



Global dynamics of a HIV transmission model

Sándor Kovács

Department of Numerical Analysis,
Eötvös Loránd University,
H-1117 Budapest, Pázmány Péter sétány 1/C,
Hungary
email: alex@ludens.elte.hu

Abstract. In this paper a simple mathematical model will be considered describing transmission dynamics of the human immunodeficiency virus in a special situation. A unique interior equilibrium is found and its stability is investigated. Results are verified by computer simulation.

1 Introduction

Acquired immunodeficiency syndrome (AIDS) was firstly recognized in 1981 among men who have sex with men (MSM) in the United States of America (USA), and shortly thereafter in populations such as injective drug users (IDU), hemophiliacs and blood transfusion recipients and infants of women with AIDS (cf. [12]). By 1983 the viral cause of AIDS, the human immunodeficiency virus (HIV) had been discovered and the basic models of transmission established: sexual transmission, parenteral exposure to blood and blood products, and perinatal transmission (cf. [7], pp. 3–17). Because HIV is primarily a sexual transmitted disease (STD), its spread reflects the social patterning of human sexual relationships. The understanding of the long-time behaviour of STD-s will help to find whether this epidemics will die out or stay in the population and to design strategies of fighting them. Various approaches for studying epidemiology of STD have been developed from time

2010 Mathematics Subject Classification: 34D23 (34C60, 92D30)

Key words and phrases: SIE model, global stability

to time. Since the famous Kermack-McKendrick model for a spread of disease (cf. [9]), differential equations have been widely used to study a disease transmission, to evaluate the spread of epidemics, and more importantly, to understand the mechanisms of epidemics in order to prevent them or minimise the transmission of this disease (cf. [1, 3, 5, 8]).

AIDS is a major public health problem in the USA. The epidemic in Europe has shown similar trends to those in the USA. Transmission has been the greatest among MSM and IDUs. More new cases of AIDS are reported each year among MSM than for any other group. The majority of MSM practice anal intercourse (cf. [10]) and this activity occurs within a community with a considerable prevalence of HIV infection (cf. [10, 14]). Within the MSM community there are high risk sexual zones (such as bathhouses) where MSM congregate for sexual activity (cf. [15]). Bathhouses are a feature of most major North American and European metropolitan areas, e.g. in New York City (NYC), Los Angeles.

Several models have been proposed in this area, e.g. one for treating IDUs and perinatal transmission in NYC (cf. [2]) and recently one for those with HIV transmission among MSM in a bathhouse (cf. [4]):

$$\begin{cases} \dot{S} &= f(S, I, E) := \pi - \beta \frac{SI}{N} - \mu S \\ \dot{I} &= g(S, I, E) := \rho - \mu I \\ \dot{E} &= h(S, I, E) := \sigma + \beta \frac{SI}{N} - \mu E \end{cases}, \quad (1)$$

where the dot means differentiation with respect to time t . The total number of visitors in the bathhouse at any time, $N(t)$, is subdivided into three parts: susceptibles (i.e., the uninfecteds), $S(t)$, the HIV-infecteds, $I(t)$, and the HIV-exposeds $E(t)$. The biological meaning of the parameters in (1) is the following: $\pi > 0$, $\rho \geq 0$ and $\sigma \geq 0$ are the inflow rates of susceptibles, infecteds and exposeds, respectively; the average time spent by individuals in the bathhouse is $1/\mu$ with $\mu > 0$. The transfer mechanism from the class of susceptibles to the class of exposeds is guided by the fraction $\beta I/N$, where the

$$\beta = c_i \beta_i (I - \eta^c \psi_i^c) (1 - \eta^m \psi_i^m) + c_r \beta_r (1 - \eta^c \psi_r^c) (1 - \eta^m \psi_r^m)$$

denotes the probability of HIV transmission. For the detailed meaning of the parameters in β we refer to the Table 1 in [4]. We note that from the section explaining the dependence on the parameters, β seems to be I -independent, while in its definition the I -dependence is present. The scalar β would essentially simplify the system while the linear I -dependence can be considered as

the fact that the probability of HIV transmission is (linearly) proportional to the size of the infected population. Thus the scalar case can be considered as a specialization of this extended version.

The aim of our study is to show that the above (extended) model is well posed and it has a unique equilibrium which is globally asymptotically stable.

2 Equilibria and their stability

We shall present some results, including the positivity and boundedness of solutions, furthermore the existence and stability of possible equilibria.

First of all, Picard–Lindelöf’s Theorem guarantees that solutions of the initial value problem for system (1) exist locally and are unique.

Clearly, the interior of the positive octant of the phase space $[S, I, E]$, denoted by $\mathbb{R}_+^3$, is an invariant region. Indeed, $f(0, I, E) \equiv \pi$, $g(S, 0, E) \equiv \rho$ and $h(S, I, 0) = \sigma + \beta S/(S + I)$, thus the time derivatives of S , I and E are positive at the boundary – provided as the inflow rates ρ, σ are positive – which implies nonnegativity. Hence, for the rest of the paper we only focus on system (1) restricted to $\mathbb{R}_+^3$.

Setting the right hand sides of the three differential equations of (1) equal to zero, we find that system (1) has only one equilibrium which lies in the interior of the positive octant of the phase space $[S, I, E]$:

$$\begin{aligned} S^* &:= \frac{\pi\mu(\pi + \rho + \sigma)}{\kappa_{ri}\mu\rho + \kappa_i\rho^2 + \mu^2(\pi + \rho + \sigma)}, \\ I^* &:= \frac{\rho}{\mu}, \\ E^* &:= \frac{\sigma}{\mu} + \frac{\pi\rho(\kappa_{ri}\mu + \kappa_i\rho)}{\kappa_{ri}\mu^2\rho + \kappa_i\mu\rho^2 + \mu^3(\pi + \rho + \sigma)}, \end{aligned} \tag{2}$$

where $\kappa_{ri} := c_r\beta_r(1 - \eta^c\psi_r^c)(1 - \eta^m\psi_r^m) - c_i\beta_i\eta^c\psi_i^c(1 - \eta^m\psi_i^m)$ and $\kappa_i := c_i\beta_i(1 - \eta^m\psi_i^m)$.

Theorem 1 (S^*, I^*, E^*) is a locally asymptotically stable equilibrium of system (1).

Proof. If we linearize the system at this equilibrium then the characteristic polynomial turns out to be

$$p(\lambda) \equiv \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0, \tag{3}$$

where

$$a_0 := \mu \left(\mu^2 + \frac{\rho(\kappa_{ri}\mu + \kappa_i\rho)}{\pi + \rho + \sigma} \right),$$

$$a_1 := 3\mu^2 + \frac{2\rho(\kappa_{ri}\mu + \kappa_i\rho)}{\pi + \rho + \sigma},$$

$$a_2 := 3\mu + \frac{\rho(\kappa_{ri}\mu + \kappa_i\rho)}{\mu(\pi + \rho + \sigma)},$$

which is stable by Routh-Hurwitz criterion, because it has only positive coefficients and

$$\begin{aligned} a_2 a_1 - a_0 &= 2 [\kappa_{ri}\rho\mu + \kappa_i\rho^2 + \mu^2(\pi + \rho + \sigma)] \\ &\quad \times \frac{\kappa_{ri}\rho\mu + \kappa_i\rho^2 + 4\mu^2(\pi + \rho + \sigma)}{\mu(\pi + \rho + \sigma)^2} > 0, \end{aligned}$$

which proves the local stability. ■

Calculating the second additive compound matrix (see e.g. [13]) of the Jacobian of the right hand side of (1), we have

$$J^{[2]}(S, I, E) = \begin{bmatrix} -2\mu - a & 0 & -b \\ c & -2\mu - a - b & -c \\ -a & 0 & -2\mu - b \end{bmatrix},$$

where the parameters a, b, c are defined for arbitrary $(S, I, E) \in \mathbb{R}_+^3$ as follows:

$$a := a(S, I, E) := \frac{I(\kappa_{ri} + \kappa_i I)(I + E)}{(S + I + E)^2} > 0,$$

$$b := b(S, I, E) := \frac{S(\kappa_{ri}(S + E) + \kappa_i I(2S + I + 2E))}{(S + I + E)^2} > 0,$$

$$c := c(S, I, E) := \frac{SI(\kappa_{ri} + \kappa_i I)}{(S + I + E)^2} > 0.$$

The stability modulus of $J^{[2]}(S, I, E)$ is negative: $s(J^{[2]}(S, I, E)) = -2\mu$. Hence, due to a result in [11] system (1) has no Hopf bifurcation from the equilibrium point.

Now, we are going to extend our local result about stability of the unique equilibrium point to a global one. For this first we examine the boundedness

of the system. Clearly, the positive octant of the $[S, I, E]$ space is positively invariant for system (1). Therefore we have to show that all solutions with positive initial conditions stay bounded in $t \in [0, +\infty)$ and there is no periodic orbit in the positive octant.

Lemma 1 *System (1) is dissipative, i.e. all solutions are bounded.*

Proof. We define the function

$$V(S, I, E) := S + I + E.$$

The time derivative along a solution of (1) is

$$\begin{aligned} \dot{V}(S, I, E) &= \dot{S} + \dot{I} + \dot{E} = \pi + \rho + \sigma - \mu(S + I + E) \\ &= \pi + \rho + \sigma - \mu V(S, I, E). \end{aligned} \tag{4}$$

Thus, for each $\epsilon \in (0, \mu)$ the sum $\dot{V} + \epsilon V$ is bounded from above, i.e. there is a $k > 0$ such that $\dot{V} + \epsilon V \leq k$. Solving this Gronwall-type inequality, we obtain the following estimate

$$0 \leq V(S, I, E) \leq \frac{k}{\epsilon} + V(S(0), I(0), E(0)) \cdot \exp(-\epsilon t) \leq \frac{k}{\epsilon} + V(S(0), I(0), E(0)),$$

which holds for all $t \geq 0$. Hence, as $t \rightarrow +\infty$ we have $0 \leq V(S, I, E) \leq \frac{k}{\epsilon} + \kappa$ for any $\kappa > 0$. Therefore all the trajectories initiated in $\mathbb{R}_+^3$ enter the region

$$\Omega := \left\{ (S, I, E) \in \mathbb{R}_+^3 \mid V(S, I, E) \leq \frac{k}{\epsilon} + \kappa, \text{ for any } \kappa > 0 \right\}.$$

This completes the proof. ■

Lemma 2 *System (1) has no nontrivial periodic solutions.*

Proof. From (4) we have

$$\dot{V}(S, I, E) = -\mu \left(V(S, I, E) - \frac{\pi + \rho + \sigma}{\mu} \right),$$

which has the consequence that the simplex

$$\Gamma := \left\{ (S, I, E) \in \mathbb{R}_+^3 : V(S, I, E) = \frac{\pi + \rho + \sigma}{\mu} \right\}$$

is positively invariant and all solutions approach to Γ with an exponential rate. Moreover, it suffices to study the dynamics of (1) on the simplex Γ . Hence, system (1) can be reduced to a planar system

$$\begin{cases} \dot{S} &= f(S, I) := \pi - \frac{\mu}{\pi + \rho + \sigma} SI (\kappa_i I + \kappa_{ri}) - \mu S, \\ \dot{I} &= g(S, I) := \rho - \mu I, \end{cases} \quad (5)$$

by dropping the third equation and making the substitution

$$E = (\pi + \rho + \sigma)/\mu - S - I$$

in the remaining two equations. Due to the Bendixon's Negative Criterion (see e.g. [6]) this reduced system has no nontrivial periodic solutions, because for $\mathbf{F} := (f, g)$

$$\begin{aligned} (\operatorname{div} \mathbf{F})(S, I) &= (\partial_1 f)(S, I) + (\partial_2 g)(S, I) \\ &= -2\mu - \frac{\mu}{\pi + \rho + \sigma} I (\kappa_i I + \kappa_{ri}) < 0 \quad (S > 0, I > 0) \end{aligned}$$

holds. ■

Thus, we can summarize our results as follows:

Theorem 2 *System (1) has only one steady state (2) which lies in the interior of the positive octant of the phase space $[S, I, E]$ and is globally asymptotically stable.*

Remark 1 *In [4] the situation $\rho = \sigma = 0$ (when there are no infected entrants) is also mentioned. In this case one has only the disease-free equilibrium $(S_0, I_0, E_0) = (\pi/\mu, 0, 0)$ which is because of the stability of the Jacobian matrix*

$$J(S_0, I_0, E_0) = \begin{bmatrix} -\mu & -\kappa_{ri} & 0 \\ 0 & -\mu & 0 \\ 0 & \kappa_{ri} & -\mu \end{bmatrix}$$

locally asymptotically stable. Its global asymptotical stability can be justified in the similar way as before.

Example 1 Set $\pi = 50.10$, $\mu = 0.50$, $\rho = 0.30$ resp. $\rho = 0$, $\sigma = 0.20$ resp. $\sigma = 0$, $\kappa_i = 0.10$ and $\kappa_{ri} = 0.11$. A Mathematica 3D plot shows (cf. Figure 1) that the trajectories of (1) converge to the unique positive resp. boundary equilibrium $(S^*, I^*, E^*) = (100, 0.6, 0.6)$ resp. $(S_0, I_0, E_0) = (100.2, 0, 0)$.

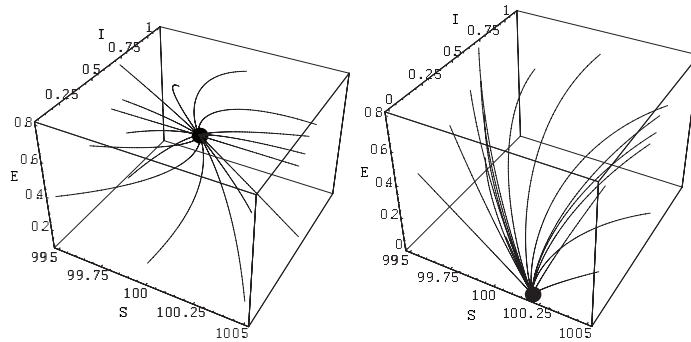


Figure 1: The unique equilibria showing their asymptotic stability.

Acknowledgment

The research was supported by the project TÁMOP-4.2.1.B-09/1/KMR-2010-003 of Eötvös Loránd University.

References

- [1] R. M. Anderson, R. M. May, *Population biology of infectious diseases*, Springer-Verlag, New York, 1982.
- [2] S. M. Blower, D. Hartel, H. Dowlatabad, R. M. Anderson and R. M. May, Drugs, sex and HIV: a mathematical model for New York City, *Philosophical Transactions: Biological Sciences*, **331(1260)** (1991), 171–187.
- [3] F. Brauer, C. Castillo-Chavez, *Mathematical models in population biology and epidemiology*, Springer, New York, 2001.
- [4] R. Breban, I. McGowan, C. Topaz, E. J. Schwartz, P. Anton and S. Blower, Modeling the potencial impact of rectal microbicides to reduce HIV transmission in bathhouses, *Math. Biosci. Eng.*, **3** (2006), 459–466.

- [5] O. Diekmann, J. A. P. Heesterbeek, *Mathematical epidemiology of infectious diseases: model building, analysis and interpretation*, Wiley, New York, 2000.
- [6] M. Farkas, *Periodic motions*, Springer-Verlag, Berlin, Heidelberg and New York, 1994.
- [7] P. Fultz, *The biology of human immunodeficiency viruses, The epidemiology of AIDS*, ed. RA Kaslow, DP Francis, New York: Oxford University Press, 1989.
- [8] H.W. Hethcote, The mathematics of infectious diseases, *SIAM Rev.*, **42** (2000), 599–653.
- [9] W.O. Kermack, A.G. McKendrick, A contribution to the mathematical theory of epidemics, *Proc. R. Soc. London A*, **115** (1927), 700–721.
- [10] B. A. Koblin, M. A. Chesney, M. J. Husnik, S. Bozeman, C. L. Celum, S. Buchbinder, K. Mayer, D. McKirnan, F. N. Judson, Y. Huang, and T. J. Coates, High-risk behaviors among men who have sex with men in 6 US cities: Baseline data from the explore study. *Am. J. Public Health*, **93** (2003), 926–932.
- [11] M. Y. Li and L. Wang, A criterion for stability of matrices, *J. Math. Anal. Appl.*, **225** (1998), 249–264.
- [12] K. MacQueen, The epidemiology of HIV transmission: trends, structure and dynamics, *Annu. Rev. Anthropol.*, **23** (1994), 509–526.
- [13] J. S. Muldowney, Compound matrices and ordinary differential equations, *Rocky Mountain J. Math.*, **20** (1990), 857–872.
- [14] D. H. Osmond, K. Page, J. Wiley, K. Garrett, H. W. Sheppard, A. R. Moss, L. Schrager, and W. Winkelstein, HIV infection in homosexual and bisexual men 18 to 29 years of age: The San Francisco young men’s health study. *Am. J. Public Health*, **8** (1994), 1933–1937.
- [15] C. A. Van Beneden, K. O’Brien, S. Modesitt, S. Yusem, A. Rose, and D. Fleming, Sexual behaviors in an urban bathhouse 15 years into the HIV epidemic. *J. Acquir. Immune Defic. Syndr.*, **30** (2002), 522–526.

Received: October 21, 2009; Revised: April 11, 2010