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Imbalances in directed multigraphs

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Abstract. In a directed multigraph, the imbalance of a vertex v_i is defined as $b_{v_i} = d^+_{v_i} - d^-_{v_i}$, where $d^+_{v_i}$ and $d^-_{v_i}$ denote the outdegree and indegree respectively of v_i. We characterize imbalances in directed multigraphs and obtain lower and upper bounds on imbalances in such digraphs. Also, we show the existence of a directed multigraph with a given imbalance set.

1 Introduction

A directed graph (shortly digraph) without loops and without multi-arcs is called a simple digraph [2]. The imbalance of a vertex v_i in a digraph as b_{v_i} (or simply b_i) $= d^+_{v_i} - d^-_{v_i}$, where $d^+_{v_i}$ and $d^-_{v_i}$ are respectively the outdegree and indegree of v_i. The imbalance sequence of a simple digraph is formed by listing the vertex imbalances in non-increasing order. A sequence of integers $F = [f_1, f_2, \ldots, f_n]$ with $f_1 \geq f_2 \geq \ldots \geq f_n$ is feasible if the sum of its elements is zero, and satisfies $\sum_{i=1}^{k} f_i \leq k(n-k)$, for $1 \leq k < n$.

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The following result [5] provides a necessary and sufficient condition for a sequence of integers to be the imbalance sequence of a simple digraph.

Theorem 1 *A sequence is realizable as an imbalance sequence if and only if it is feasible.*

The above result is equivalent to saying that a sequence of integers $B = [b_1, b_2, \ldots, b_n]$ with $b_1 \geq b_2 \geq \ldots \geq b_n$ is an imbalance sequence of a simple digraph if and only if

$$\sum_{i=1}^{k} b_i \leq k(n-k),$$

for $1 \leq k < n$, with equality when $k = n$.

On arranging the imbalance sequence in non-decreasing order, we have the following observation.

Corollary 1 *A sequence of integers* $B = [b_1, b_2, \ldots, b_n]$ *with* $b_1 \leq b_2 \leq \ldots \leq b_n$ *is an imbalance sequence of a simple digraph if and only if*

$$\sum_{i=1}^{k} b_i \geq k(k-n),$$

for $1 \leq k < n$ *with equality when* $k = n$.

Various results for imbalances in simple digraphs and oriented graphs can be found in [6], [7].

2 Imbalances in r-graphs

A multigraph is a graph from which multi-edges are not removed, and which has no loops [2]. If $r \geq 1$ then an r-digraph (shortly r-graph) is an orientation of a multigraph that is without loops and contains at most r edges between the elements of any pair of distinct vertices. Clearly 1-digraph is an oriented graph. Let D be an f-digraph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$, and let d_v^+ and d_v^- respectively denote the outdegree and indegree of vertex v. Define b_{v_i} (or simply b_i) $= d_{v_i}^+ - d_{u_i}^-$ as imbalance of v_i. Clearly, $-r(n-1) \leq b_{v_i} \leq r(n-1)$. The imbalance sequence of D is formed by listing the vertex imbalances in non-decreasing order.

We remark that r-digraphs are special cases of (a, b)-digraphs containing at least a and at most b edges between the elements of any pair of vertices. Degree sequences of (a, b)-digraphs are studied in [3, 4].

Let u and v be distinct vertices in D. If there are f arcs directed from u to v and g arcs directed from v to u, we denote this by $u(f - g)v$, where $0 \leq f, g, f + g \leq r$.

A double in D is an induced directed subgraph with two vertices u, and v having the form $u(f_1 f_2)v$, where $1 \leq f_1$, $f_2 \leq r$, and $1 \leq f_1 + f_2 \leq r$, and f_1 is the number of arcs directed from u to v, and f_2 is the number of arcs directed from v to u. A triple in D is an induced subgraph with tree vertices u, v, and w having the form $u(f_1 f_2)v(g_1 g_2)w(h_1 h_2)u$, where $1 \leq f_1$, f_2, g_1, g_2, h_1, $h_2 \leq r$, and $1 \leq f_1 + f_2$, $g_1 + g_2$, $h_1 + h_2 \leq r$, and the meaning of f_1, f_2, g_1, g_2, h_1, h_2 is similar to the meaning in the definition of doubles. An oriented triple in D is an induced subdigraph with three vertices. An oriented triple is said to be transitive if it is of the form $u(1 - 0)v(1 - 0)w(0 - 1)u$, or $u(1 - 0)v(0 - 1)w(0 - 0)u$, or $u(1 - 0)v(0 - 0)w(0 - 1)u$, or $u(1 - 0)v(0 - 0)w(0 - 0)u$, or $u(0 - 0)v(0 - 0)w(0 - 0)u$, otherwise it is intransitive. An r-graph is said to be transitive if all its oriented triples are transitive. In particular, a triple C in an r-graph is transitive if every oriented triple of C is transitive.

The following observation can be easily established and is analogues to Theorem 2.2 of Avery [1].

Lemma 1 *If* D_1 *and* D_2 *are two* r*-graphs with same imbalance sequence, then* D_1 *can be transformed to* D_2 *by successively transforming (i) appropriate oriented triples in one of the following ways, either (a) by changing the intransitive oriented triple* $u(1 - 0)v(1 - 0)w(1 - 0)u$ *to a transitive oriented triple* $u(0-0)v(0-0)w(0-0)u$*, which has the same imbalance sequence or vice versa, or (b) by changing the intransitive oriented triple* $u(1 - 0)v(1 - 0)w(0 - 0)u$ *to a transitive oriented triple* $u(0 - 0)v(0 - 0)w(0 - 1)u$*, which has the same imbalance sequence or vice versa; or (ii) by changing a double* $u(1 - 1)v$ *to a double* $u(0 - 0)v$*, which has the same imbalance sequence or vice versa.*

The above observations lead to the following result.

Theorem 2 *Among all* r*-graphs with given imbalance sequence, those with the fewest arcs are transitive.*

Proof. Let B be an imbalance sequence and let D be a realization of B that is not transitive. Then D contains an intransitive oriented triple. If it is of the form $u(1-0)v(1-0)w(1-0)u$, it can be transformed by operation *i(a)* of Lemma 3 to a transitive oriented triple $u(0-0)v(0-0)w(0-0)u$ with the same imbalance sequence and three arcs fewer. If D contains an intransitive oriented triple of the form $u(1-0)v(1-0)w(0-0)u$, it can be transformed by operation *i(b)* of Lemma 3 to a transitive oriented triple $u(0-0)v(0-0)w(0-1)u$ same imbalance sequence but one arc fewer. In case D contains both types of intransitive oriented triples, they can be transformed to transitive ones with certainly lesser arcs. If in D there is a double $u(1-1)v$, by operation *(ii)* of Lemme 4, it can be transformed to $u(0-0)v$, with same imbalance sequence but two arcs fewer. □

The next result gives necessary and sufficient conditions for a sequence of integers to be the imbalance sequence of some r-graph.

Theorem 3 *A sequence* $B = [b_1, b_2, \ldots, b_n]$ *of integers in non-decreasing order is an imbalance sequence of an* r*-graph if and only if*

$$\sum_{i=1}^{k} b_i \geq rk(k-n), \tag{1}$$

with equality when $k = n$.

Proof. Necessity. A multi subdigraph induced by k vertices has a sum of imbalances $rk(k-n)$.

Sufficiency. Assume that $B = [b_1, b_2, \ldots, b_n]$ be the sequence of integers in non-decreasing order satisfying conditions (1) but is not the imbalance sequence of any r-graph. Let this sequence be chosen in such a way that n is the smallest possible and b_1 is the least with that choice of n. We consider the following two cases.

Case (i). Suppose equality in (1) holds for some $k \leq n$, so that

$$\sum_{i=1}^{k} b_i = rk(k-n),$$

for $1 \leq k < n$.

By minimality of n, $B_1 = [b_1, b_2, \ldots, b_k]$ is the imbalance sequence of some r-graph D_1 with vertex set, say V_1. Let $B_2 = [b_{k+1}, b_{k+2}, \ldots, b_n]$.
Consider,

$$\begin{aligned}\sum_{i=1}^{f} b_{k+i} &= \sum_{i=1}^{k+f} b_i - \sum_{i=1}^{k} b_i \\ &\geq r(k+f)[(k+f)-n] - rk(k-n) \\ &= r(k_2 + kf - kn + fk + f_2 - fn - k_2 + kn) \\ &\geq r(f_2 - fn) \\ &= rf(f-n),\end{aligned}$$

for $1 \leq f \leq n-k$, with equality when $f = n-k$. Therefore, by the minimality for n, the sequence B_2 forms the imbalance sequence of some r-graph D_2 with vertex set, say V_2. Construct a new r-graph D with vertex set as follows.

Let $V = V_1 \cup V_2$ with, $V_1 \cap V_2 = \phi$ and the arc set containing those arcs which are in D_1 and D_2. Then we obtain the r-graph D with the imbalance sequence B, which is a contradiction.

Case (ii). Suppose that the strict inequality holds in (1) for some $k < n$, so that

$$\sum_{i=1}^{k} b_i > rk(k-n),$$

for $1 \leq k < n$. Let $B_1 = [b_1 - 1, b_2, \ldots, b_{n-1}, b_n + 1]$, so that B_1 satisfy the conditions (1). Thus by the minimality of b_1, the sequences B_1 is the imbalances sequence of some r-graph D_1 with vertex set, say V_1). Let $b_{v_1} = b_1 - 1$ and $b_{v_n} = a_n + 1$. Since $b_{v_n} > b_{v_1} + 1$, there exists a vertex $v_p \in V_1$ such that $v_n(0-0)v_p(1-0)v_1$, or $v_n(1-0)v_p(0-0)v_1$, or $v_n(1-0)v_p(1-0)v_1$, or $v_n(0-0)v_p(0-0)v_1$, and if these are changed to $v_n(0-1)v_p(0-0)v_1$, or $v_n(0-0)v_p(0-1)v_1$, or $v_n(0-0)v_p(0-0)v_1$, or $v_n(0-1)v_p(0-1)v_1$ respectively, the result is an r-graph with imbalances sequence B, which is again a contradiction. This proves the result. $\square$

Arranging the imbalance sequence in non-increasing order, we have the following observation.

Corollary 2 *A sequence* $B = [b_1, b_2, \ldots, b_n]$ *of integers with* $b_1 \geq b_2 \geq \ldots \geq b_n$ *is an imbalance sequence of an* r*-graph if and only if*

$$\sum_{i=1}^{k} b_i \leq rk(n-k),$$

for $1 \leq k \leq n$*, with equality when* $k = n$*.*

The converse of an r-graph D is an r-graph D', obtained by reversing orientations of all arcs of D. If $B = [b_1, b_2, \ldots, b_n]$ with $b_1 \le b_2 \le \ldots \le b_n$ is the imbalance sequence of an r-graph D, then $B' = [-b_n, -b_{n-1}, \ldots, -b_1]$ is the imbalance sequence of D.

The next result gives lower and upper bounds for the imbalance b_i of a vertex v_i in an r-graph D.

Theorem 4 *If* $B = [b_1, b_2, \ldots, b_n]$ *is an imbalance sequence of an* r*-graph* D*, then for each* i

$$r(i-n) \le b_i \le r(i-1).$$

Proof. Assume to the contrary that $b_i < r(i-n)$, so that for $k < i$,

$$b_k \le b_i < r(i-n).$$

That is,

$$b_1 < r(i-n), b_2 < r(i-n), \ldots, b_i < r(i-n).$$

Adding these inequalities, we get

$$\sum_{k=1}^{i} b_k < ri(i-n),$$

which contradicts Theorem 3.

Therefore, $r(i-n) \le b_i$.

The second inequality is dual to the first. In the converse r-graph with imbalance sequence $B = [b'_1, b'_2, \ldots, b'_n]$ we have, by the first inequality

$$\begin{aligned} b'_{n-i+1} &\ge r[(n-i+1)-n] \\ &= r(-i+1). \end{aligned}$$

Since $b_i = -b'_{n-i+1}$, therefore

$$b_i \le -r(-i+1) = r(i-1).$$

Hence, $b_i \le r(i-1)$. □

Now we obtain the following inequalities for imbalances in r-graphs.

Theorem 5 *If* $B = [b_1, b_2, \ldots, b_n]$ *is an imbalance sequence of an* r*-graph with* $b_1 \ge b_2 \ge \ldots \ge b_n$*, then*

$$\sum_{i=1}^{k} b_i^2 \le \sum_{i=1}^{k} (2rn - 2rk - b_i)^2,$$

for $1 \le k \le n$ *with equality when* $k = n$*.*

Proof. By Theorem 3, we have for $1 \leq k \leq n$ with equality when $k = n$

$$rk(n-k) \geq \sum_{i=1}^{k} b_i,$$

implying

$$\sum_{i=1}^{k} b_i^2 + 2(2rn-2rk)rk(n-k) \geq \sum_{i=1}^{k} b_i^2 + 2(2rn-2rk)\sum_{i=1}^{k} b_i,$$

from where

$$\sum_{i=1}^{k} b_i^2 + k(2rn-2rk)^2 - 2(2rn-2rk)\sum_{i=1}^{k} b_i \geq \sum_{i=1}^{k} b_i^2,$$

and so we get the required

$$\begin{aligned} b_1^2 + b_2^2 + \ldots + b_k^2 + (2rn-2rk)^2 + (2rn-2rk)^2 + \ldots + (2rn-2rk)^2 \\ - 2(2rn-2rk)b_1 - 2(2rn-2rk)b_2 - \ldots - 2(2rn-2rk)b_k \\ \geq \sum_{i=1}^{k} b_i^2, \end{aligned}$$

or

$$\sum_{i=1}^{k} (2rn-2rk-b_i)^2 \geq \sum_{i=1}^{k} b_i^2.$$

□

The set of distinct imbalances of vertices in an r-graph is called its imbalance set. The following result gives the existence of an r-graph with a given imbalance set. Let $(p_1, p_2, \ldots, p_m, q_1, q_2, \ldots, q_n)$ denote the greatest common divisor of $p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_n$.

Theorem 6 *If* $P = \{p_1, p_2, \ldots, p_m\}$ *and* $Q = \{-q_1, -q_2, \ldots, -q_n\}$ *where* $p_1, p_2, \ldots, p_m, q_1, q_2, \ldots, q_n$ *are positive integers such that* $p_1 < p_2 < \ldots < p_m$ *and* $q_1 < q_2 < \ldots < q_n$ *and* $(p_1, p_2, \ldots, p_m, q_1, q_2, \ldots, q_n) = t$, $1 \leq t \leq r$, *then there exists an* r*-graph with imbalance set* $P \cup Q$.

Proof. Since $(p_1, p_2, \ldots, p_m, q_1, q_2, \ldots, q_n) = t$, $1 \leq t \leq r$, there exist positive integers $f_1, f_2, \ldots, f_m$ and $g_1, g_2, \ldots, g_n$ with $f_1 < f_2 < \ldots < f_m$ and $g_1 < g_2 < \ldots < g_n$ such that

$$p_i = tf_i$$

for $1 \leq i \leq m$ and
$$q_i = tg_i$$
for $1 \leq j \leq n$.

We construct an r-graph D with vertex set V as follows.

Let
$$V = X_1^1 \cup X_2^1 \cup \ldots \cup X_m^1 \cup X_1^2 \cup X_1^3 \cup \ldots \cup X_1^n \cup Y_1^1 \cup Y_2^1 \cup \ldots \cup Y_m^1 \cup Y_1^2 \cup Y_1^3 \cup \ldots \cup Y_1^n,$$
with $X_i^j \cap X_k^l = \phi$, $Y_i^j \cap Y_k^l = \phi$, $X_i^j \cap Y_k^l = \phi$ and

$|X_i^1| = g_1$, for all $1 \leq i \leq m$,

$|X_1^i| = g_i$, for all $2 \leq i \leq n$,

$|Y_i^1| = f_i$, for all $1 \leq i \leq m$,

$|Y_1^i| = f_1$, for all $2 \leq i \leq n$.

Let there be t arcs directed from every vertex of X_i^1 to each vertex of Y_i^1, for all $1 \leq i \leq m$ and let there be t arcs directed from every vertex of X_1^i to each vertex of Y_1^i, for all $2 \leq i \leq n$ so that we obtain the r-graph D with imbalances of vertices as under.

For $1 \leq i \leq m$, for all $x_i^1 \in X_i^1$
$$b_{x_i^1} = t|Y_i^1| - 0 = tf_i = p_i,$$
for $2 \leq i \leq n$, for all $x_1^i \in X_1^i$
$$b_{x_1^i} = t|Y_1^i| - 0 = tf_1 = p_1,$$
for $1 \leq i \leq m$, for all $y_i^1 \in Y_i^1$
$$b_{y_i^1} = 0 - t|X_i^1| = -tg_i = -q_i,$$
and for $2 \leq i \leq n$, for all $y_1^i \in Y_1^i$
$$b_{y_1^i} = 0 - t|X_1^i| = -tg_i = -q_i.$$

Therefore imbalance set of D is $P \cup Q$. □

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