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On logarithm of circular and hyperbolic functions and bounds for $\exp(\pm x^2)$

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Abstract. We show that certain known or new inequalities for the logarithm of circular hyperbolic functions imply bounds for $\exp(\pm x^2)$ proved in [1].

1 Introduction

In the recent paper [1], the following sharp bounds for $\exp(\pm x^2)$ are proved (see Theorem 1, resp. Theorem 2 of [1]).

For $x \in (0, \pi/2)$ one has

$$\left(\frac{1+\cos x}{2}\right)^a < \exp(-x^2) < \left(\frac{1+\cos x}{2}\right)^b \tag{1}$$

where $a = 4$, $b = \pi^2/4\ln 2$ are best possible; and

$$\left(\frac{2+\cos x}{3}\right)^c < \exp(-x^2) < \left(\frac{2+\cos x}{3}\right)^d, \tag{2}$$

with $c = \pi^2/4\ln(3/2)$, $d = 6$ best possible;

$$\left(\frac{1+\cosh x}{2}\right)^\alpha < \exp(x^2) < \left(\frac{1+\cosh x}{2}\right)^\beta \tag{3}$$

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where $\alpha = 4$, $\beta = \pi^2/4\ln[(1+\cosh(\pi/2))/2]$ are best possible;

$$\left(\frac{2+\cosh x}{3}\right)^{\theta} < \exp(x^2) < \left(\frac{2+\cosh x}{3}\right)^{\gamma} \tag{4}$$

with $\theta = 6$ and

$$\gamma = \pi^2/4\ln[(2+\cosh(\pi/2))/3]$$

are best possible.

We first want to point out that relations (1), (2) and (3) are essentially known (but not stated explicitely), and that (4) can be deduced in a similar way, using the Jensen integral inequality.

2 Proofs

First remark that, as

$$\frac{1+\cos x}{2} = \cos^2\frac{x}{2} \quad \text{and} \quad \frac{1+\cosh x}{2} = \cosh^2\frac{x}{2};$$

by letting $\frac{x}{2} = t$, where $t \in (0, \pi/4)$, to prove (1) it is sufficient to show that the function

$$f_1(t) = \frac{\ln(\cos t)}{t^2} \tag{5}$$

is strictly decreasing. As

$$t^3 f_1'(t) = -t\cdot\tan t + 2\ln(1/\cos t),$$

this follows by the inequality

$$\ln\left(\frac{1}{\cos t}\right) < \frac{t}{2}\cdot\tan t. \tag{6}$$

This is proved in [2] (see Corollary 3.8, right side of (1)).

Now, relation (1) with best possible a and b follow by

$$f_1(0+) > f_1(t) > f_1(\pi/4).$$

In a similar manner for (3) it is sufficient to prove that the function

$$f_2(t) = \frac{\ln(\cosh t)}{t^2} \tag{7}$$

is strictly decreasing. As

$$t_3 f_2'(t) = t \cdot \tanh t - 2\ln(\cosh t),$$

this follows by the inequality

$$\ln(\cosh t) > \frac{t}{2} \cdot \tanh t, \tag{8}$$

proved in [5] and [2] (see Lemma 2.1 in [5] and Corollary 3.8, left side of (2) in [2]).

For improvements of (8) and related inequalities, see [9].

Now (3), with best possible α and β follow from

$$f_2(0+) > f_2(t) > f_2(\pi/4).$$

We note that inequalities (6) and (8) are simple consequences of the Jensen integral inequality ([3]):

$$\int_a^b F(x)dx \underset{(>)}{<} (b-a)\left[\frac{F(a)+F(b)}{2}\right] \tag{9}$$

with inequality $<$ when $F(x)$ is strictly convex, and $(>)$, when $F(x)$ is strictly concave on $[a,b]$. Inequality (9) is called also as one of the Hermite-Hadamard inequalities. By letting the convex function $F_1(x) = \tan x$ and $[a,b] = [0,t]$, we get (6). Similarly, by letting $F_2(x) = \tanh x$ and $[a,b] = [0,t]$, we get relation (8).

Now, let

$$F_3(x) = \frac{\sinh t}{2+\cosh t},$$

remarking that

$$\int_0^t F_3(x)dx = \ln\left(\frac{\cosh x + 2}{3}\right).$$

It is immediate that

$$F_3'(x) = \frac{1+2\cosh x}{(\cosh x+2)^2}$$

and

$$F_3''(x) \cdot (\cosh x + 2) = 2(\sinh x) \cdot (1-\cosh x) < 0,$$

we get that $F_3(x)$ is strictly concave. Thus, by (9) we get the inequality

$$\ln\left(\frac{\cosh t + 2}{3}\right) > \frac{t}{2} \cdot \frac{\sinh t}{\cosh t + 2}. \tag{10}$$

Similarly, by remarking that

$$\int_0^t \frac{\sin x}{2+\cos x}dx = \ln\left(\frac{3}{2+\cos x}\right)$$

and that the function

$$F_4(x) = \frac{\sin x}{2+\cos x}$$

is strictly concave, we can deduce the inequality

$$\ln\frac{3}{2+\cos t} > \frac{t}{2}\cdot\frac{\sin t}{\cos t+2}. \tag{11}$$

This inequality, with another proof, appears also in [8] (see relation (2.2)).

Now, to prove (2), let

$$f_3(t) = \frac{\left[\ln\left(\frac{\cos t+2}{3}\right)\right]}{t^2}.$$

It is immediate that

$$t^3 f_3'(t) = -\frac{\sin t}{2+\cos t}\cdot t + 2\ln\left(\frac{3}{\cos t+2}\right) > 0$$

by (11). Thus $f_3(t)$ is strictly increasing, and (2) with best possible c and d follow by $f_3(0+) < f_3(t) < f_3(\pi/2)$.

Finally, inequality (4) follows in the same manner by considering

$$f_4(t) = \left[\ln\left(\frac{\cosh t+2}{3}\right)\right]/t^2,$$

and applying inequality (10).

Remarks 1) In [1], the L'Hospital's rule of monotonicity, as well as the following lemma is used:

$$\frac{\sin x}{x} > \frac{1+2\cos x}{2+\cos x}, \quad x\in(0,\pi/2) \tag{12}$$

and

$$\frac{x}{\sinh x} + \cosh x > 2, \quad x > 0. \tag{13}$$

We note that relation (12), with strong improvements, appears in paper [7] (see relation (4.7) and Theorem 4.2). We point out also, that (13) is weaker than the Neumann-Sándor inequality [4]:

$$\frac{\sinh x}{x} < \frac{2+\cosh x}{3}. \tag{14}$$

Indeed, as

$$\frac{x}{\sinh x} > \frac{3}{2+\cosh x},$$

and letting $u = \mathrm{ch}x$, one has

$$u + \frac{3}{2+u} > 2.$$

Indeed, this is equivalent to $u^2 + 2u + 3 > 4 + 2u$, or $u > 1$, which is true. Therefore, one has

$$\frac{x}{\sinh x} + \cosh x > \frac{3}{2+\cosh x} + \cosh x > 2. \tag{15}$$

2) Other inequalities for the logarithm of circular and hyperbolic functions can be found in papers [2, 5, 6, 8, 9]. For various applications in the theory of means, see [10].

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