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Analysis of semigroups with soft intersection ideals

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Abstract. In this paper, semisimple semigroups, duo semigroups, right (left) zero semigroups, right (left) simple semigroups, semilattice of left (right) simple semigroups, semilattice of left (right) groups and semilattice of groups are characterized in terms of soft intersection semigroups, soft intersection ideals of semigroups. Moreover, soft normal semigroups are defined and some characterizations of semigroups with soft normality are given.

1 Introduction

In 1999, the concept of soft sets was introduced by Molodtsov [31] for modeling vagueness and uncertainty. Many complex problems of social science and science involve uncertainties. To be able to deal with these uncertainties and incomplete information, some theories have been proposed such as the theory of probability, as is well known, the most successful theoretical approaches are undoubtedly fuzzy set [1] and interval mathematics [2] . Despite all these

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developments, Molodtsov [3] pointed out that each of these theories have inherent limitations in insufficient parameterization tools, and he introduced the soft set theory for modeling vagueness and uncertainty. Since the soft set theory is very convenient and easy to apply in practice, researches focused on soft sets that have been growing rapidly, and which has some potential applications in many different fields; such as extended theories [4], combination forecast [5], data mining [6], medical diagnosis [7] and decision making [8] . Meanwhile, many related concepts with soft sets, especially soft set operations, have recently undergone tremendous studies. Maji et al. [30] presented some definitions on soft sets and based on the analysis of several operations on soft sets Ali et al. [12] introduced several operations of soft sets and Sezgin and Atagün [36] and Ali et al. [13] studied on soft set operations as well. Soft set theory have found its wide-ranging applications in the mean of algebraic structures such as groups [11, 37], semirings [18], rings [9], BCK/BCI-algebras [24, 25, 26], BL-algebras [42], near-rings [35] and soft substructures and union soft substructures [14, 38], hemirings [29, 43] and so on [18, 19, 21].

In [20], Feng et al. applied soft relations to semigroups. In [39], Sezer et al. made a new approach to the classical semigroup theory via soft set theory with the concept of soft intersection semigroups. They defined soft intersection semigroups, soft intersection left (right, two-sided) ideals and bi-ideals and soft semiprime ideals of semigroups and obtained their basic properties. As a following study of [39], Sezer et al. [40] defined soft intersection interior ideals, quasi-ideals, generalized bi-ideals and investigate the interrelations of them. Moreover, they characterized regular, intra-regular, completely regular, weakly regular and quasi-regular semigroups by the properties of these ideals in [39, 40].

In this paper, certain classes of semigroups, such as semisimple semigroups, duo semigroups, right (left) zero semigroups, right (left) simple semigroups, semilattice of left (right) simple semigroups, semilattice of left (right) groups and semilattice of groups in terms of soft intersection ideals, bi-ideals, interior ideals, quasi-ideals, generalized bi-ideals are characterized. Moreover, soft normal semigroups are defined and discussed on the relation of this concept with semigroups.

2 Preliminaries

In this section, some notions relevant to semigroups and soft sets are recalled. A *semigroup* S is a nonempty set with an associative binary operation. Throughout this paper, S denotes a semigroup. A nonempty subset A of S is called a *right ideal* of S if $AS \subseteq A$ and is called a *left ideal* of S if $SA \subseteq A$. By *two-sided ideal* (or simply *ideal*), we mean a subset of S, which is both a left and right ideal of S. A subsemigroup X of S is called a *bi-ideal* of S if $XSX \subseteq X$. A nonempty subset A of S is called an *interior ideal* of S if $SAS \subseteq A$. A nonempty subset Q of S is called a *quasi-ideal* of S if $QS \cap SQ \subseteq Q$.

We denote by $L[a](R[a], J[a], B[a]Q[a], I[a])$, the principal left (right, two-sided, bi-ideal, quasi-ideal, interior ideal) of a semigroup S generated by $a \in S$, that is,

$$\begin{aligned} L[a] &= \{a\} \cup Sa, \\ R[a] &= \{a\} \cup aS, \\ J[a] &= \{a\} \cup Sa \cup aS \cup SaS \\ Q[a] &= \{a\} \cup (aS \cap Sa) \\ I[a] &= \{a\} \cup \{a^2\} \cup SaS. \end{aligned}$$

A semigroup S is called *regular* if for every element a of S, there exists an element x in S such that $a = axa$ or equivalently $a \in aSa$. An element a of S is called a *completely regular* if there exists an element $x \in S$ such that $a = axa$ and $ax = xa$. A semigroup S is called *completely regular* if every element of S is completely regular. A semigroup S is called *left (right) regular* if for each element a of S, there exists an element $x \in S$ such that $a = xa^2$ $(a = a^2x)$. A semigroup is called *left (right) regular* if for each element a of S, there exists an element $x \in S$ such that

$$a = xa^2 \ (a = a^2x).$$

A *semilattice* is a structure $S = (S, .)$, where "." is an infix binary operation, called the *semilattice operation*, such that "." is associative, commutative and idempotent. For all undefined concepts and notions about semigroups, see [22, 33].

Definition 1 [15, 31] *A soft set* f_A *over* U *is a set defined by*

$$f_A : E \to P(U) \ \textit{such that} f_A(x) = \emptyset \ \textit{if} \ x \notin A.$$

Here f_A *is also called an approximate function. A soft set over* U *can be represented by the set of ordered pairs*

$$f_A = \{(x, f_A(x)) : x \in E, f_A(x) \in P(U)\}.$$

Definition 2 [15] *Let* $f_A, f_B \in S(U)$*. Then,* f_A *is called a soft subset of* f_B *and denoted by* $f_A \tilde{\subseteq} f_B$*, if* $f_A(x) \subseteq f_B(x)$ *for all* $x \in E$.

Definition 3 [15] *Let* f_A, $f_B \in S(U)$. *Union of* f_A *and* f_B, *denoted by* $f_A \widetilde{\cup} f_B$, *is defined as* $f_A \widetilde{\cup} f_B = f_{A \widetilde{\cup} B}$, *where* $f_{A \widetilde{\cup} B}(x) = f_A(x) \cup f_B(x)$ *for all* $x \in E$. *Intersection of* f_A *and* f_B, *denoted by* $f_A \widetilde{\cap} f_B$, *is defined as* $f_A \widetilde{\cap} f_B = f_{A \widetilde{\cap} B}$, *where* $f_{A \widetilde{\cap} B}(x) = f_A(x) \cap f_B(x)$ *for all* $x \in E$.

Definition 4 [39] *Let* S *be a semigroup and* f_S *and* g_S *be soft sets over the common universe* U. *Then, soft intersection product* $f_S \circ g_S$ *is defined by*

$$(f_S \circ g_S)(x) = \begin{cases} \bigcup_{x=yz}\{f_S(y) \cap g_S(z)\}, & \text{if } \exists y, z \in S \text{ such that } x = yz, \\ \emptyset, & \text{otherwise} \end{cases}$$

for all $x \in S$.

Definition 5 [39] *Let* X *be a subset of* S. *We denote by* $\mathcal{S}_X$ *the soft characteristic function of* X *and define as*

$$\mathcal{S}_X(x) = \begin{cases} U, & \text{if } x \in X, \\ \emptyset, & \text{if } x \notin X \end{cases}$$

Definition 6 [39] *Let* S *be a semigroup and* f_S *be a soft set over* U. *Then,* f_S *is called a soft intersection semigroup of* S, *if*

$$f_S(xy) \supseteq f_S(x) \cap f_S(y)$$

for all $x, y \in S$.

Definition 7 [39] *A soft set over* U *is called a soft intersection left (right) ideal of* S *over* U *if*

$$f_S(ab) \supseteq f_S(b) \ (f_S(ab) \supseteq f_S(a))$$

for all $a, b \in S$. *A soft set over* U *is called a soft intersection two-sided ideal (soft intersection ideal) of* S *if it is both soft intersection left and soft intersection right ideal of* S *over* U.

Definition 8 [39] *A soft intersection semigroup* f_S *over* U *is called a soft intersection bi-ideal of* S *over* U *if*

$$f_S(xyz) \supseteq f_S(x) \cap f_S(z)$$

for all $x, y, z \in S$.

Definition 9 [40] *A soft set over* U *is called a soft intersection interior of* S *over* U *if* $f_S(xyz) \supseteq f_S(y)$, *soft intersection generalized bi-ideal of* S *over* U *if* $f_S(xyz) \supseteq f_S(x) \cap f_S(z)$ *for all* $x, y, z \in S$.

For the sake of brevity, soft intersection semigroup, soft intersection right (left, two-sided, interior, generalized bi-) ideal are abbreviated by SI-semigroup, SI-right (left, two-sided, quasi, generalized bi-) ideal, respectively.

It is easy to see that if $f_S(x) = U$ for all $x \in S$, then f_S is an SI-semigroup (right ideal, left ideal, ideal, bi-ideal, interior ideal, quasi-ideal, generalized bi-ideal) of S over U. We denote such a kind of SI-semigroup (right ideal, left ideal, ideal, bi-ideal) by $\widetilde{\mathbb{S}}$ [39].

Definition 10 [40] *A soft set over* U *is called a soft intersection quasi-ideal of* S *over* U *if*

$$(f_S \circ \widetilde{\mathbb{S}}) \widetilde{\cap} (\widetilde{\mathbb{S}} \circ f_S) \widetilde{\subseteq} f_S.$$

Definition 11 [39] *A soft set* f_S *over* U *is called soft semiprime if for all* $a \in S$,

$$f_S(a) \supseteq f_S(^2).$$

Theorem 1 [39, 40] *Let* X *be a nonempty subset of a semigroup* S. *Then,* X *is a subsemigroup (left, right, two-sided ideal, bi-ideal, interior ideal, quasi-ideal, generalized bi-ideal) of* S *if and only if* $\mathcal{S}_X$ *is an* SI*-semigroup (left, right, two-sided ideal, bi-ideal, interior ideal, quasi-ideal, generalized bi-ideal) of* S.

Proposition 1 [39, 40] *Let* f_S *be a soft set over* U. *Then,*

i) f_S *is an* SI*-semigroup over* U *if and only if* $f_S \circ f_S \widetilde{\subseteq} f_S$.

ii) f_S *is an* SI*-left (right) ideal of* S *over* U *if and only if* $\widetilde{\mathbb{S}} \circ f_S \widetilde{\subseteq} f_S$ $(f_S \circ \widetilde{\mathbb{S}} \widetilde{\subseteq} f_S)$.

iii) f_S *is an* SI*-bi-ideal of* S *over* U *if and only if* $f_S \circ f_S \widetilde{\subseteq} f_S$ *and* $f_S \circ \widetilde{\mathbb{S}} \circ f_S \widetilde{\subseteq} f_S$.

iv) f_S *is an* SI*-interior ideal of* S *over* U *if and only if* $\widetilde{\mathbb{S}} \circ f_S \circ \widetilde{\mathbb{S}} \widetilde{\subseteq} f_S$.

v) f_S *is an* SI*-generalized bi-ideal of* S *over* U *if and only if* $f_S \circ \widetilde{\mathbb{S}} \circ f_S \widetilde{\subseteq} f_S$.

Theorem 2 [39] *Every* SI*-left (right, two sided) ideal of a semigroup* S *over* U *is an* SI*-bi-ideal of* S *over* U.

Proposition 2 [40] *For a semigroup* S, *the following conditions are equivalent:*

1) *Every* SI*-ideal of a semigroup* S *over* U *is an* SI*-interior ideal of* S *over* U*.*

2) *Every* SI*-quasi ideal of* S *is an* SI*-semigroup of* S*.*

3) *Every one-sided* SI*-ideal of* S *is an* SI*-quasi-ideal of* S*.*

4) *Every* SI*-quasi-ideal of* S *is an* SI*-bi-ideal of* S*.*

Theorem 3 [39] *For a semigroup* S *the following conditions are equivalent:*

1) S *is regular.*

2) $f_S \circ g_S = f_S \widetilde{\cap} g_S$ *for every* SI*-right ideal* f_S *of* S *over* U *and* SI*-left ideal* g_S *of* S *over* U*.*

Theorem 4 [39] *For a semigroup* S *the following conditions are equivalent:*

1) S *is regular.*

2) *For every* SI*-quasi-ideal of* S*,* $f_S = f_S \circ \widetilde{\mathbb{S}} \circ f_S$*.*

Theorem 5 [40] *Let* f_S *be a soft set over* U*, where* S *is a regular semigroup. Then, the following conditions are equivalent:*

1) f_S *is an* SI*-ideal of* S *over* U*.*

2) f_S *is an* SI*-interior ideal of* S *over* U*.*

Theorem 6 [39] *For a left regular semigroup* S*, the following conditions are equivalent:*

1) *Every left ideal of* S *is a two-sided ideal of* S*.*

2) *Every* SI*-left ideal of* S *is an* SI*-ideal of* S*.*

For more on soft intersection semigroups and ideals, we refer [39, 40].

3 Semisimple semigroups

In this section, semisimple semigroups with respect to SI-ideals of semigroups are characterized. A semigroup S is called *semisimple* if $J^2 = J$ holds for every ideal J of S, that is, every ideal of S is idempotent.

Proposition 3 [41] *For a semigroup S, the following conditions are equivalent:*

1) S *is semisimple.*

2) $a \in (SaS)(SaS)$ *for every element* a *of* S*, that is, there exist elements* $x, y, z \in S$ *such that* $a = xayaz$*.*

Proposition 4 *Every* SI*-interior ideal of a semisimple semigroup* S *is an* SI*-ideal of* S*.*

Proof. Let f_S be an SI-interior ideal of S. Let a and b be any elements of S. Then, since S is semisimple, there exist elements $x, y, z \in S$ such that

$$a = xayaz.$$

Thus,

$$f_S(ab) = f_S((xayaz)b) = f_S(xay)a(zb)) \supseteq f_S(a)$$

Hence, f_S is an SI-right ideal of S. Similarly, one can prove that f_S is an SI-left ideal of S. Thus, f_S is an SI-ideal of S. □

Now a characterization of a semisimple semigroup by SI-ideals is given.

Theorem 7 *For a semigroup* S*, the following conditions are equivalent:*

1) S *is semisimple.*

2) $f_S \circ f_S = f_S$ *for every* SI*-ideal* f_S *of* S*. (That is, every* SI*-ideal is idempotent).*

3) $f_S \circ f_S = f_S$ *for every* SI*-interior* f_S *of* S*. (That is, every* SI*-interior ideal is idempotent).*

4) $f_S \tilde{\cap} g_S = f_S \circ g_S$ *for every* SI*-ideals* f_S *and* g_S *of* S*.*

5) $f_S \tilde{\cap} g_S = f_S \circ g_S$ *for every* SI*-ideal* f_S *and every* SI*-interior ideal* g_S *of* S*.*

6) $f_S \tilde{\cap} g_S = f_S \circ g_S$ *for every* SI*-interior ideal* f_S *and every* SI*-ideal* g_S *of* S*.*

7) $f_S \tilde{\cap} g_S = f_S \circ g_S$ *for every* SI*-interior ideals* f_S *and* g_S *of* S*.*

8) *The set of all* SI*-ideals of a semisimple semigroup* S *is a semilattice under the soft intersection product, that is,* $f_S \circ (g_S \circ h_S) = f_S \circ (g_S \circ h_S)$*,* $f_S \circ g_S = g_S \circ f_S$ *and* $f_S \circ f_S = f_S$ *for all* SI*-ideals* f_S *and* g_S *of* S*.*

9) *The set of all* SI*-interior ideals of a semisimple semigroup* S *is a semilattice under the soft intersection product.*

Proof. First assume that (1) holds. Let f_S and g_S be any SI-interior ideals of S. Since, $\widetilde{\mathbb{S}}$ itself is an SI-interior ideal of S and since f_S is an SI-ideal of S by Proposition 4:

$$f_S \circ g_S \widetilde{\subseteq} f_S \circ \widetilde{\mathbb{S}} \widetilde{\subseteq} f_S \quad \text{and} \quad f_S \circ g_S \widetilde{\subseteq} \widetilde{\mathbb{S}} \circ g_S \widetilde{\subseteq} g_S.$$

Thus, $f_S \circ g_S \widetilde{\subseteq} f_S \widetilde{\cap} g_S$.

Now, let a be any element of S. Since there exist elements $x, y, z, w \in S$ such that

$$a = (xay)(zaw),$$

$$(f_S \circ g_S)(a) \neq \emptyset.$$

And since f_S and g_S are SI-interior ideals of S,

$$\begin{aligned} (f_S \circ g_S)(a) &= \bigcup_{a=pq} (f_S(p) \cap g_S(q)) \\ &\supseteq f_S(xay) \cap g_S(zaw) \\ &\supseteq f_S(a) \cap g_S(a) \\ &= (f_S \widetilde{\cap} g_S)(a) \end{aligned}$$

and so $f_S \circ g_S \widetilde{\supseteq} f_S \widetilde{\cap} g_S$. Hence,

$$f_S \circ g_S = f_S \widetilde{\cap} g_S.$$

So, (1) implies (7). (7) implies (6), (6) implies (4), (7) implies (5), (5) implies (4), (4) implies (2), (7) implies (3), (3) implies (2) and (7) implies (9), (9) implies (8), (8) implies (2).

Assume that (2) holds. Let a be any element of S. Since the soft characteristic function $\mathcal{S}_{J[a]}$ of the principal ideal $J[a]$ of S is an SI-ideal of S,

$$\mathcal{S}_{J[a]J[a]}(a) = (\mathcal{S}_{J[a]} \circ \mathcal{S}_{J[a]})(a) = \mathcal{S}_{J[a]}(a) = U$$

and so,

$$\begin{gathered} a \in J[a]J[a] = (\{a\} \cup aS \cup Sa \cup SaS)(\{a\} \cup aS \cup Sa \cup SaS) = \\ \{a^2\} \cup a^2S \cup aSa \cup aSaS \cup aSa \cup aSaS \cup aSSa \cup aSSaS \cup Sa^2 \cup Sa^2S \cup \\ SaSa \cup SaSaS \cup SaSa \cup SaSaS \cup SaSSa \cup SaSSaS \subseteq (SaS)(SaS) \end{gathered}$$

Hence, S is semisimple and so, (2) implies (1). $\square$

4 Regular duo semigroups

In this section, a left (right) duo semigroup in terms of SI-ideals is charachterized. A semigroup S is called *left (right) duo* if every left (right) ideal of S is a two-sided ideal of S. A semigroup S is *duo* if it is both left and right duo.

Definition 12 *A semigroup S is called soft left (right) duo if every SI-left (right) ideal of S is an SI-ideal of S and is called soft duo, if it is both soft left and soft right duo.*

Theorem 8 *For a regular semigroup S, the following conditions are equivalent:*

1) S *is left (right) duo.*

2) S *is soft left (right) duo.*

Proof. First assume that S is left duo. Let f_S be any SI-left ideal of S and a and b be any elements of S. It is known that Sa is a left-ideal of S. And so, by hypothesis, it is a two-sided ideal of S. Since S is regular,

$$ab \in (aSa)b \subseteq (Sa)S \subseteq Sa$$

This implies that there exists an element $x \in S$ such that

$$ab = xa.$$

Thus, since f_S is an SI-left ideal of S,

$$f_S(ab) = f_S(xa) \supseteq f_S(a)$$

This means that f_S is an SI-right ideal of S and so f_S is an SI-ideal of S. Thus, S is soft left duo and (1) implies (2).

Conversely, assume that S is soft left duo. Let A be any left ideal of S. Then, the soft characteristic function $\mathcal{S}_A$ of A is an SI-left ideal of S. By assumption, $\mathcal{S}_A$ is an SI-ideal of S and so A is a two-sided ideal of S. Thus, S is left duo and (2) implies (1). The right dual of the proof can be seen similarly. So, the proof is completed. □

Theorem 9 *For a regular semigroup S, the following conditions are equivalent:*

1) S *is duo.*

2) S *is soft duo.*

Every SI-right (left) ideal of S is an SI-bi-ideal of S ([39]). Moreover, we have the following:

Theorem 10 *Let* S *be a regular duo semigroup. Then, every* SI*-bi-ideal of* S *is an* SI*-ideal of* S.

Proof. Let f_S be any SI-bi-ideal of S and a, b be any elements of S. It is known that Sa is a left ideal of S. Since S is a duo semigroup, Sa is a right ideal of S. And since S is regular,

$$ab \in (aSa)b \subseteq a((Sa)S) \subseteq aSa$$

This implies that there exists an element $x \in S$ such that

$$ab = axa.$$

Then, since f_S is an SI-bi-ideal of S,

$$f_S(ab) = f_S(axa) \supseteq f_S(a) \cap f_S(a) = f_S(a).$$

This means that f_S is an SI-right ideal of S. It can be seen in a similar way that f_S is an SI-left ideal of S. Therefore, f_S is an SI-ideal of S. This completes the proof. □

Theorem 11 [17, 32] *For a semigroup* S*, the following conditions are equivalent:*

1) S *is a regular duo semigroup.*

2) $A \cap B = AB$ *for every left ideal* A *and every right ideal* B *of* S.

3) $Q^2 = Q$ *for every quasi-ideal of* S. *(That is, every quasi-ideal is idempotent.)*

4) $EQE = E \cap Q \cap E$ *for every ideal* E *and every quasi-ideal* Q *of* S.

Theorem 12 *For a semigroup* S*, the following conditions are equivalent:*

1) S *is a regular duo semigroup.*

2) S *is a regular soft duo semigroup.*

3) $f_S \circ g_S = f_S \widetilde{\cap} g_S$ *for all* SI-*bi-ideals* f_S *and* g_S *of* S.

4) $f_S \circ g_S = f_S \widetilde{\cap} g_S$ *for all* SI-*bi-ideal* f_S *and for all* SI-*quasi-ideal* g_S *of* S.

5) $f_S \circ g_S = f_S \widetilde{\cap} g_S$ *for all* SI-*bi-ideal* f_S *and and for all* SI-*right ideal* g_S *of* S

6) $f_S \circ g_S = f_S \widetilde{\cap} g_S$ *for all* SI-*quasi-ideal* f_S *and for all* SI-*bi-ideal* g_S *of* S.

7) $f_S \circ g_S = f_S \widetilde{\cap} g_S$ *for all* SI-*quasi-ideals* f_S *and* g_S *of* S.

8) $f_S \circ g_S = f_S \widetilde{\cap} g_S$ *for all* SI-*quasi-ideal* f_S *and for all* SI-*right ideal* g_S *of* S.

9) $f_S \circ g_S = f_S \widetilde{\cap} g_S$ *for all* SI-*left ideal* f_S *and for all* SI-*bi-ideal* g_S *of* S.

10) $f_S \circ g_S = f_S \widetilde{\cap} g_S$ *for all* SI-*left ideal* f_S *and for all* SI-*right ideal* g_S *of* S.

11) $f_S \circ g_S = f_S \widetilde{\cap} g_S$ *and* $h_S \circ k_S = h_S \widetilde{\cap} k_S$ *for all* SI-*right ideals* f_S *and* g_S *of* S *and for all* SI-*left ideal* h_S *and* k_S *of* S.

12) *Every* SI-*quasi-ideal of* S *is idempotent.*

Proof. The equivalence of (1) and (2) follows from Theorem 9. Assume that (2) holds. Let f_S and g_S be any SI-bi-ideals of S. Then, by Theorem 10, f_S is an SI-right ideal of S and g_S is an SI-left ideal of S. Since S is regular, it follows by Theorem 3 that

$$f_S \circ g_S = f_S \widetilde{\cap} g_S$$

Thus, (2) implies (3). It is clear that (3) implies (4), (4) implies (5), (5) implies (8), (8) implies (11), (11) implies (3), (3) implies (6), (6) implies (7), (7) implies (8) and (6) implies (9), (9) implies (10), (10) implies (11).

Assume that (11) holds. Let A and B be any left ideal and right ideal of S, respectively. Let a be any element of $A \cap B$. Then, $a \in A$ and $b \in B$ and so,

$$\mathcal{S}_A(a) = \mathcal{S}_B(a) = U.$$

Since $\mathcal{S}_A$ and $\mathcal{S}_B$ is an SI-left ideal and SI-right ideal of S, respectively, by assumption

$$\mathcal{S}_{AB}(a) = (\mathcal{S}_A \circ \mathcal{S}_B)(a) = (\mathcal{S}_A \widetilde{\cap} \mathcal{S}_B)(a) = \mathcal{S}_A(a) \cap \mathcal{S}_B(a) = U,$$

so $a \in AB$. Thus, $A \cap B \subseteq AB$. For the converse inclusion, let a be any element of AB. Thus,

$$\mathcal{S}_{A\cap B}(a) = (\mathcal{S}_A \widetilde{\cap} \mathcal{S}_B)(a) = (\mathcal{S}_A \circ \mathcal{S}_B)(a) = \mathcal{S}_{AB}(a) = U$$

This implies that $a \in A \cap B$ and that $AB \subseteq A \cap B$. Thus, $AB = A \cap B$. It follows by Theorem 11 that S is a regular duo semigroup. Thus (11) implies (1). It is clear that (7) implies (12) by taking $g_S = f_S$.

Conversely, assume that (12) holds. Let Q be any quasi-ideal of S and a be any element of Q. Then, $\mathcal{S}_Q$ is an SI-quasi-ideal of S. Then,

$$\mathcal{S}_{Q^2}(a) = (\mathcal{S}_Q \circ \mathcal{S}_Q)(a) = \mathcal{S}_Q(a) = U$$

Thus, $a \in Q^2$ and $Q \subseteq Q^2$. Since the converse inclusion always holds, $Q = Q^2$. It follows by Theorem 11 that S is a regular duo semigroup and that (12) implies (1). This completes the proof. □

Theorem 13 *For a semigroup S, the following conditions are equivalent:*

1) S *is a regular duo semigroup.*

2) $f_S \circ g_S \circ f_S = f_S \widetilde{\cap} g_S$ *for every* SI*-ideal* f_S *and every* SI*-bi-ideal* g_S *of* S.

3) $f_S \circ g_S \circ f_S = f_S \widetilde{\cap} g_S$ *for every* SI*-ideal* f_S *and every* SI*-quasi-ideal* g_S *of* S.

Proof. First assume that (1) holds. Let f_S and g_S be any SI-bi-ideal and any SI-ideal of S, respectively. Then,

$$f_S \circ g_S \circ f_S \widetilde{\subseteq} (f_S \circ \widetilde{\mathbb{S}}) \circ \widetilde{\mathbb{S}} = f_S \circ (\widetilde{\mathbb{S}} \circ \widetilde{\mathbb{S}}) \widetilde{\subseteq} f_S \circ \widetilde{\mathbb{S}} \widetilde{\subseteq} f_S$$

On the other hand, since S is regular and duo, f_S is an SI-ideal of S by Theorem 10. Hence,

$$f_S \circ g_S \circ f_S \widetilde{\subseteq} (\widetilde{\mathbb{S}} \circ g_S) \circ \widetilde{\mathbb{S}} \widetilde{\subseteq} g_S \circ \widetilde{\mathbb{S}} \widetilde{\subseteq} g_S$$

and so

$$f_S \circ g_S \circ f_S \widetilde{\subseteq} f_S \widetilde{\cap} g_S$$

In order to show the converse inclusion, let a be any element of S. Then, since S is regular, there exists an element x in S such that

$$a = axa = (axa)xa$$

Thus,

$$\begin{aligned}(f_S \circ g_S \circ f_S)(a) &= [f_S \circ (g_S \circ f_S)](a) \\ &= \bigcup_{a=pq} [f_S(a) \cap (g_S \circ f_S)(q)]\end{aligned}$$

$$\begin{aligned}
&\supseteq f_S(ax)\cap(g_S\circ f_S)(axa)\\
&= f_S(ax)\cap\{\bigcup_{axa=bc}[g_S(b)\circ f_S(c)]\}\\
&\supseteq f_S(ax)\cap(g_S(a)\cap f_S(xa))\\
&\supseteq f_S(a)\cap(g_S(a)\cap f_S(a))\\
&= f_S(a)\cap g_S(a)\\
&= (f_S\tilde{\cap}g_S)(a)
\end{aligned}$$

and so $f_S\circ g_S\circ f_S\tilde{\supseteq}f_S\tilde{\cap}g_S$ Thus,

$$f_S\circ g_S\circ f_S=f_S\tilde{\cap}g_S.$$

Hence, (1) implies (2). It is clear that (2) implies (3).

Assume that (3) holds. Let E and Q any two-sided ideal and quasi-ideal of S, respectively and a be any element of $E\cap Q$. Then,

$$\mathcal{S}_E(a)=\mathcal{S}_Q(a)=U.$$

Since $\mathcal{S}_E$ and $\mathcal{S}_Q$ is an SI-ideal and an SI-quasi-ideal of S, respectively,

$$\mathcal{S}_{EQE}(a)=(\mathcal{S}_E\circ\mathcal{S}_Q\circ\mathcal{S}_E)(a)=(\mathcal{S}_E\tilde{\cap}\mathcal{S}_Q)(a)=\mathcal{S}_E(a)\cap\mathcal{S}_Q(a)=U$$

and so $a\in EQE$. Thus, $E\cap Q\subseteq EQE$. For the converse inclusion, let a be any element of EQE. Thus,

$$\mathcal{S}_{E\cap Q}(a)=(\mathcal{S}_E\tilde{\cap}\mathcal{S}_Q)(a)=(\mathcal{S}_E\circ\mathcal{S}_Q\circ\mathcal{S}_E)(a)=\mathcal{S}_{EQE}(a)=U$$

and so $a\in EQE$. Thus, $EQE\subseteq E\cap Q$ and so $EQE=E\cap Q$. It follows from Proposition 11 that S is regular duo. Hence, (3) implies (1). This completes the proof. □

5 Right (left) zero semigroup

In this section, right (left) zero semigroups are charachterized in terms of SI-ideals of S. A semigroup S is called *right (left) zero* if $xy=y$ $(xy=x)$ for all $x,y\in S$.

Proposition 5 *For a semigroup S, the following conditions are equivalent:*

1) *The set of all idempotent elements of S forms a left (right) zero subsemigroup of S.*

2) *For every* SI*-left (right) ideal* f_S *of* S, $f_S(e) = f_S(f)$ *for all idempotent elements* e *and* f *of* S.

Proof. First assume that the set I_S of all idempotent elements of S is a left zero subsemigroup of S. Let $e, f \in I_S$ and f_S be an SI-left ideal of S. Then, since

$$ef = e \quad \text{and} \quad fe = f$$

$$f_S(e) = f_S(ef) \supseteq f_S(f) = f_S(fe) \supseteq f_S(e)$$

and so

$$f_S(e) = f_S(f).$$

Thus, (1) implies (2).

Conversely, assume that (2) holds. Since S is regular, it is obvious that $I_S \neq \emptyset$. Moreover, the soft characteristic function $\mathcal{S}_{L[f]}$ of the left ideal $L[f]$ of S is an SI-left ideal of S. Thus, by assumptio,n

$$\mathcal{S}_{L[f]}(e) = \mathcal{S}_{L[f]}(f) = U$$

and so $e \in L[f] = Sf$. (Here note that, if S is a regular semigroup, $L[a] = Sa$ for every $a \in S$ ([17]). Thus, for some $x \in S$,

$$e = xf = x(ff) = (xf)f = ef$$

This means that I_S is a left zero semigroup. Thus (2) implies (1). The case when S is right zero, the proof can be seen similarly. This completes the proof. □

Corollary 1 *For an idempotent semigroup* S, *the following conditions are equivalent:*

1) S *is left (right) zero.*

2) *For every* SI*-left (right) ideal* f_S *of* S, $f_S(e) = f_S(f)$ *for all elements* $e, f \in S$.

Proposition 6 *Let* S *be a group. Then, every* SI*-bi-ideal of* S *is a constant function.*

Proof. Let S be a group with identity e and f_S be any SI-bi-ideal of S and a be any element of S. Then,

$$f_S(a) = f_S(eae) \supseteq f_S(e) \cap f_S(e) = f_S(e) = f_S(ee) = f_S((aa^{-1})(a^{-1}a)) = f_S(a(a^{-1}a^{-1})a) \supseteq f_S(a) \cap f_S(a) = f_S(a)$$

and so $f_S(e) = f_S(a)$. This implies that f_S is a constant function. □

Proposition 7 *For a regular semigroup* S*, the following conditions are equivalent:*

1) S *is a group.*

2) *For every* SI-*bi-ideal* f_S *of* S*,* $f_S(e) = f_S(f)$ *for all idempotent elements* $e, f \in S$.

Proof. Assume that (1) holds. Let f_S be any SI-bi-ideal of S. Then, it follows from Proposition 6 that f_S is a constant function. This implies that

$$f_S(e) = f_S(f)$$

for all idempotent elements $e, f \in S$. Thus (1) implies (2).

Conversely, assume that (2) holds. Let e and f be any idempotent elements of S. As is well-known, if S is a regular semigroup, B[x], the principal ideal of S generated by $x \in S$ is $B[x] = xSx$ ([17]). Moreover, since the soft characteristic function $\mathcal{S}_{B[f]}$ of the bi-ideal $B[f]$ of S is an SI-bi-ideal of S and since $f \in B[f]$,

$$\mathcal{S}_{B[f]}(e) = \mathcal{S}_{B[f]}(f) = U$$

and so $e \in B[f] = fsf$, which means that $e = fxf$ for some $x \in S$. One can similarly obtain that $f = eye$ for some $y \in S$. Thus,

$$e = fxf = fx(ff) = (fxf)f = ef = e(eye) = (ee)ye = eye = f$$

Since S is regular, $I_S \neq \emptyset$ and S contains exactly one idempotent. Thus, it follows from ([17], p.33) that S is a group. Thus (2) implies (1). This completes the proof. □

6 Right (left) simple semigroups

In this section, soft simple semigroup is defined and the relation of soft simple semigroup with simple semigroup is given. A semigroup S is called *left (right) simple* if it contains no proper left (right) ideal of S and is called *simple* if it contains no proper ideal.

Definition 13 *A semigroup* S *is called soft left (right) simple if every* SI-*left (right) ideal of* S *is a constant function and is called soft simple if every* SI-*ideal of* S *is a constant function.*

Theorem 14 *For a semigroup* S*, the following conditions are equivalent:*

1) S *is left (right) simple.*

2) S *is soft left (right) simple.*

Proof. First assume that S is left simple. Let f_S be any SI-left ideal of S and a and b be any element of S. Then, it follows from ([17], p. 6) that there exist elements $x, y \in S$ such that $b = xa$ and $a = yb$. Hence, since S is an SI-left ideal of S,

$$f_S(a) = f_S(yb) \supseteq f_S(b) = f_S(xa) \supseteq f_S(a)$$

and so $f_S(a) = f_S(b)$. Since a and b be any elements of S, this means that f_S is a constant function. Thus, it is obtained that S is soft left simple and (1) implies (2).

Conversely, assume that (2) holds. Let A be any left ideal of S. Then, $\mathcal{S}_A$ is an SI-left ideal of S. By assumption, $\mathcal{S}_A$ is a constant function. Let x be any element of S. Then, since $A \neq \emptyset$,

$$\mathcal{S}_A(x) = U$$

and so $x \in A$. This implies that $S \subseteq A$, and so $S = A$. Hence, S is left simple and (2) implies (1). In the case, when S is soft right simple, the proof follows similarly. □

Theorem 15 *For a semigroup* S*, the following conditions are equivalent:*

1) S *is simple.*

2) S *is soft simple.*

As is well-known, a semigroup S is a group if it is left and right simple. From this, the following theorem:

Proposition 8 *For a semigroup* S*, the following conditions are equivalent:*

1) S *is a group.*

2) S *is both soft left and soft right simple.*

Proposition 9 *Let* S *be a left simple semigroup. Then, every* SI*-bi-ideal of* S *is an* SI*-right ideal of* S*.*

Proof. Let f_S be an SI-bi-ideal of S and a and b be any elements of S. Then, since S is left simple, there exists an element x in S such that

$$b = xa.$$

Then, since f_S is an SI-bi-ideal of S,

$$f_S(ab) = f_S(a(xa)) = f_S(a) \cap f_S(a) = f_S(a)$$

which means that f_S is an SI-right ideal of S. This completes the proof. □

7 Semilattices of left (right) simple semigroups

In this section, a semigroup that is a semilattice of left (right) simple semigroups is characterized by SI-ideals. A semigroup S is a *semilattice of left simple semigroups* if it is the set-theoretical union of the family of left simple semigroups S_i $(i \in M)$ such that,

$$S = \bigcup_{i \in M} S_i$$

such that the products S_iS_j and S_jS_i are both contained in the same S_k $(k \in M)$.

Theorem 16 [17, 34] *For a semigroup S, the following conditions are equivalent:*

1) *S is a semilattice of left simple semigroups.*

2) *S is left regular and every left ideal of S is two-sided.*

3) *S is left regular and $AB = BA$ for any left ideals A and B of S.*

Theorem 17 [39] *For a left regular semigroup S, the following conditions are equivalent:*

1) *Every left ideal of S is a two-sided ideal of S.*

2) *Every SI-left ideal of S is an SI-ideal of S.*

Theorem 18 *For a semigroup S, the following conditions are equivalent:*

1) *S is a semilattice of left simple semigroups.*

2) *S is left regular and every SI-left ideal of S is an SI-ideal of S.*

3) $f_S \circ g_S = f_S \widetilde{\cap} g_S$ *for every SI-left ideals of S.*

4) *The set of all SI-left ideals of S is a semilattice under the soft int-product.*

5) *The set of all left ideals of S is a semilattice under the multiplication of subsets.*

Proof. The equivalence of (1) and (2) follows from Theorem 16 and Theorem 17. Assume that (2) holds. Let f_S and g_S be any SI-left ideals of S and a be any element of S. Then, since S is left regular, there exists an element $x \in S$ such that $a = xa^2$. By assumption, f_S is also an SI-right ideal of S. So,

$$\begin{aligned}(f_S \circ g_S)(a) &= \bigcup_{a=yz} (f_S(y) \cap g_S(z)) \\ &\supseteq (f_S(xa) \cap g_S(a)) \\ &\supseteq (f_S(a) \cap g_S(a)) \\ &= (f_S \widetilde{\cap} g_S)(a)\end{aligned}$$

Thus, $f_S \circ g_S \widetilde{\supseteq} f_S \widetilde{\cap} g_S$. On the other hand, by assumption, g_S is SI-right ideal of S, and so

$$\begin{aligned}(f_S \circ g_S)(a) &= \bigcup_{a=yz} (f_S(y) \cap g_S(z)) \\ &\subseteq (f_S(yz) \cap g_S(yz)) \\ &= f_S(a) \cap g_S(a) \\ &= (f_S \widetilde{\cap} g_S)(a)\end{aligned}$$

Thus, $f_S \circ g_S \widetilde{\subseteq} f_S \widetilde{\cap} g_S$. Thus, $f_S \circ g_S = f_S \widetilde{\cap} g_S$ and so (2) implies (3).

(3) implies (4) is clear. Assume that (4) holds. Let A and B be any left ideals of S and a be any element of BA. Since the soft characteristic function $\mathcal{S}_A$ and $\mathcal{S}_B$ are SI-left ideals of S,

$$\mathcal{S}_{AB}(a) = (\mathcal{S}_A \circ \mathcal{S}_B)(a) = (\mathcal{S}_B \circ \mathcal{S}_A)(a) = \mathcal{S}_{BA}(a) = U$$

which implies that $a \in AB$. Thus, $BA \subseteq AB$. Similarly, $AB \subseteq BA$. Thus, $AB = BA$.

In order to see that any left ideal A of S is idempotent, let a be any element of A. Since $\mathcal{S}_A$ is an SI-left ideal of S,

$$\mathcal{S}_{A^2}(a) = (\mathcal{S}_A \circ \mathcal{S}_A)(a) = \mathcal{S}_A = U$$

and so $a \in A^2$. Thus, $A \subseteq A^2$ and so $A = A^2$. Therefore (4) implies (5).

Finally, assume that (5) holds. Let A be any left ideal of S and a be any element of S. Then, since S itself is a left ideal, by assumption

$$AS = SA \subseteq A$$

Thus, A is a right ideal of S, and so A is a two-sided ideal of S.

Let a be any element of S. Then, since the left ideal $L[a]$ of S is idempotent by assumption and since $a \in L[a]$,

$$a \in L[a]L[a] = (\{a\} \cup Sa)(\{a\} \cup Sa) = \{a^2\} \cup aSa \cup Sa^2 \cup SaSa \subseteq \\ \{a^2\} \cup (aS)aSa \cup Sa^2 \cup SaSa \subseteq \{a^2\} \cup SaSa \cup Sa^2 \subseteq \{a^2\} \cup Sa^2$$

which implies that S is left-regular. Thus, it follows by Theorem 16-(2) that S is a semilattice of left simple groups. That is to say (5) implies (1). This completes the proof. $\square$

The left-right dual of Theorem 18 reads as follows:

Theorem 19 *For a semigroup S, the following conditions are equivalent:*

1) *S is a semilattice of right simple semigroups.*

2) *S is right regular and every SI-right ideal of S is an SI-ideal of S.*

3) $f_S \circ g_S = f_S \tilde{\cap} g_S$ *for every SI-right ideals of S.*

4) *The set of all SI-right ideals of S is a semilattice under the soft int-product.*

5) *The set of all right ideals of S is a semilattice under the multiplication of subsets.*

Theorem 20 [39] *For a semigroup S, the following conditions are equivalent:*

1) *S is left regular.*

2) *For every SI-left ideal f_S of S, $f_S(a) = f_S(a^2)$ for all $a \in S$.*

Theorem 21 *For a semigroup S, the following conditions are equivalent:*

1) *S is a semilattice of left simple semigroups.*

2) *For every SI-left ideal f_S of S, $f_S(a) = f_S(a^2)$ and $f_S(ab) = f_S(ba)$ for all $a, b \in S$.*

Proof. Assume that S is a semilattice of left simple semigroups. Let f_S be any SI-left ideal of S. Then, by Theorem 16-(2), S is left regular and f_S is an SI-ideal of S. Let a be any element of S. Thus, by Theorem 20,

$$f_S(ab) = f_S((ab)^2) = f_S(a(ba)b) \supseteq f_S(ba).$$

Similarly, $f_S(ba) \supseteq f_S(ab)$. Hence,

$$f_S(ab) = f_S(ba).$$

Thus, (1) implies (2).

Conversely, assume that (2) holds. Let f_S be any SI-ideal of S. Since $f_S(a) = f_S(a^2)$ for all $a \in S$, it follows from Theorem 20 that S is left regular. Let A and B be any left ideal of S and ab be any element of AB. Since the soft characteristic function $\mathcal{S}_{L[ba]}$ of the the left ideal $L[ba]$ is an SI-left ideal of S and since $ba \in L[ba]$,

$$\mathcal{S}_{L[ba]}(ab) = \mathcal{S}_{L[ba]}(ba) = U$$

This implies that

$$ab \in L[ba] = \{ba\} \cup Sba \subseteq BA \cup SBA \subseteq BA$$

and so $AB \subseteq BA$. Similarly, it can be seen that the converse inclusion holds. Thus,

$$AB = BA$$

Then, it follows by Theorem 16-(3) that S is a semilattice of left simple semigroups. Therefore (3) implies (1). This completes the proof. □

The right dual of Theorem 21 is as following:

Theorem 22 *For a semigroup S, the following conditions are equivalent:*

1) S *is a semilattice of right simple semigroups.*

2) *For every SI-right ideal f_S of S, $f_S(a) = f_S(a^2)$ and $f_S(ab) = f_S(ba)$ for all $a, b \in S$.*

8 A semilattice of left (right) groups

In this section, a semigroup that is a semilattice of left (right) simple groups is characterized by SI-ideals. An element a of S is said to be *left (right) cancellable* if, for any $x, y \in S$ $ax = ay$ $(xa = ya)$ implies $x = y$. A semigroup S is called *left (right) cancellative* if every element of S is left (right) cancellative. A semigroup S is called a *left group* if it is left simple and right cancellable ([17]), that is, for all $a \in S$, there exists a unique element $x \in S$ such that $xa^2 = a$ ([33]). Dually, a semigroup S is called a *right group* if it is right simple and left cancellable.

Theorem 23 [33] *For a semigroup S, the following conditions are equivalent:*

1) *S is a semilattice of left groups.*

2) *S is regular and $aS \subseteq Sa$ for every $a \in S$.*

Theorem 24 *Let S be a semigroup that is a semilattice of left groups. Then, every SI-(generalized) bi-ideal of S is an SI-right ideal of S.*

Proof. Let f_S be any SI-bi-ideal of S, and a and b any elements of S. Then, it follows from Theorem 23 that there exist elements $x, y \in S$ such that

$$a = axa \text{ and } ab = ya.$$

Thus,

$$ab = (axa)b = (ax)(ab) = (ax)(ya) = a(xy)a.$$

Since f_S is an SI-bi-ideal of S,

$$f_S(ab) = f_S(a(xy)a) \supseteq f_S(a) \cap f_S(a) = f_S(a).$$

Hence, f_S is an SI-right ideal of S. □

Corollary 2 *Let S be a semigroup that is a semilattice of left groups. Then, every SI-left ideal of S is an SI-right ideal of S, that is to say, S is soft left duo.*

Theorem 25 *Let S be a semigroup that is a semilattice of left groups. Then, every SI-interior ideal of S is an SI-left ideal of S.*

Proof. Let f_S be any SI-interior ideal of S, and a and b any elements of S. Then, it follows from Theorem 23 that there exist element $z \in S$ such that

$$b = bzb.$$

Thus,

$$ab = (axa)b = (ax)(ab) = (ax)(ya) = a(xy)a.$$

Since f_S is an SI-bi-ideal of S,

$$f_S(ab) = f_S(a(bzb)) = f_S((a)b(zb)) \supseteq f_S(b).$$

Hence, f_S is an SI-left ideal of S. $\square$

Theorem 26 [40] *For a semigroup S the following conditions are equivalent:*

1) S *is regular.*

2) $f_S \tilde{\cap} g_S = f_S \circ g_S \circ f_S$ *for every* SI*-quasi-ideal* f_S *of* S *and* SI*-ideal* g_S *of* S *over* U.

Theorem 27 *For a semigroup S, the following conditions are equivalent:*

1) S *is a semilattice of left groups.*

2) $f_S \tilde{\cap} g_S = f_S \circ g_S$ *for every* SI*-quasi-ideal* f_S *and* SI*-left ideal* g_S *of* S.

3) $f_S \tilde{\cap} g_S = f_S \circ g_S$ *for every* SI*-quasi-ideal* f_S *and* SI*-ideal* g_S *of* S.

4) $f_S \tilde{\cap} g_S = f_S \circ g_S$ *for every* SI*-quasi-ideal* f_S *and* SI*-interior ideal* g_S *of* S.

5) $f_S \tilde{\cap} g_S = f_S \circ g_S$ *for every* SI*-bi-ideal* f_S *and* SI*-left ideal* g_S *of* S.

6) $f_S \tilde{\cap} g_S = f_S \circ g_S$ *for every* SI*-bi-ideal* f_S *and* SI*-ideal* g_S *of* S.

7) $f_S \tilde{\cap} g_S = f_S \circ g_S$ *for every* SI*-bi-ideal* f_S *and* SI*-interior ideal* g_S *of* S.

8) $f_S \tilde{\cap} g_S = f_S \circ g_S$ *for every* SI*-bi-ideal* f_S *and* SI*-left ideal* g_S *of* S.

9) $f_S \tilde{\cap} g_S = f_S \circ g_S$ *for every* SI*-generalized bi-ideal* f_S *and* SI*-left ideal* g_S *of* S.

10) $f_S \tilde{\cap} g_S = f_S \circ g_S$ *for every* SI*-generalized bi-ideal* f_S *and* SI*-ideal* g_S *of* S.

11) $f_S \tilde{\cap} g_S = f_S \circ g_S$ *for every* SI*-generalized bi-ideal* f_S *and* SI*-interior ideal* g_S *of* S.

12) $f_S \tilde{\cap} g_S = f_S \circ g_S$ *for every* SI*-one-sided ideal* f_S *and* SI*-ideal* g_S *of* S.

13) $f_S \tilde{\cap} g_S = f_S \circ g_S$ *for every* SI*-one-sided ideal* f_S *and* SI*-interior ideal* g_S *of* S.

14) *is regular left duo.*

Proof. First assume that (1) holds. Let f_S and g_S be any SI-generalize bi-ideal of S and SI-interior ideal of S, respectively and a be any element of S. Then, since S is regular by Theorem 23, there exists an element $x \in S$ such that

$$a = axa (= axaxa).$$

Since g_S is an SI-interior ideal of S, $g_S((x)a(xa)) \supseteq g_S(a)$. Thus,

$$\begin{aligned}
(f_S \circ g_S)(a) &= \bigcup_{a=pq} (f_S(p) \cap g_S(q)) \\
&\supseteq f_S(a) \cap g_S((x)a(xa)) \\
&\supseteq f_S(a) \cap g_S(a) \\
&= (f_S \widetilde{\cap} g_S)(a)
\end{aligned}$$

and so $f_S \circ g_S \widetilde{\supseteq} f_S \widetilde{\cap} g_S$. Moreover, it follows by Theorem 24 that f_S is an SI-right ideal of S. Thus,

$$\begin{aligned}
(f_S \circ g_S)(a) &= \bigcup_{a=pq} (f_S(p) \cap g_S(q)) \\
&\subseteq \bigcup_{a=pq} (f_S(pq) \cap g_S(pq)) \\
&= \bigcup_{a=pq} (f_S(a) \cap g_S(a)) \\
&= f_S(a) \cap g_S(a) \\
&= (f_S \widetilde{\cap} g_S)(a)
\end{aligned}$$

and so $f_S \circ g_S \widetilde{\subseteq} f_S \widetilde{\cap} g_S$. Therefore, $f_S \circ g_S = f_S \widetilde{\cap} g_S$ and that (1) implies (10). It is clear that (10) implies (9), (9) implies (8), (8) implies (5), (5) implies (2), (10) implies (7), (7) implies (6), (6) implies (5), (5) implies (2), (7) implies (4), (4) implies (3), (3) implies (2) and (4) implies (12), (12) implies (11).

Assume that (2) holds. Then, it follows by Theorem 26 that S is regular. Let Q be any quasi-ideal of S. Then, the soft characteristic function $\mathcal{S}_Q$ is an SI-quasi-ideal of S. Since $\widetilde{\mathbb{S}}$ itself is an SI-left ideal of S and so by assumption,

$$\mathcal{S}_Q = \mathcal{S}_Q \widetilde{\cap} \widetilde{\mathbb{S}} = \mathcal{S}_Q \circ \widetilde{\mathbb{S}}.$$

Thus, $\mathcal{S}_Q$ is an SI-right ideal of S, and so Q is a right ideal of S. Thus, any quasi-ideal of S is a right ideal of S. Let $a \in S$. Then, the quasi-ideal Sa is a right ideal of S. Since S is regular,

$$aS \subseteq (aSa)S = ((aS)a)S \subseteq (Sa)S \subseteq Sa.$$

Thus, $aS \subseteq Sa$ and since S is regular, S is a semilattice of left groups by Theorem 23. Thus, (2) implies (1).

Assume that (11) holds. Let f_S and g_S be any SI-right ideal and any SI-left ideal of S, respectively. Then, since $\widetilde{\mathbb{S}}$ itself is an SI-ideal of S and so by assumption,

$$g_S = g_S \widetilde{\cap} \widetilde{\mathbb{S}} = g_S \circ \widetilde{\mathbb{S}}$$

Thus, g_S is an SI-right ideal of S, that is, g_S is an SI-ideal of S. Thus, by assumption, $f_S \circ g_S = f_S \widetilde{\cap} g_S$ for every SI-right ideal f_S of S over U and SI-left ideal g_S of S over U. It follows by Theorem 3 that S is regular. As is proved in (2) implies (1), $aS \subseteq Sa$. Thus, S is a semilattice of left groups, so (11) implies (1).

Assume that (1) holds. Then, it follows by Theorem 23 that S is regular. Moreover, it follows by Corollary 2 that S is soft left duo and so by Theorem 8, S is left duo. Thus (1) implies (13).

Conversely assume that (13) holds. Then, it follows by Theorem 8 that S is left duo, that is, every left ideal of S is a right ideal of S. In order to prove that S is semilattice of left groups, by Theorem 23, it suffices to show that $aS \subseteq Sa$ for all $a \in S$. As is proved in (2) implies (1), $aS \subseteq Sa$. Thus, S is a semilattice of left groups, so (13) implies (1). This completes the proof. □

The left-right dual of Theorem 27 is as following:

Theorem 28 *For a semigroup S, the following conditions are equivalent:*

1) S *is a semilattice of right groups.*

2) $f_S \widetilde{\cap} g_S = f_S \circ g_S$ *for every* SI-*quasi-ideal* f_S *and* SI-*right ideal* g_S *of* S.

3) $f_S \widetilde{\cap} g_S = f_S \circ g_S$ *for every* SI-*quasi-ideal* f_S *and* SI-*ideal* g_S *of* S.

4) $f_S \widetilde{\cap} g_S = f_S \circ g_S$ *for every* SI-*quasi-ideal* f_S *and* SI-*interior ideal* g_S *of* S.

5) $f_S \widetilde{\cap} g_S = f_S \circ g_S$ *for every* SI-*bi-ideal* f_S *and* SI-*right ideal* g_S *of* S.

6) $f_S \widetilde{\cap} g_S = f_S \circ g_S$ *for every* SI-*bi-ideal* f_S *and* SI-*ideal* g_S *of* S.

7) $f_S \widetilde{\cap} g_S = f_S \circ g_S$ *for every* SI-*bi-ideal* f_S *and* SI-*interior ideal* g_S *of* S.

8) $f_S \widetilde{\cap} g_S = f_S \circ g_S$ *for every* SI-*bi-ideal* f_S *and* SI-*right ideal* g_S *of* S.

9) $f_S \tilde{\cap} g_S = f_S \circ g_S$ *for every* SI-*generalized bi-ideal* f_S *and* SI-*right ideal* g_S *of* S.

10) $f_S \tilde{\cap} g_S = f_S \circ g_S$ *for every* SI-*generalized bi-ideal* f_S *and* SI-*ideal* g_S *of* S.

11) $f_S \tilde{\cap} g_S = f_S \circ g_S$ *for every* SI-*generalized bi-ideal* f_S *and* SI-*interior ideal* g_S *of* S.

12) $f_S \tilde{\cap} g_S = f_S \circ g_S$ *for every* SI-*one-sided ideal* f_S *and* SI-*ideal* g_S *of* S.

13) $f_S \tilde{\cap} g_S = f_S \circ g_S$ *for every* SI-*one-sided ideal* f_S *and* SI-*interior ideal* g_S *of* S.

14) S *is regular right duo.*

Theorem 29 *For a semigroup* S*, the following conditions are equivalent:*

1) S *is a semilattice of left groups.*

2) $f_S \tilde{\cap} g_S = f_S \circ g_S \circ f_S$ *for every* SI-*quasi-ideal* f_S *and* SI-*left ideal* g_S *of* S.

3) $f_S \tilde{\cap} g_S = f_S \circ g_S \circ f_S$ *for every* SI-*bi-ideal* f_S *and* SI-*left ideal* g_S *of* S.

4) $f_S \tilde{\cap} g_S = f_S \circ g_S \circ f_S$ *for every* SI-*generalized bi-ideal* f_S *and* SI-*left ideal* g_S *of* S.

Proof. First assume that (1) holds. Let f_S and g_S be any SI-generalized bi-ideal of S. Then,

$$f_S \circ g_S \circ f_S \tilde{\subseteq} f_S \circ \tilde{\mathbb{S}} \circ f_S \tilde{\subseteq} f_S$$

On the other hand, since the SI-left ideal g_S is an SI-bi-ideal of S,

$$f_S \circ g_S \circ f_S \tilde{\subseteq} (\tilde{\mathbb{S}} \circ g_S) \circ \tilde{\mathbb{S}} \tilde{\subseteq} g_S \circ \tilde{\mathbb{S}} \tilde{\subseteq} g_S$$

Therefore,

$$f_S \circ g_S \circ f_S \tilde{\subseteq} f_S \tilde{\cap} g_S.$$

Let a be any element of S. Then, it follows by Theorem 23 that there exist elements $x, y \in S$ such that $a = axa$ and $ax = ya$. Hence,

$$ax = axaxax = axax(ya) = (axa)(xya).$$

Thus,

$$(f_S \circ g_S \circ f_S)(a) \quad = \quad [(f_S \circ g_S) \circ f_S](a)$$

$$
\begin{aligned}
&= \bigcup_{a=pq} [(f_S \circ g_S)(p) \circ f_S(q)] \\
&\supseteq (f_S \circ g_S)(ax) \cap f_S(a) \\
&= \{ \bigcup_{ax=pq} (f_S(p) \cap g_S(q)) \cap f_S(a) \\
&\supseteq (f_S(axa) \cap g_S(xya)) \cap f_S(a) \\
&\supseteq (f_S(a) \cap g_S(a)) \cap f_S(a) \\
&= (f_S \widetilde{\cap} g_S)(a)
\end{aligned}
$$

and so, $f_S \circ g_S \circ f_S \widetilde{\supseteq} f_S \widetilde{\cap} g_S$. Thus, $f_S \circ g_S \circ f_S = f_S \widetilde{\cap} g_S$ and (1) implies (4). It is clear that (4) implies (3) and (3) implies (2).

Assume that (2) holds. Let f_S be any SI-quasi ideal of S. Then, $\widetilde{\mathbb{S}}$ is an SI-left ideal of S and so by assumption,

$$f_S = f_S \widetilde{\cap} \widetilde{\mathbb{S}} = f_S \circ \widetilde{\mathbb{S}} \circ f_S$$

Thus,it follows by Theorem 4 that S is regular. On the other hand, let g_S be any SI-left ideal of S. then, by assumption,

$$g_S = \widetilde{\mathbb{S}} \widetilde{\cap} g_S = \widetilde{\mathbb{S}} \circ g_S \circ \widetilde{\mathbb{S}}$$

Thus, g_S is an SI-interior ideal of S. Since S is regular, g_S is an SI-ideal of S by Theorem 5. Therefore, every SI-left ideal of S is an ideal of S. It follows by Theorem 6 that every SI-left ideal of S is an SI-ideal of S. Let $a \in S$. Since S is regular, the left ideal Sa is an ideal of S. Thus,

$$aS \subseteq (aSa)S \subseteq a((Sa)S) \subseteq a(Sa) = (aS)a \subseteq Sa.$$

Thus, $aS \subseteq Sa$ and since S is regular, S is a semilattice of left groups by Theorem 23. Thus (2) implies (1). $\square$

The left-right dual of Theorem 29 is as following:

Theorem 30 *For a semigroup S, the following conditions are equivalent:*

1) *S is a semilattice of right groups.*

2) $f_S \widetilde{\cap} g_S = f_S \circ g_S \circ f_S$ *for every SI-quasi-ideal f_S and SI-right ideal g_S of S.*

3) $f_S \widetilde{\cap} g_S = f_S \circ g_S \circ f_S$ *for every SI-bi-ideal f_S and SI-right ideal g_S of S.*

4) $f_S \widetilde{\cap} g_S = f_S \circ g_S \circ f_S$ *for every SI-generalized bi-ideal f_S and SI-right ideal g_S of S.*

Theorem 31 *For a semigroup S, the following conditions are equivalent:*

1) *S is a semilattice of left groups.*

2) *$f_S \tilde{\cap} g_S = f_S \circ \tilde{\mathbb{S}} \circ g_S$ for every SI-quasi-ideal f_S and SI-left ideal g_S of S.*

3) *$f_S \tilde{\cap} g_S = f_S \circ \tilde{\mathbb{S}} \circ g_S$ for every SI-bi-ideal f_S and SI-left ideal g_S of S.*

4) *$f_S \tilde{\cap} g_S = f_S \circ \tilde{\mathbb{S}} \circ g_S$ for every SI-generalized bi-ideal f_S and SI-left ideal g_S of S.*

Proof. First assume that (1) holds. Let f_S and g_S be any SI-generalized bi-ideal and SI-left ideal of S, respectively. Then,

$$f_S \circ \tilde{\mathbb{S}} \circ g_S = f_S \circ (\tilde{\mathbb{S}} \circ g_S) \tilde{\subseteq} f_S \circ g_S \tilde{\subseteq} \tilde{\mathbb{S}} \circ g_S \tilde{\subseteq} g_S.$$

Moreover, by Theorem 24 that f_S is an SI-right ideal of S. Thus,

$$f_S \circ \tilde{\mathbb{S}} \circ g_S = (f_S \circ \tilde{\mathbb{S}}) \circ g_S \tilde{\subseteq} f_S \circ g_S \tilde{\subseteq} f_S \circ \tilde{\mathbb{S}} \tilde{\subseteq} f_S.$$

Thus, $f_S \circ \tilde{\mathbb{S}} \circ g_S \tilde{\subseteq} f_S \tilde{\cap} g_S$.

Let a be any element of S. Then, it follows by Theorem 23 that there exist elements $x, y \in S$ such that $a = axa$ and $ax = ya$. Hence,

$$ax = axaxax = axax(ya) = (axa)(xya).$$

Thus,

$$\begin{aligned}
(f_S \circ \tilde{\mathbb{S}} \circ g_S)(a) &= [(f_S \circ \tilde{\mathbb{S}}) \circ g_S](a) \\
&= [\bigcup_{a=pq} (f_S \circ \tilde{\mathbb{S}})(p)] \circ g_S(q) \\
&\supseteq (f_S \circ \tilde{\mathbb{S}})(ax) \cap g_S(a) \\
&= \{\bigcup_{ax=pq} (f_S(p) \cap \tilde{\mathbb{S}}(q))\} \cap g_S(a) \\
&\supseteq (f_S(axa) \cap \tilde{\mathbb{S}}(aya)) \cap g_S(a) \\
&= (f_S(a) \cap U) \cap g_S(a) \\
&\supseteq f_S(a) \cap g_S(a) \\
&= (f_S \tilde{\cap} g_S)(a)
\end{aligned}$$

and so, $f_S \circ \tilde{\mathbb{S}} \circ g_S \tilde{\supseteq} f_S \tilde{\cap} g_S$. And so, $f_S \circ \tilde{\mathbb{S}} \circ g_S = f_S \tilde{\cap} g_S$. Thus, (1) implies (4).

It is clear that (4) implies (3) and (3) implies (2).

Assume that (2) holds. Let f_S and g_S be any SI-quasi-ideal and SI-left ideal of S, respectively. Then, by assumption,

$$f_S \widetilde{\cap} g_S = f_S \circ \widetilde{\mathbb{S}} \circ g_S = f_S \circ (\widetilde{\mathbb{S}} \circ g_S) \widetilde{\subseteq} f_S \widetilde{\circ} g_S.$$

Hence, it follows by Theorem 3 that S is regular. Let g_S be any SI-left ideal of S. Then, since g_S is an SI-quasi-ideal of S and since $\widetilde{\mathbb{S}}$ itself is an SI-left ideal of S,

$$g_S = g_S \widetilde{\cap} \widetilde{\mathbb{S}} = g_S \circ \widetilde{\mathbb{S}} \circ \widetilde{\mathbb{S}}.$$

Let L be any left ideal of S and $a \in L$. Then, the soft characteristic function $\mathcal{S}_L$ is an SI-left ideal of S. Thus,

$$\mathcal{S}_{LSS}(a) = (\mathcal{S}_L \circ \mathcal{S}_S \circ \mathcal{S}_S)(a) = \mathcal{S}_L(a) = U$$

which means that $a \in LSS$. Thus, $L \subseteq LSS$. Moreover, let $a \in LSS$. Then,

$$\mathcal{S}_L(a) = (\mathcal{S}_L \circ \mathcal{S}_S \circ \mathcal{S}_S)(a) = \mathcal{S}_{LSS}(a) = U$$

and so $a \in L$. Thus, $LSS \subseteq L$, and so $LSS = L$. Since Sa is a left ideal of S, $(Sa)SS = Sa$ and so,

$$aS \subseteq (aSa)S = a(Sa)S = a((Sa)SS)S \subseteq a((Sa)SS) \subseteq a(Sa) = (aS)a \subseteq Sa.$$

It follows by Theorem 23 that S is a semilattice of left groups and so (2) implies (1). $\square$

The left-right dual of Theorem 31 is as following:

Theorem 32 *For a semigroup S, the following conditions are equivalent:*

1) *S is a semilattice of right groups.*

2) *$f_S \widetilde{\cap} g_S = f_S \circ \widetilde{\mathbb{S}} \circ g_S$ for every SI-quasi-ideal f_S and SI-right ideal g_S of S.*

3) *$f_S \widetilde{\cap} g_S = f_S \circ \widetilde{\mathbb{S}} \circ g_S$ for every SI-bi-ideal f_S and SI-right ideal g_S of S.*

4) *$f_S \widetilde{\cap} g_S = f_S \circ \widetilde{\mathbb{S}} \circ g_S$ for every SI-generalized bi-ideal f_S and SI-right ideal g_S of S.*

Theorem 33 *For a semigroup S, the following conditions are equivalent:*

1) *S is a semilattice of left groups.*

2) *$f_S \widetilde{\cap} h_S \widetilde{\cap} g_S = f_S \circ h_S \circ g_S$ for every SI-quasi-ideal f_S, for every SI-ideal h_S and every SI-left ideal g_S of S.*

3) $f_S \tilde{\cap} h_S \tilde{\cap} g_S = f_S \circ h_S \circ g_S$ *for every* SI-*bi-ideal* f_S, *for every* SI-*ideal* h_S *and every* SI-*left ideal* g_S *of* S.

4) $f_S \tilde{\cap} h_S \tilde{\cap} g_S = f_S \circ h_S \circ g_S$ *for every* SI-*generalized bi-ideal* f_S, *for every* SI-*ideal* h_S *and every* SI-*left ideal* g_S *of* S.

Proof. First assume that (1) holds. Let f_S be any SI-generalized bi-ideal of S, h_S be any SI-ideal of S and g_S be any SI-left ideal of S. Then,

$$f_S \circ h_S \circ g_S \tilde{\subseteq} \tilde{\mathbb{S}} \circ (\tilde{\mathbb{S}} \circ g_S) \tilde{\subseteq} \tilde{\mathbb{S}} \circ g_S \tilde{\subseteq} g_S$$

and

$$f_S \circ h_S \circ g_S \tilde{\subseteq} \tilde{\mathbb{S}} \circ h_S \circ \tilde{\mathbb{S}} \tilde{\subseteq} h_S.$$

Moreover, by Theorem 24, since SI-generalized bi-ideal f_S of S is an SI-right ideal of S,

$$f_S \circ h_S \circ g_S \tilde{\subseteq} (f_S \circ \tilde{\mathbb{S}}) \circ \tilde{\mathbb{S}} \tilde{\subseteq} f_S \circ \tilde{\mathbb{S}} \tilde{\subseteq} f_S.$$

Hence,

$$f_S \circ h_S \circ g_S \tilde{\subseteq} f_S \tilde{\cap} h_S \tilde{\cap} g_S.$$

Let $a \in S$. Then, by Theorem 23, $a = axa$ and $ax = ya$ for some $x, y \in S$. Then,

$$ax = axaxax = axax(ya) = (axa)(xya).$$

Hence,

$$\begin{aligned}
(f_S \circ h_S \circ g_S)(a) &= [(f_S \circ h_S) \circ g_S](a) \\
&= [\bigcup_{a=pq} (f_S \circ h_S)(p)] \circ g_S(q) \\
&\supseteq (f_S \circ h_S)(ax) \cap g_S(a) \\
&= \{\bigcup_{ax=pq} (f_S(p) \cap h_S(q))\} \cap g_S(a) \\
&\supseteq (f_S(axa) \cap h_S(xya)) \cap g_S(a) \\
&\supseteq (f_S(a) \cap h_S(a)) \cap g_S(a) \\
&= (f_S \tilde{\cap} h_S \tilde{\cap} g_S)(a)
\end{aligned}$$

and so, $f_S \circ h_S \circ g_S \tilde{\supseteq} f_S \tilde{\cap} h_S \tilde{\cap} g_S$. Thus, $f_S \circ h_S \circ g_S = f_S \tilde{\cap} h_S \tilde{\cap} g_S$ and (1) implies (4).

It is clear that (4) implies (3) and (3) implies (2).

Conversely, assume that (2) holds. Let f_S be any SI-quasi-ideal and g_S be any SI-left ideal of S. Then, since $\widetilde{\mathbb{S}}$ itself is an SI-ideal of S, by assumption that

$$f_S \widetilde{\cap} g_S = f_S \widetilde{\cap} \widetilde{\mathbb{S}} \widetilde{\cap} g_S = f_S \circ \widetilde{\mathbb{S}} \circ g_S = f_S \circ (\widetilde{\mathbb{S}} \circ g_S) \widetilde{\subseteq} f_S \circ g_S.$$

It follows by Theorem 3 that S is regular. As in the above Theorem, one can easily show that $aS \subseteq Sa$. Thus, S is a semilattice of left groups. Thus, (2) implies (1). This completes the proof. □

The left-right dual of Theorem 33 is as following:

Theorem 34 *For a semigroup S, the following conditions are equivalent:*

1) S *is a semilattice of right groups.*

2) $f_S \widetilde{\cap} h_S \widetilde{\cap} g_S = f_S \circ h_S \circ g_S$ *for every* SI-*quasi-ideal* f_S, *for every* SI-*ideal* h_S *and every* SI-*right ideal* g_S *of* S.

3) $f_S \widetilde{\cap} h_S \widetilde{\cap} g_S = f_S \circ h_S \circ g_S$ *for every* SI-*bi-ideal* f_S, *for every* SI-*ideal* h_S *and every* SI-*right ideal* g_S *of* S.

4) $f_S \widetilde{\cap} h_S \widetilde{\cap} g_S = f_S \circ h_S \circ g_S$ *for every* SI-*generalized bi-ideal* f_S, *for every* SI-*ideal* h_S *and every* SI-*right ideal* g_S *of* S.

9 A semilattice of groups

Let S be a semigroup. We shall say that S is a *semilattice of groups* if it is the set-theoretical union of a family of mutually disjoint subgroups G_i ($i \in M$) such that, for any pair i, j in M, the products G_iG_j and G_jG_i are both contained in the same subgroups G_k ($k \in M$). The following is due to [17, 28, 33].

Proposition 10 [17, 28, 33] *For a semigroup S, the following conditions are equivalent:*

1) S *is a semilattice of groups.*

2) S *is regular and* $aS = Sa$ *for all* $a \in S$.

3) $LR = L \cap R$ *for every left ideal* L *and every right ideal* R *of* S.

4) $LB = L \cap B$ *for every left ideal* L *and every bi-ideal* B *of* S.

5) $BR = B \cap R$ *for every bi-ideal* B *and every right ideal* R *of* S.

6) S *is regular and every one-sided ideal of* S *is two-sided.*

Proposition 11 *Let* S *be a semigroup that is a semilattice of groups. Then, every* SI*-(generalized) bi-ideal of* S *is an* SI*-ideal of* S*.*

Proof. Let f_S be any SI-bi-ideal of S and a and b be any elements of S. Then, it follows by Proposition 10 that

$$ab \in (aSa)S = (aS)(aS) = (aS)(Sa) = a(SS)a \subseteq aSa$$

Thus, there exists an element $x \in S$ such that $ab = axa$. Hence,

$$f_S(ab) = f_S(axa) \supseteq f_S(a) \cap f_S(a) = f_S(a).$$

Hence, f_S is an SI-right ideal of S. Similarly,

$$ab \in S(bSb) = (Sb)(Sb) = (bS)(Sb) = b(SS)b \subseteq bSb$$

Thus, there exists an element $x \in S$ such that $ab = bxb$. Hence,

$$f_S(ab) = f_S(bxb) \supseteq f_S(b) \cap f_S(a) = f_S(b).$$

Therefore, f_S is an SI-left ideal of S. That is to say, f_S is an SI-ideal of S. □

Proposition 12 [28] *For a semigroup* S*, the following conditions are equivalent:*

1) S *is a semilattice of groups.*

2) *The set of all (generalized) bi-ideals of* S *is a semilattice under the multiplication of subsets.*

Now, see the characterization of a semigroup which s a semilattice of groups in terms of SI-ideals of semigroups.

Theorem 35 *For a semigroup* S*, the following conditions are equivalent:*

1) S *is a semilattice of groups.*

2) $f_S \circ g_S = f_S \tilde{\cap} g_S$ *for every* SI*-left ideal* f_S *and every* SI*-right ideal* g_S *of* S*.*

3) $f_S \circ g_S = f_S \tilde{\cap} g_S$ *for every* SI*-left ideal* f_S *and every* SI*-quasi ideal* g_S *of* S*.*

4) $f_S \circ g_S = f_S \tilde{\cap} g_S$ *for every* SI*-left ideal* f_S *and every* SI*-bi-ideal* g_S *of* S*.*

5) $\mathsf{f}_S \circ \mathsf{g}_S = \mathsf{f}_S \widetilde{\cap} \mathsf{g}_S$ *for every* SI*-left ideal* f_S *and every* SI*-generalized bi-ideal* g_S *of* S.

6) $\mathsf{f}_S \circ \mathsf{g}_S = \mathsf{f}_S \widetilde{\cap} \mathsf{g}_S$ *for every* SI*-quasi-ideal* f_S *and every* SI*-right ideal* g_S *of* S.

7) $\mathsf{f}_S \circ \mathsf{g}_S = \mathsf{f}_S \widetilde{\cap} \mathsf{g}_S$ *for all* SI*-quasi-ideals* f_S *and* g_S *of* S.

8) $\mathsf{f}_S \circ \mathsf{g}_S = \mathsf{f}_S \widetilde{\cap} \mathsf{g}_S$ *for every* SI*-quasi-ideal* f_S *and every* SI*-bi-ideal* g_S *of* S.

9) $\mathsf{f}_S \circ \mathsf{g}_S = \mathsf{f}_S \widetilde{\cap} \mathsf{g}_S$ *for every* SI*-quasi-ideal* f_S *and every* SI*-generalized bi-ideal* g_S *of* S.

10) $\mathsf{f}_S \circ \mathsf{g}_S = \mathsf{f}_S \widetilde{\cap} \mathsf{g}_S$ *for every* SI*-bi-ideal* f_S *and every* SI*-right ideal* g_S *of* S.

11) $\mathsf{f}_S \circ \mathsf{g}_S = \mathsf{f}_S \widetilde{\cap} \mathsf{g}_S$ *for every* SI*-bi-ideal* f_S *and every* SI*-quasi-ideal* g_S *of* S.

12) $\mathsf{f}_S \circ \mathsf{g}_S = \mathsf{f}_S \widetilde{\cap} \mathsf{g}_S$ *for all* SI*-bi-ideals* f_S *and* g_S *of* S.

13) $\mathsf{f}_S \circ \mathsf{g}_S = \mathsf{f}_S \widetilde{\cap} \mathsf{g}_S$ *for every* SI*-bi-ideal* f_S *and every* SI*-generalized bi-ideal* g_S *of* S.

14) $\mathsf{f}_S \circ \mathsf{g}_S = \mathsf{f}_S \widetilde{\cap} \mathsf{g}_S$ *for every* SI*-generalized bi-ideal* f_S *and every* SI*-right ideal* g_S *of* S.

15) $\mathsf{f}_S \circ \mathsf{g}_S = \mathsf{f}_S \widetilde{\cap} \mathsf{g}_S$ *for every* SI*-generalized bi-ideal* f_S *and every* SI*-quasi-ideal* g_S *of* S.

16) $\mathsf{f}_S \circ \mathsf{g}_S = \mathsf{f}_S \widetilde{\cap} \mathsf{g}_S$ *for every* SI*-generalized bi-ideal* f_S *and every* SI*-bi-ideal* g_S *of* S.

17) $\mathsf{f}_S \circ \mathsf{g}_S = \mathsf{f}_S \widetilde{\cap} \mathsf{g}_S$ *for all* SI*-generalized bi-ideals* f_S *and* g_S *of* S.

18) S *is regular and every* SI*-one-sided ideal of* S *is an* SI*-ideal of* S.

19) *The set of all* SI*-quasi-ideals of* S *is a semilattice under the multiplication of soft int-product.*

20) *The set of all* SI*-bi-ideals of* S *is a semilattice under the multiplication of soft int-product.*

21) *The set of all* SI*-generalized-bi-ideals of* S *is a semilattice under the multiplication of soft int-product.*

Proof. First assume that (1) holds. In order to prove that (17) holds, let f_S and g_S be any SI-generalized bi-ideals of S. Then, it follows by Proposition 11 that f_S and g_S are SI-ideals of S. Since S is regular by Proposition 10, it follows from Theorem 3 that $f_S \circ g_S = f_S \widetilde{\cap} g_S$. Hence, it is obtained that (1) implies (17). It is clear that (17) implies (16), (16) implies (15), (15) implies (14), (14) implies (10), (10) implies (6), (6) implies (2), (17) implies (13), (13) implies (12), (12) implies (11), (11) implies (10), (13) implies (9), (9) implies (8), (8) implies (7), (7) implies (6) and (9) implies (5), (5) implies (4), (4) implies (3) and (3) implies (2).

Assume that (2) holds. Let L and R be any left and right ideal of S, respectively. Then, the soft characteristic functions $\mathcal{S}_L$ and $\mathcal{S}_R$ are SI-left and SI-right ideal of S, respectively. Let a be any element of $L \cap R$. Then,

$$\mathcal{S}_{LR}(a) = (\mathcal{S}_L \circ \mathcal{S}_R)(a) = (\mathcal{S}_L \widetilde{\cap} \mathcal{S}_R)(a) = (\mathcal{S}_{L\cap R})(a) = U$$

and so $a \in LR$. Thus, $L \cap R \subseteq LR$.

Conversely, let a be any element of LR. Then,

$$(\mathcal{S}_{L\cap R})(a) = (\mathcal{S}_L \widetilde{\cap} \mathcal{S}_R)(a) = (\mathcal{S}_L \circ \mathcal{S}_R)(a) = \mathcal{S}_{LR}(a) = U,$$

and so $a \in L \cap R$. Thus, $LR \subseteq L \cap R$, hence $LR = L \cap R$. It follows by Proposition 10 that S is a semilattice of groups and so (2) implies (1).

Assume that (1) holds. Then, as shown above, (17) holds and (21) holds. It is obvious that (21) implies (20) and (20) implies (19). Assume that (19) holds. Then, since every SI-quasi-ideal of S is idempotent, it follows that S is regular ([40]. Let L and R be any left and right ideal of S, respectively. Then, since L and R are quasi-ideal of S, soft characteristic functions $\mathcal{S}_L$ and $\mathcal{S}_R$ are SI-quasi-ideal of S. Thus,

$$\mathcal{S}_{LR} = (\mathcal{S}_L \circ \mathcal{S}_R) = \mathcal{S}_R \circ \mathcal{S}_L) = \mathcal{S}_{RL}.$$

This implies that $LR = L \cap R$. Then, since S is regular,

$$R \cap L = RL = LR.$$

It follows by Proposition 12 that S is a semilattice of groups. Thus (19) implies (1).

Now assume that (2) holds. To see that (18) holds, let f_S be any SI-left ideal of S. Since $\widetilde{\mathbb{S}}$ is an SI-right ideal of S,

$$f_S = f_S \widetilde{\cap} \widetilde{\mathbb{S}} = f_S \circ \widetilde{\mathbb{S}}$$

Thus, f_S is an SI-right ideal of S. One can similarly show that every SI-right ideal of S is an SI-left ideal of S. As shown above, S is regular. Thus, (2) implies (18). Assume that (17) holds. In order to show that (1) holds, let A and B be any generalized bi-ideals of S and a be any element of AB. Then, the soft characteristic functions $\mathcal{S}_A$ and $\mathcal{S}_B$ are SI-generalized bi-ideals of S. Thus, by assumption,

$$(\mathcal{S}_B \circ \mathcal{S}_A)(a) = (\mathcal{S}_A \circ \mathcal{S}_B)(a) = \mathcal{S}_{AB}(a) = U$$

implying that $a \in BA$. Thus, $AB \subseteq BA$. It can be seen in a similar way that the converse inclusion holds. Thus, $AB = BA$. Now, let prove that any generalized bi-ideal of S is idempotent. Let B be any generalized bi-ideal of S and $a \in B$. Then, since the soft characteristic function $\mathcal{S}_B$ is an SI-generalized bi-ideal of S, by assumption

$$\mathcal{S}_{BB}(a) = (\mathcal{S}_B \circ \mathcal{S}_B)(a) = \mathcal{S}_B(a) = U$$

implying that $a \in BB$. Thus, $B \subseteq BB$. Similarly, one can show that $BB \subseteq B$. Hence, $B = BB$. This means that the set of all generalized bi-ideals of S is a semilattice under the multiplication of subsets. It follows by Proposition 12 that S is a semilattice of groups. Thus (2) implies (1). This completes the proof. □

Theorem 36 [39] *For a semigroup S the following conditions are equivalent:*

1) S *is completely regular.*

2) *Every bi-ideal of S is semiprime.*

3) *Every SI-bi-ideal of S is soft semiprime.*

4) $f_S(a) = f_S(a^2)$ *for every SI-bi-ideal f_S of S and for all $a \in S$.*

Theorem 37 *For a semigroup S, the following conditions are equivalent:*

1) S *is a semilattice of groups.*

2) *For every SI-quasi-ideal f_S of S, $f_S(a) = f_S(a^2)$ and $f_S(ab) = f_S(ba)$ for all $a, b \in S$.*

3) *For every SI-bi-ideal f_S of S, $f_S(a) = f_S(a^2)$ and $f_S(ab) = f_S(ba)$ for all $a, b \in S$.*

4) *For every* SI-*generalized bi-ideal* f_S *of* S, $f_S(a) = f_S(a^2)$ *and* $f_S(ab) = f_S(ba)$ *for all* $a, b \in S$.

Proof. First assume that (1) holds. Let f_S be any SI-generalized bi-ideal of S and a and b be any elements of S. Then, since S is regular by Proposition 10, there exists an element x in S such that $a = axa = axaxaxa$. Since $aS \subseteq Sa$ by Proposition 10, there exist elements $y, z \in S$ such that $xa = ya$ and $ax = za$. Thus,

$$a = axa = a(xaxaxa) = a(xa)x(ax)a = a(ya)x(za)a = a^2(yxz)a^2.$$

Hence, since f_S is an SI-generalized bi-ideal of S,

$$f_S(a) = f_S(a^2(yxz)a^2) \supseteq f_S(a^2) \cap f_S(a^2) = f_S(a^2) = f_S(a(axa)) = f_S(a(ax)a) \supseteq f_S(a) \cap f_S(a) = f_S(a)$$

and so $f_S(a) = f_S(a^2)$. Moreover, by Proposition 10,

$$(ab)^4 = a(ba)ba(ba)b \in (Sba)S(baS) = (baS)S(Sba).$$

Hence, there exists an element $u \in S$ such that $(ab)^4 = bauba$. Thus,

$$f_S(ab) = f_S((ab)^2) = f_S((ab)^4) = f_S((ba)u(ba)) \supseteq f_S(ba) \cap f_S(ba) = f_S(ba).$$

Similarly, $f_S(ba) \supseteq f_S(ab)$ and so $f_S(ab) = f_S(ba)$. Thus, (1) implies (4).

It is clear that (4) implies (3) and (3) implies (2).

Conversely, assume that (2) holds. Then, it follows by Theorem 36 that S is completely regular and so regular. Let a be any element of S. To see that $aS = Sa$, let ax be any element of aS. Since the soft characteristic function $\mathcal{S}_{B[xa]}$ of the principal bi-ideal $B[xa]$ is an SI-bi-ideal of S, by assumption,

$$\mathcal{S}_{B[xa]}(ax) = \mathcal{S}_{B[xa]}(xa) = U$$

and so $ax \in B[xa] = \{xa\} \cup (xa)^2 \cup (xa)S(xa)$. If $ax = xa$, then $ax = xa \in Sa$, and so $aS \subseteq Sa$. If $ax = (xa)^2$, then $ax = (xax)a \in Sa$. Hence, $aS \subseteq Sa$. If $ax \in (xa)S(xa)$, then

$$ax \in (xa)S(xa) = (xaSx)a \in Sa$$

and so $aS \subseteq Sa$. In any case, $aS \subseteq Sa$. Similarly, $Sa \subseteq aS$. Thus, $aS = Sa$. Hence, it follows by Proposition 10 that S is a semilattice of groups. Thus, (2) implies (1). This completes the proof. $\square$

Theorem 38 *For a semigroup S, the following conditions are equivalent:*

1) S *is a semilattice of groups.*

2) $f_S \tilde{\cap} g_S = g_S \circ f_S \circ g_S$ *for every* SI-*quasi-ideal* f_S *of* S *and for all* SI-*ideal* g_S *of* S.

3) $f_S \tilde{\cap} g_S = g_S \circ f_S \circ g_S$ *for every* SI-*quasi-ideal* f_S *of* S *and for all* SI-*interior ideal* g_S *of* S.

4) $f_S \tilde{\cap} g_S = g_S \circ f_S \circ g_S$ *for every* SI-*bi-ideal* f_S *of* S *and for all* SI-*ideal* g_S *of* S.

5) $f_S \tilde{\cap} g_S = g_S \circ f_S \circ g_S$ *for every* SI-*bi-ideal* f_S *of* S *and for all* SI-*interior ideal* g_S *of* S.

6) $f_S \tilde{\cap} g_S = g_S \circ f_S \circ g_S$ *for every* SI-*generalized bi-ideal* f_S *of* S *and for all* SI-*ideal* g_S *of* S.

7) $f_S \tilde{\cap} g_S = g_S \circ f_S \circ g_S$ *for every* SI-*generalized bi-ideal* f_S *of* S *and for all* SI-*interior ideal* g_S *of* S.

Proof. First assume that (1) holds. Let f_S be any SI-generalized bi-ideal and g_S be any SI-interior ideal of S. It follows by Proposition 11 that f_S is an SI-ideal of S. Thus,

$$g_S \circ f_S \circ g_S \tilde{\subseteq} \tilde{\mathbb{S}} \circ f_S \circ \tilde{\mathbb{S}} \tilde{\subseteq} f_S.$$

Moreover, $g_S \circ f_S \circ g_S \tilde{\subseteq} g_S \circ (\tilde{\mathbb{S}} \circ g_S) \tilde{\subseteq} g_S \circ g_S \tilde{\subseteq} g_S \circ \tilde{\mathbb{S}} \tilde{\subseteq} g_S$. Therefore,

$$g_S \circ f_S \circ g_S \tilde{\subseteq} f_S \tilde{\cap} g_S.$$

Now, let a be any element of S. Since S is regular by Proposition 10, there exists an element $x \in S$ such that $a = axa$. Hence

$$\begin{aligned}
(g_S \circ f_S \circ g_S)(a) &= [(g_S \circ f_S) \circ g_S](a) \\
&= [\bigcup_{a=pq} (g_S \circ f_S)(p)] \circ g_S(q) \\
&\supseteq (g_S \circ f_S)(a) \cap g_S(xa) \\
&= \{\bigcup_{a=uv} (g_S(u) \cap f_S(v))\} \cap g_S(a) \\
&\supseteq (g_S(ax) \cap f_S(a)) \cap g_S(a) \\
&\supseteq f_S(a) \cap g_S(a)
\end{aligned}$$

$$= \quad (f_S \tilde{\cap} g_S)(a)$$

and so, $g_S \circ f_S \circ g_S \tilde{\supseteq} f_S \tilde{\cap} g_S$. Thus, $g_S \circ f_S \circ g_S = f_S \tilde{\cap} g_S$, so, (1) implies (7). It is clear that (7) implies (6), (6) implies (4), (4) implies (2) and (7) implies (5), (5) implies (3) and (3) implies (2).

Assume that (2) holds. Let Q and J be any quasi-ideal and ideal of S, respectively. Thus, the soft characteristic function $\mathcal{S}_Q$ and $\mathcal{S}_J$ are SI-quasi-ideal and SI-ideal of S, respectively. Hence, by assumption,

$$\mathcal{S}_{JQJ}(a) = (\mathcal{S}_J \circ \mathcal{S}_Q \circ \mathcal{S}_J)(a) = (\mathcal{S}_J \cap \mathcal{S}_Q)(a) = \mathcal{S}_{J \cap Q}(a) = U$$

which implies that $a \in JQJ$. Thus, $J \cap Q \subseteq JQJ$.

Now, let a be any element of JQJ. Then,

$$\mathcal{S}_{J \cap Q}(a) = (\mathcal{S}_J \cap \mathcal{S}_Q)(a) = (\mathcal{S}_J \circ \mathcal{S}_Q \circ \mathcal{S}_J)(a) = \mathcal{S}_{JQJ}(a) = U$$

which implies that $a \in J \cap Q$. Thus, $JQJ \subseteq J \cap Q$. Therefore, that $JQJ = J \cap Q$ for every quasi-ideal Q and ideal J of S, which implies that S is regular and (2) implies (1). This completes the proof. □

10 Soft normal semigroups

In this section, the concepts of soft normality in a semigroup is introduced. It is known that a semigroup S is called *normal* if $aS = Sa$ for all $a \in S$.

Definition 14 *An* SI-*quasi-ideal* f_S *of* S *is called* $Q-\mathrm{normal}$ *if* $f_S(ab) = f_S(ba)$ *for all* $a, b \in S$.

Definition 15 *An* SI-*bi-ideal* f_S *of* S *is called* $B-\mathrm{normal}$ *if* $f_S(ab) = f_S(ba)$ *for all* $a, b \in S$.

Definition 16 *A semigroup* S *is called soft* $B^\star-\mathrm{normal}$ *if every* SI-*bi ideal of* S *is* $B-\mathrm{normal}$.

Definition 17 *A semigroup* S *is called soft* $Q^\star-\mathrm{normal}$ *if every* SI-*quasi-ideal of* S *is* $Q-\mathrm{normal}$.

Theorem 39 *Any soft* $Q^\star-\mathrm{normal}$ *semigroup is normal.*

Proof. Let f_S be an SI-quasi-ideal of a soft $Q^\star-\mathrm{normal}$ semigroup of S. Let a be any element of S. To see that $aS = Sa$, let ax be any element of aS.

Since the soft characteristic function $\mathcal{S}_{Q[xa]}$ of the principal bi-ideal $Q[xa]$ is an SI-quasi-ideal of S, by assumption,

$$\mathcal{S}_{Q[xa]}(ax) = \mathcal{S}_{Q[xa]}(xa) = U$$

which implies that

$$ax \in Q[xa] = \{xa\} \cup (xaS \cap Sxa) \subseteq Sa$$

Thus, $aS \subseteq Sa$. Similarly, $Sa \subseteq aS$ holds. Thus, $aS = Sa$ and S is normal. This completes the proof. □

The following theorem shows that the converse of Theorem 39 holds for a regular semigroup.

Theorem 40 *For a regular semigroup S, the following conditions are equivalent:*

1) S *is soft* $Q^\star - \mathrm{normal}$.

2) S *is normal.*

Proof. It suffices to prove that (2) implies (1). Assume that (2) holds. Let f_S be any SI-quasi-ideal of S and a and b be any elements of S. Since S is regular and normal,

$$ab \in (aSa)(bSb) = (aS)(ab)(Sb) \subseteq (aS)(abSab)(Sb) = (aSa)b(Sa)(bSb) \subseteq (Sb)(Sa)S = (Sb)(aS)S = S(ba)SS = (ba)SSS \subseteq baS$$

This implies that there exists an element $x \in S$ such that $ab = bax$. Thus, since f_S is an SI-bi-ideal of S,

$$(f_S \circ \widetilde{\mathbb{S}})(ab) = \bigcup_{ab=pq} \{(f_S(p) \cap \widetilde{\mathbb{S}}(q)\} \supseteq f_S(ba) \cap \widetilde{\mathbb{S}}(x) = f_S(ba).$$

One can similarly show that

$$(\widetilde{\mathbb{S}} \circ f_S)(ab) \supseteq f_S(ba)$$

Since, f_S is an SI-quasi-ideal of S,

$$f_S(ab) \supseteq ((f_S \circ \widetilde{\mathbb{S}}) \widetilde{\cap} (\widetilde{\mathbb{S}} \circ f_S))(ab) = (f_S \circ \widetilde{\mathbb{S}})(ab) \cap (\widetilde{\mathbb{S}} \circ f_S)(ab) \supseteq f_S(ba) \cap f_S(ba) = f_S(ba)$$

Similarly, it can be proved that $f_S(ba) \supseteq f_S(ab)$. Thus, $f_S(ba) = f_S(ab)$, and so S is soft $Q^\star-\mathtt{normal}$ and that (2) implies (1). This completes the proof. □

Theorem 41 *Any soft* $B^\star - \mathtt{normal}$ *semigroup is normal.*

Proof. Let f_S be an SI-bi-ideal of a soft $B^\star - \mathtt{normal}$ semigroup of S. Let a be any element of S and ax be any element of aS. Since the soft characteristic function $\mathcal{S}_{B[xa]}$ of the principal bi-ideal $B[xa]$ is an SI-bi-ideal of S, by assumption,

$$\mathcal{S}_{B[xa]}(ax) = \mathcal{S}_{B[xa]}(xa) = U$$

which implies that

$$ax \in B[xa] = \{xa\} \cup \{xaxa\} \cup (xa)S(xa) \subseteq Sa$$

Thus, $aS \subseteq Sa$. Similarly, $Sa \subseteq aS$ holds. Thus, $aS = Sa$ and S is normal. This completes the proof. □

The following theorem shows that the converse of Theorem 41 holds for a regular semigroup.

Theorem 42 *For a regular semigroup* S*, the following conditions are equivalent:*

1) S *is soft* $B^\star - \mathtt{normal}$.

2) S *is normal.*

Proof. It suffices to prove that (2) implies (1). Assume that (2) holds. Let f_S be any SI-bi-ideal of S and a and b be any elements of S. Since S is regular,

$$ab \in (aSa)(bSb) = (aS)(ab)(Sb) \subseteq (aS)(abSab)(Sb) = (aSa)b(Sa)(bSb) \subseteq (Sb)(aS)S = S(ba)SS = (ba)SSS \subseteq baS = (baSba)S = (baS)(Sba) = ba(SS)ba \subseteq baSba.$$

This implies that there exists an element $x \in S$ such that $a = baxba$. Thus, since f_S is an SI-bi-ideal of S,

$$f_S(ab) = f_S((ba)x(ba)) \supseteq f_S(ba) \cap f_S(ba) = f_S(ba).$$

One can similarly show that $f_S(ba) \supseteq f_S(ab)$. Hence $f_S(ab) = f_S(ba)$ which implies that S is soft $B^\star - \mathtt{normal}$ and that (2) implies (1). This completes the proof. □

Proposition 13 *For an idempotent semigroup* S*, the following conditions are equivalent:*

1) S *is commutative.*

2) S *is soft* $Q^{\star}-\mathtt{normal}$.

3) S *is soft* $B^{\star}-\mathtt{normal}$.

Proof. (1) implies (3) and (3) implies (2) is obvious. Assume that (2) holds. Then, S is normal. Let $a, b \in S$. Then, $ab \in Sb = bS$. Thus, there exists an element x in S such that $ab = bx$. Similarly, $ba = yb$ for some $b \in S$. Hence, since S is idempotent,

$$ab = bx = (bb)x = b(bx) = b(ab) = (ba)b = (yb)b = yb = ba$$

which implies that S is commutative. Hence (2) implies (1). □

Definition 18 [33] *A semigroup* S *is called Archimedean if for all* $a, b \in S$*, there exists a positive integer* n *such that* $a^n \in SbS$.

Definition 19 [33] *A semigroup* S *is called weakly commutative if for all* $a, b \in S$*, there exists a positive integer* n *such that* $(ab)^n \in bSa$.

Proposition 14 [33] *Every weakly commutative semigroup is a semilattice of archimedean semigroups.*

Proposition 15 *Any soft* $B^{\star}-\mathtt{normal}$ *semigroup is a semilattice of Archimedean semigroups.*

Proof. Let S be any soft $B^{\star}-\mathtt{normal}$ semigroup. Let a and b be any elements of S, and f_S be any SI-bi-ideal of S. Since the soft characteristic function $\mathcal{S}_{B[ba]}$ of the principal bi-ideal $B[ba]$ is an SI-bi-ideal of S, by assumption,

$$\mathcal{S}_{B[ba]}(ab) = \mathcal{S}_{B[ba]}(ba) = U$$

and so

$$ab \in B[ba] = \{ba\} \cup \{baba\} \cup (baSba) \subseteq Sa$$

Thus, $(ab)^2 \in baSba \subseteq bSa$. Therefore, S is weakly commutative. Hence by Proposition 14, S is a semilattice of Archimedean semigroups. □

One can similarly prove the following proposition.

Proposition 16 *Any soft* $Q^\star-\mathtt{normal}$ *semigroup is a semilattice of Archimedean semigroups.*

Theorem 43 *For a completely regular semigroup* S*, the following conditions are equivalent:*

1) S *is soft* $Q^\star-\mathtt{normal}$.

2) S *is soft* $B^\star-\mathtt{normal}$.

3) *For each elements* a *and* b *of* S*, there exists a positive integer* n *such that* $(ab)^n \in baSba$.

Proof. It is obvious that (2) implies (1). Assume that (1) holds. Then, S is normal. Let a and b be any elements of S. Thus,

$$\begin{aligned}(ab)^3 &= ababab = a(ba)bab \subseteq (Sba)(baS) = (baS)(Sba)\\ &= (ba)SS(ba) \subseteq baSba\end{aligned}$$

which shows that (1) implies (3).

Conversely, assume that (3) holds. To see that (2) holds, l et f_S be any SI-bi-ideal of S and a and b be any elements of S. Then, by assumption, there exists a positive integer n such that $(ab)^n = baxba$. Since S is completely regular, for this positive integer, there exists an element $y \in S$ such that $ab = (ab)^n y (ab)^n$. Then, since f_S is an SI-bi-ideal of S,

$$f_S(ab) = f_S((ab)^n y (ab)^n) \supseteq f_S((ab)^n) \cap f_S((ab)^n) = f_S((ab)^n)) = \\ f_S(baxba) \supseteq f_S(ba) \cap f_S(ba) = f_S(ba).$$

One can similarly show that $f_S(ba) \supseteq f_S(ab)$. Hence, $f_S(ab) = f_S(ba)$ which implies that f_S is soft $B^\star-\mathtt{normal}$. Thus, (3) implies (2). □

11 Conclusion

In this paper, certain classes of semigroups are characterized with regards to different soft intersection ideals of semigroups and soft normal semigroups are defined and the relation of this concept are studied with semigroups. Based on these results, some further work can be done on the properties of soft intersection semigroups and different classes of soft union ideals, which may be useful to characterize the classical semigroups.

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