radius $1/2$.*

Proof. According to (15) we have $d(m, A) \leq 1/2$ for any $m \in \mathcal{M}$. Inversely, assume that m is a binary map satisfying (15). This is equivalent to

$$|m(x,y) - A(x,y)| \leq \frac{|x-y|}{2} \tag{21}$$

and so,

$$\min(x,y) := \frac{x+y-|x-y|}{2} \leq m(x,y) \leq \frac{x+y+|x-y|}{2} := \max(x,y).$$

The desired result follows and the proof is finished. □

Remark 5 *In a geometrical point of view, the inequality (21) implies that every mean $m(x,y)$ lies on the sphere centered at the arithmetic mean and with radius equal to the half of the euclidian distance between x and y. So, according to the definition given by Dinh et al [6], every bivariate mean satisfies the in-sphere property.*

An important topological property for $(\mathcal{M}, d)$ is quoted in the following theorem.

Theorem 1 *The metric space $(\mathcal{M}, d)$ is complete.*

Proof. Let (m_p) be a Cauchy sequence in $(\mathcal{M}, d)$. Let $a, b > 0$ with $a \neq b$ be fixed. For a given $\epsilon > 0$ there is $\eta_\epsilon \in \mathbb{N}$ such that,

$$p, q \geq \eta_\epsilon \Longrightarrow d(m_p, m_q) \leq \frac{\epsilon}{|a-b|}$$

or equivalently

$$p, q \geq \eta_\epsilon \Longrightarrow \sup_{0<x,y} \left| \frac{m_p(x,y) - m_q(x,y)}{x-y} \right| \leq \frac{\epsilon}{|a-b|}.$$

We then deduce that

$$p, q \geq \eta_\epsilon \Longrightarrow \left| \frac{m_p(a,b) - m_q(a,b)}{a-b} \right| \leq \frac{\epsilon}{|a-b|}.$$

This latter inequality holds for each $a, b > 0$ with $a \neq b$. It follows that, for any $x, y > 0$ there exists an integer N_ϵ such that we get

$$p, q \geq N_\epsilon \Longrightarrow |m_p(x, y) - m_q(x, y)| \leq \epsilon,$$

which means that $(m_p(x, y))_p$ is a Cauchy sequence in $\mathbb{R}$. By completeness of $\mathbb{R}$, for the standard metric, the sequence $\big(m_n(x, y)\big)_n$ is convergent in $\mathbb{R}$ and we put $\lim_{p\uparrow\infty} m_p(x, y) = m(x, y)$ for any $x, y > 0$.

Since $\min(x, y) \leq m_n(x, y) \leq \max(x, y)$ we then deduce, when $p \uparrow \infty$, that $\min(x, y) \leq m(x, y) \leq \max(x, y)$. So we can confirm that $m \in \mathcal{M}$ and it remains to prove that $(m_p)_p$ converges to m in $(\mathcal{M}, d)$.

Since $(m_p(x, y))_p$ is a Cauchy sequence in $\mathbb{R}$ then, for any $x, y > 0$ with $x \neq y$, $\big(m_p(x, y)/(x - y)\big)_p$ is also a Cauchy sequence in $\mathbb{R}$. This means that,

$$\forall \epsilon > 0 \quad \exists N_\epsilon \in \mathbb{N} \quad \forall p, q \geq N_\epsilon \quad \forall x, y > 0,\ x \neq y \quad \left|\frac{m_p(x, y) - m_q(x, y)}{x - y}\right| \leq \epsilon.$$

By letting q to ∞ in this latter inequality we obtain,

$$\left|\frac{m_p(x, y) - m(x, y)}{x - y}\right| \leq \epsilon \ \text{ for all } x, y > 0,\ x \neq y$$

and so $d(m_p, m) \leq \epsilon$ which gives the convergence of $(m_p)_p$ to m in $(\mathcal{M}, d)$. □

In the aim to give more topological properties for $(\mathcal{M}, d)$, we need the following lemma.

Lemma 1 *For every* $p \geq 1$, *we have the following*

$$d\big(L_p, \max\big) = 1.$$

Proof. The means L_p and max are homogenous so we can use the formulae (12). It is easy to check that

$$\forall x \in (0, 1) \qquad \mathcal{T}(L_p, \max)(x, 1) = \frac{x^{p-1}}{1 + x^{p-1}}. \tag{22}$$

For $p = 1$, there is nothing to prove. For $p \neq 1$, we set $z = x^{p-1}$ and the right hand-side of (22) remains a simple homographic function in z which is so monotone for $z \in (0, +\infty)$. For both cases $p > 1$ or $p < 1$, the variable z describes the interval $(0, +\infty)$. Passing to the supremum over z in (22) we obtain the desired result, by simple arguments of real analysis. □

Now, we are in a position to state the following result.

Theorem 2 *The space* $(\mathcal{M}, d)$ *is not compact.*

Proof. Since $(\mathcal{M}, d)$ is a metric space, it suffices to show that there exists a sequence which have no convergent subsequence.
Let's at first recall that if a sequence (m_p) converges in $(\mathcal{M}, d)$ to a mean m then we have $\lim\limits_{p\to+\infty} m_p(x,y) = m(x,y)$ for any $x, y > 0$.
On one hand, by Lemma 1, we have $d(L_p, \max) = 1$ for all integer $p > 1$, and so the sequence $(L_p)_{p>1}$ does not converge to max in $(\mathcal{M}, d)$.
On the other hand, it is not hard to see that $\lim\limits_{p\to+\infty} L_p(a,b) = \max(a,b)$, see [2] for instance. This shows that we can not extract a convergent subsequence from the bounded sequence $(L_p)_{p>1}$. The proof of the theorem is completed.□

Proposition 6 $\mathcal{M}_s$ *and* $\mathcal{M}_h$ *are closed in* $(\mathcal{M}, d)$.

Proof. We show that $\mathcal{M}_s$ is closed. For this, let $(m_n)_n$ be a sequence of symmetric means converging to a mean m in $(\mathcal{M}, d)$. As already mentioned before, for any $x, y > 0$, the two sequences $\big(m_n(x,y)\big)_n$ and $\big(m_n(y,x)\big)_n$ converge in $\mathbb{R}$ to $m(x,y)$ and $m(y,x)$, respectively. Since $m_n(x,y) = m_n(y,x)$ then by letting $n \uparrow \infty$ we obtain $m(x,y) = m(y,x)$ and so $m \in \mathcal{M}_s$.
In a similar way, we prove the closeness of $\mathcal{M}_h$. □

Another topological property of $\big(\mathcal{M}, d\big)$ is recited in the following result.

Theorem 3 *The metric space* $\big(\mathcal{M}, d\big)$ *is path-wise connected and so, it is connected.*

Proof. Let $m_1, m_2 \in \mathcal{M}$ and consider the map $f : [0,1] \longrightarrow \mathcal{M}$ such that,

$$\forall t \in [0,1] \quad f(t) = (1-t)m_1 + tm_2.$$

Since $\mathcal{M}$ is convex then f is well defined. We will show that f is a path (i.e. continuous function) with endpoints m_1 and m_2. Indeed, for all $t_1, t_2 \in [0,1]$ we have

$$\begin{aligned} d\big(f(t_1), f(t_2)\big) &= \sup_{0<x,y} \left| \frac{f(t_1)(x,y) - f(t_2)(x,y)}{x-y} \right| \\ &= \sup_{0<x,y} \left| \frac{(t_1-t_2)(m_1(x,y) - m_2(x,y))}{x-y} \right| \\ &= |t_1 - t_2| d(m_1, m_2) \\ &\leq |t_1 - t_2| \end{aligned}$$

We then infer that f is uniformly continuous on $[0,1]$ and so, it is continuous. Moreover, $f(0) = m_1$ and $f(1) = m_2$. In summary, f is a path with endpoints m_1 and m_2. The proof is finished. $\square$

References

[1] K. M. R. Audenaert, In-betweenness, a geometrical monotonicity property for operator means, *Linear Algebra Appl, 16th ILAS Conference Proceedings*, **438/4** (2013), 1769–1778.

[2] J. M. Borwein and P. B. Borwein, *Pi and the AGM, A study in Analytic Number Theory and Computational Complexity*, John Wiley and Sons Inc., New York, 1987.

[3] J. M. Borwein and P. B. Borwein, The way of all means, *Amer. Math. Monthly*, **94/6** (1987), 519–522.

[4] P. S. Bullen, *Handbook of Means and their Inequalities*, Mathematics and its Applications, Springer, Second edition, 1987.

[5] A. L. Cauchy, *Cours d'Analyse de l'Ecole Royale Polytechnique*, 1^{ere} Partie: Analyse Algébrique. Imprimerie Royale, Paris, 1821.

[6] T. H. Dinh, B. K. T. Vo, T. Y. Tam, In-sphere property and reverse inequalities for matrix means, *Electron. J. Linear Algebra*, **35/1** (2019), 35–41.

[7] B. Farhi, Algebraic and topological structures on the set of means functions and generalization of the AGM mean, *Colloquium Mathematicum*, **132** (2013), 139–149.

[8] W. Janous, A note on generalized Heronian means, *Math. Inequal. Appl.* **4/3** (2001), 369–375.

[9] M. Raïssouli and M. Chergui, On some inequalities involving three or more means, *Abstract and Applied Analysis*, (2016), Art. ID 1249604, 8 pages.

Received: February 16, 2021