The connected vertex detour number of a graph

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Abstract. For a connected graph G of order $p \geq 2$ and a vertex x of G, a set $S \subseteq V(G)$ is an x-detour set of G if each vertex $v \in V(G)$ lies on an $x - y$ detour for some element y in S. The minimum cardinality of an x-detour set of G is defined as the x-detour number of G, denoted by $d_x(G)$. An x-detour set of cardinality $d_x(G)$ is called a d_x-set of G. A connected x-detour set of G is an x-detour set S such that the subgraph $G[S]$ induced by S is connected. The minimum cardinality of a connected x-detour set of G is defined as the connected x-detour number of G and is denoted by $cd_x(G)$. A connected x-detour set of cardinality $cd_x(G)$ is called a cd_x-set of G. We determine bounds for the connected x-detour number and find the same for some special classes of graphs. If a, b and c are positive integers such that $3 \leq a \leq b+1 < c$, then there exists a connected graph G with detour number $dn(G) = a$, $d_x(G) = b$ and $cd_x(G) = c$ for some vertex x in G. For positive integers R, D and $n \geq 3$ with $R < D \leq 2R$, there exists a connected graph G with $rad_D G = R$, $diam_D G = D$ and $cd_x(G) = n$ for some vertex x in G. Also, for each triple D, n and p of integers with $4 \leq D \leq p - 1$ and $3 \leq n \leq p$, there is a connected graph G of order p, detour diameter D and $cd_x(G) = n$ for some vertex x of G.

1 Introduction

By a graph $G = (V, E)$ we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q

respectively. For basic graph theoretic terminology we refer to Harary [6]. For vertices x and y in a connected graph G, the distance $d(x, y)$ is the length of a shortest $x - y$ path in G. An $x - y$ path of length $d(x, y)$ is called an $x - y$ geodesic. For a cut-vertex v in a connected graph G and a component H of $G - v$, the subgraph H and the vertex v together with all edges joining v and $V(H)$ is called a branch of G at v. The closed interval $I[x, y]$ consists of all vertices lying on some $x - y$ geodesic of G, while for $S \subseteq V$, $I[S] = \bigcup_{x,y \in S} I[x, y]$.

A set S of vertices is a geodetic set if $I[S] = V$, and the minimum cardinality of a geodetic set is the geodetic number $g(G)$. A geodetic set of cardinality $g(G)$ is called a g-set. The geodetic number of a graph was introduced in [1, 7] and further studied in [3].

The concept of vertex geodomination number was introduced by Santhakumaran and Titus in [8] and further studied in [9]. Let x be a vertex of a connected graph G. A set S of vertices of G is an x-geodominating set of G if each vertex v of G lies on an $x - y$ geodesic in G for some element y in S. The minimum cardinality of an x-geodominating set of G is defined as the x-geodomination number of G and is denoted by $g_x(G)$. An x-geodominating set of cardinality $g_x(G)$ is called a g_x-set. The connected vertex geodomination number was introduced and studied by Santhakumaran and Titus in [11]. A connected x-geodominating set of G is an x-geodominating set S such that the subgraph $G[S]$ induced by S is connected. The minimum cardinality of a connected x-geodominating set of G is the connected x-geodomination number of G and is denoted by $cg_x(G)$. A connected x-geodominating set of cardinality $cg_x(G)$ is called a cg_x-set of G.

For vertices x and y in a connected graph G, the detour distance $D(x, y)$ is the length of a longest $x - y$ path in G. For any vertex u of G, the detour eccentricity of u is $e_D(u) = \max \{D(u, v) : v \in V\}$. A vertex v of G such that $D(u, v) = e_D(u)$ is called a detour eccentric vertex of u. The detour radius R and detour diameter D of G are defined by $R = rad_D G = \min \{e_D(v) : v \in V\}$ and $D = diam_D G = \max \{e_D(v) : v \in V\}$ respectively. An $x - y$ path of length $D(x, y)$ is called an $x - y$ detour. The closed interval $I_D[x, y]$ consists of all vertices lying on some $x - y$ detour of G, while for $I_D[S] = \bigcup_{x,y \in S} I_D[x, y]$. A set S of vertices is a detour set if $I_D[S] = V$, and the minimum cardinality of a detour set is the detour number $dn(G)$. A detour set of cardinality $dn(G)$ is called a minimum detour set. The detour number of a graph was introduced in [4] and further studied in [5].

The concept of vertex detour number was introduced by Santhakumaran

and Titus in [10]. Let x be a vertex of a connected graph G. A set S of vertices of G is an x-detour set if each vertex v of G lies on an $x - y$ detour in G for some element y in S. The minimum cardinality of an x-detour set of G is defined as the x-detour number of G and is denoted by $d_x(G)$. An x-detour set of cardinality $d_x(G)$ is called a d_x-set of G.
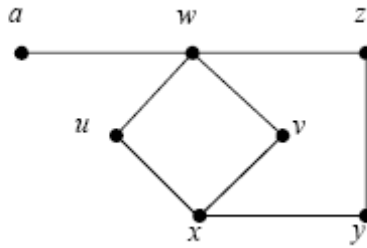

Figure 1

For the graph G given in Figure 1, $\{a, y\}$ and $\{a, z\}$ are the minimum x-detour sets of G and so $d_x(G) = 2$. It was proved in [10] that for any vertex x in G, $1 \le d_x(G) \le p - 1$. An elaborate study of results in vertex detour number with several interesting applications is given in [10].

The following theorems will be used in the sequel.

Theorem 1 [6] *Let v be a vertex of a connected graph G. The following statements are equivalent:*

(i) v is a cut vertex of G.

(ii) There exist vertices u and w distinct from v such that v is on every $u - w$ path.

(iii) There exists a partition of the set of vertices $V - \{v\}$ into subsets U and W such that for any vertices $u \in U$ and $w \in W$, the vertex v is on every $u - w$ path.

Theorem 2 [4] *Every end-vertex of a nontrivial connected graph G belongs to every detour set of G.*

Theorem 3 [4] *If T is a tree with k end-vertices, then $dn(T) = k$.*

Theorem 4 [10] *Let x be any vertex of a connected graph G. Then every end-vertex of G other than the vertex x (whether x is end-vertex or not) belongs to every d_x-set.*

Theorem 5 [10] *Let* T *be a tree with* k *end-vertices. Then* $d_x(T) = k - 1$ *or* $d_x(T) = k$ *according as* x *is an end-vertex or not.*

Theorem 6 [10] *For any vertex* x *in* G, $dn(G) \leq d_x(G) + 1$.

Theorem 7 [10] *If* G *is the complete graph* K_p ($p \geq 2$), *the cycle* C_p ($p \geq 3$), *the complete bipartite graph* $K_{m,n}$ ($m, n \geq 2$), *the* n-*cube* Q_n ($n \geq 2$) *or the wheel* $W_n = K_1 + C_{n-1}$ ($n \geq 4$), *then* $d_x(G) = 1$ *for every vertex* x *in* G.

Throughout this paper G denotes a connected graph with at least two vertices.

2 Connected vertex detour number

Definition 1 *Let* x *be any vertex of a connected graph* G. *A* connected x-detour set *of* G *is an* x-detour set S *such that the subgraph* $G[S]$ *induced by* S *is connected. The minimum cardinality of a connected* x-detour set of G is the connected x-detour number *of* G *and is denoted by* $cd_x(G)$. *A connected* x-detour set of cardinality $cd_x(G)$ is called a cd_x-set *of* G.

Example 1 *For the graph* G *given in Figure 2, the minimum vertex detour sets, the vertex detour numbers, the minimum connected vertex detour sets and the connected vertex detour numbers are given in Table 1.*

It is observed in [10] that x is not an element of any d_x-set of G. However, x may belong to a cd_x-set of G. For the graph G given in Figure 2, the vertex v is an element of a cd_v-set and the vertex t is not an element of any cd_t-set.
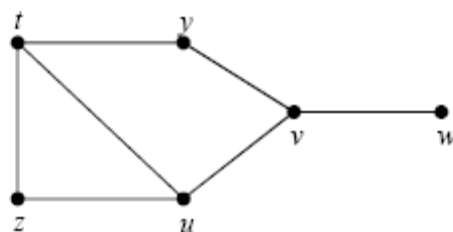

Figure 2

Table 1

Vertex x	d_x-sets	$d_x(G)$	cd_x-sets	$cd_x(G)$
t	$\{y,w\},\{z,w\},\{u,w\}$	2	$\{y,v,w\},\{u,v,w\}$	3
y	$\{w\}$	1	$\{w\}$	1
z	$\{w\}$	1	$\{w\}$	1
u	$\{w\}$	1	$\{w\}$	1
v	$\{y,w\},\{z,w\},\{u,w\}$	2	$\{y,v,w\},\{u,v,w\}$	3
w	$\{y\},\{z\},\{u\}$	1	$\{y\},\{z\},\{u\}$	1

Theorem 8 *Let x be any vertex of a connected graph G. If $y \neq x$ is an end vertex of G, then y belongs to every x-detour set of G.*

Proof. Let x be any vertex of G and let $y \neq x$ be an end-vertex of G. Then y is the terminal vertex of an $x - y$ detour and y is not an internal vertex of any detour so that y belongs to every x-detour set of G. $\qquad\square$

Theorem 9 *Let G be a connected graph with cut vertices and let S_x be a connected x-detour set of G. If v is a cut vertex of G, then every component of $G - \{v\}$ contains an element of $S_x \bigcup \{x\}$.*

Proof. Suppose that there is a component B of $G - \{v\}$ such that B contains no vertex of $S_x \bigcup \{x\}$. Then clearly, $x \in V - V(B)$. Let $u \in V(B)$. Since S_x is a connected x-detour set, there exists an element $y \in S_x$ such that u lies in some $x - y$ detour $P : x = u_0, u_1, \ldots, u, \ldots, u_n = y$ in G. By Theorem 1, the $x - u$ subpath of P and the $u - y$ subpath of P both contain v, it follows that P is not a path, contrary to assumption. $\qquad\square$

Corollary 1 *Let G be a connected graph with cut vertices and let S_x be a connected x-detour set of G. Then every branch of G contains an element of $S_x \bigcup \{x\}$.*

Theorem 10 (i) *If T is any tree, then $cd_x(T) = p$ for any cut vertex x of T.*
  (ii) *If T is any tree which is not a path, then for an end vertex x, $cd_x(T) = p - D(x,y)$, where y is the vertex of T with $\deg y \geq 3$ such that $D(x,y)$ is minimum.*
  (iii) *If T is a path, then $cd_x(T) = 1$ for any end vertex x of T.*

Proof. (i) Let x be a cut vertex of T and let S be any connected x-detour set of T. By Theorem 8, every connected x-detour set of T contains all end vertices. If $S \neq V(T)$, there exists a cut vertex v of T such that $v \notin S$. Let u and w be two end vertices belonging to different components of $T - \{v\}$. Since v lies on the unique path joining u and w, it follows that the subgraph $G[S]$ induced by S is disconnected, which is a contradiction. Hence $cd_x(T) = p$.

(ii) Let T be a tree which is not a path and x an end vertex of T. Let $S = (V(T) - I_D[x, y]) \bigcup \{y\}$. Clearly S is a connected x-detour set of T and so $cd_x(T) \leq | S | = p - D(x, y)$. We claim that $cd_x(T) = p - D(x, y)$. Otherwise, there is a connected x-detour set M of T with $| M | < p - D(x, y)$. By Theorem 8, every connected x-detour set of T contains all end vertices except possibly x and hence there exists a cut vertex v of T such that $v \in S$ and $v \notin M$. Let $B_1, B_2, \ldots, B_m (m \geq 3)$ be the components of $T - \{y\}$. Assume that x belongs to B_1.

Case 1. Suppose $v = y$. Let $z \in B_2$ and $w \in B_3$ be two end vertices of T. By Theorem 1, v lies on the unique $z - w$ detour. Since z and w belong to M and $v \notin M$, $G[M]$ is not connected, which is a contradiction.

Case 2. Suppose $v \neq y$. Let $v \in B_i (i \neq 1)$. Now, choose an end vertex $u \in B_i$ such that v lies on the $y - u$ detour. Let $a \in B_j (j \neq i, 1)$ be an end vertex of T. By Theorem 1, y lies on the $u - a$ detour. Hence it follows that v lies on the $u - a$ detour. Since u and a belong to M and $v \notin M$, $G[M]$ is not connected, which is a contradiction.

(iii) Let T be a path. For an end vertex x in T, let y be the eccentric vertex of x. Clearly every vertex of T lies on the $x - y$ detour and so $\{y\}$ is a connected x-detour set of T so that $cd_x(T) = 1$. $\qquad \square$

Corollary 2 *For any tree* T, $cd_x(T) = p$ *if and only if* x *is a cut vertex of* T.

Proof. This follows from Theorem 10. $\qquad \square$

Theorem 11 *For any vertex* x *in a connected graph* G,

$$1 \leq d_x(G) \leq cd_x(G) \leq p.$$

Proof. It is clear from the definition of x-detour number that $d_x(G) \geq 1$. Since every connected x-detour set is also an x-detour set, it follows that $d_x(G) \leq cd_x(G)$. Also, since $V(G)$ induces a connected x-detour set of G, it is clear that $cd_x(G) \leq p$. $\qquad \square$

Remark 1 *The bounds in Theorem 11 are sharp. For the cycle C_n, $d_x(C_n) =$ 1 for every vertex x in C_n. For any non-trivial tree T with $p \geq 3$, $cd_x(T) = p$ for any cut vertex x in T. For the graph G given in Figure 3, $d_x(G) = cd_x(G) = 2$ for the vertex x. Also, all the inequalities in the theorem are strict. For an end vertex x in the star $G = K_{1,n}(n \geq 3)$, $d_x(G) = n - 1$, $cd_x(G) = n$ and $p = n + 1$ so that $1 < d_x(G) < cd_x(G) < p$.*
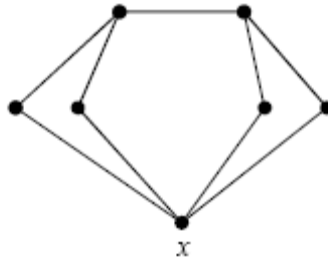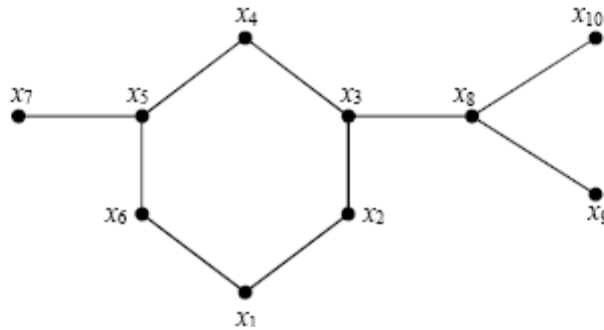

Figure 3

Figure 4

The following theorem gives a characterization for $cd_x(G) = 1$. For this, we introduce the following definition. Let x be any vertex in G. A vertex y in G is said to be an x-*detour superior vertex* if for any vertex z with $D(x, y) < D(x, z)$, z lies on an $x - y$ detour. For the graph G given in Figure 4, x_9 and x_{10} are the only x_1-detour superior vertices.

Theorem 12 *Let x be any vertex of a connected graph G. Then the following are equivalent:*

(i) $cd_x(G) = 1$

(ii) $d_x(G) = 1$

(iii) *There exists an x-detour superior vertex y in G such that every vertex of G is on an $x - y$ detour.*

Proof.

(i) $\Rightarrow$ (ii) Let $cd_x(G) = 1$. Then it follows from Theorem 11 that $d_x(G) = 1$.

(ii) $\Rightarrow$ (iii) Let $d_x(G) = 1$ and $S_x = \{y\}$ be a d_x-set of G. If y is not an x-detour superior vertex, then there is a vertex z in G with $D(x, y) < D(x, z)$ and z does not lie on any $x - y$ detour. Thus S_x is not a d_x-set of G, which is a contradiction.

(iii) $\Rightarrow$ (i) Let y be an x-detour superior vertex of G such that every vertex of G is on an $x - y$ detour. Then $\{y\}$ is a connected x-detour set of G so that $cd_x(G) = 1$. $\qquad\square$

Corollary 3 (i) *For the complete graph K_p, $cd_x(K_p) = 1$ for any vertex x in K_p.*

(ii) *For any cycle C_p, $cd_x(C_p) = 1$ for any vertex x in C_p.*

(iii) *For the wheel $W_p = K_1 + C_{p-1}(p \geq 5)$, $cd_x(W_p) = 1$ for any vertex x in W_p.*

(iv) *For any cube Q_n, $cd_x(Q_n) = 1$ for any vertex x in Q_n.*

(v) *For the complete bipartite graph $K_{m,n}(m, n \geq 2)$, $cd_x(K_{m,n}) = 1$ for any vertex x in $K_{m,n}$.*

Proof. This follows from Theorems 7 and 12. $\qquad\square$

Theorem 13 *For any vertex x in a connected graph G, $dn(G) \leq d_x(G) + 1 \leq cd_x(G) + 1$.*

Proof. This follows from Theorem 6 and Theorem 11. $\qquad\square$

The following theorem gives a realization for the detour number, the vertex detour number and the connected vertex detour number when

$$3 \leq a \leq b + 1 < c.$$

Theorem 14 *For any three integers a, b and c with $3 \leq a \leq b + 1 < c$, there exists a connected graph G with $dn(G) = a, d_x(G) = b$ and $cd_x(G) = c$ for some vertex x in G.*

Proof. We prove this theorem by considering two cases.

Case 1. $3 \leq a = b + 1 < c$. Let $k > c$ be any integer and let P_{k-a+2} : $u_1, u_2, \ldots, u_{k-a+2}$ be a path of order $k - a + 2$. Add $a - 2$ new vertices $v_1, v_2, \ldots, v_{a-2}$ to P_{k-a+2} and join these to u_{k-c+1}, thereby producing the graph G of Figure 5. Then G is a tree of order k with a end vertices. By Theorem 3, $dn(G) = a$ and it follows from Theorem 5 and Theorem 10 (ii) that $d_x(G) = b$ and $cd_x(G) = c$ respectively, for the vertex $x = u_1$.
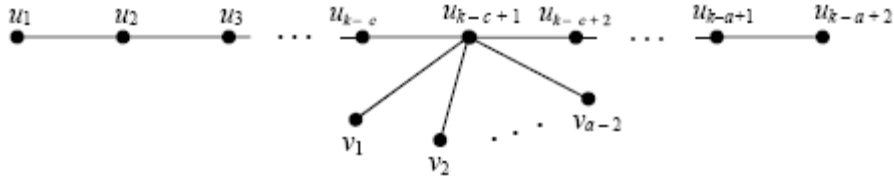

Figure 5

Case 2. $3 \leq a < b + 1 < c$. Let $F = (K_3 \bigcup P_2 \bigcup (b - a + 1)K_1) + \overline{K_2}$, where $U = V(K_3) = \{u_1, u_2, u_3\}$, $W = V(P_2) = \{w_1, w_2\}$, $X = V((b - a + 1)K_1) = \{x_1, x_2, \ldots, x_{b-a+1}\}$ and $V(\overline{K_2}) = \{x, y\}$. Let $P_{c-b-1} : v_1, v_2, \ldots, v_{c-b-1}$ be the path of order $c - b - 1$. Let H be the graph obtained from P_{c-b-1} by adding $a - 1$ new vertices $z_1, z_2, \ldots, z_{a-1}$ and joining each $z_i (1 \leq i \leq a - 1)$ to v_1. Now, let G be the graph obtained from F and H by identifying u_1 in F and v_{c-b-1} in H. The graph G is shown in Figure 6. Let $Z = \{z_1, z_2, \ldots, z_{a-1}\}$ be the set of all end vertices of G.

First, we show that $dn(G) = a$. By Theorem 2, every detour set of G contains Z. Since $I_D[Z] = Z \bigcup \{v_1\} \neq V(G)$, it follows that Z is not a detour set of G and so $dn(G) >| Z |= a - 1$. On the other hand, let $S = Z \bigcup \{w_1\}$. Then, for each i with $1 \leq i \leq b - a + 1$, the path $P : z_1, v_1, v_2, \ldots, v_{c-b-2}, u_1, u_2, u_3, y, x_i, x, w_2, w_1$ is a $z_1 - w_1$ detour in G of length $c - b + 6$. Hence S is a detour set of G and so $dn(G) \leq| S |= a$. Therefore, $dn(G) = a$.

Next, we show that $d_x(G) = b$ for the vertex x. Let S_x be any x-detour set of G. By Theorem 8, $Z \subseteq S_x$. It is clear that $D(x, z_i) = c - b + 5$ for $1 \leq i \leq a - 1$ and no $x_j (1 \leq j \leq b - a + 1)$ lies on an $x - z_i$ detour for any $z_i \in Z$. Thus Z is not an x-detour set of G. Now we claim that $X \subseteq S_x$. Assume, to the contrary, $X \supset S_x$. Then there exists an $x_i \in X$ such that $x_i \notin S_x (1 \leq i \leq b - a + 1)$. Now, it is clear that this x_i does not lie on any $x - v$ detour for any $v \in S_x$, which is a contradiction to S_x is an x-detour set. Hence $X \subseteq S_x$. Thus we see that every x-detour set S_x contains $X \bigcup Z$. Now, since $X \bigcup Z$ is an x-detour set
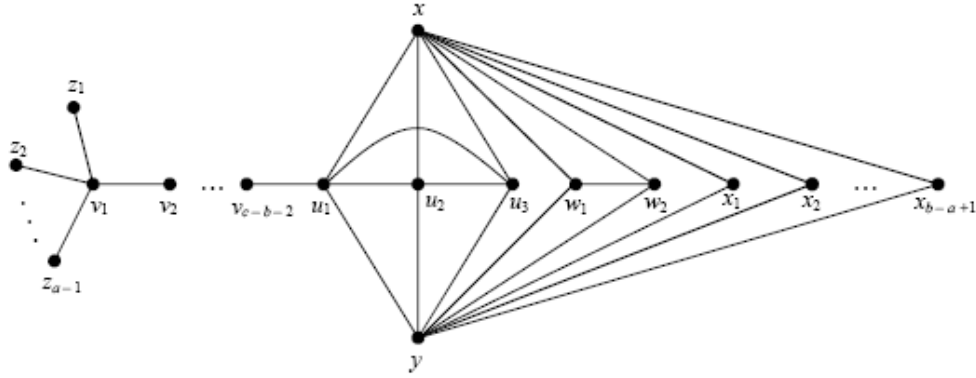
Figure 6

of G, it follows that $X \bigcup Z$ is the unique minimum x-detour set of G so that $d_x(G) = |X \bigcup Z| = b$.

Now, we show that $cd_x(G) = c$. Let T_x be any connected x-detour set of G. Since any connected x-detour set of G is also an x-detour set of G, it follows that T_x contains $X \bigcup Z$ as in the above paragraph. Now, since the induced subgraph $G[T_x]$ is connected, $M = \{v_1, v_2, \ldots, v_{c-b-1}\} \subseteq T_x$. Thus $M \bigcup X \bigcup Z \subseteq T_x$. It is clear that $M \bigcup X \bigcup Z$ is an x-detour set of G and the induced subgraph $G[M \bigcup X \bigcup Z]$ is not connected. Let $T = M \bigcup X \bigcup Z \bigcup \{x\}$. It is clear that T is a minimum connected x-detour set of G and so $cd_x(G) = c$. $\square$

For every connected graph G, $rad_D G \leq diam_D G \leq 2rad_D G$. Chartrand, Escuadro and Zhang [2] showed that every two positive integers a and b with $a \leq b \leq 2a$ are realizable as the detour radius and detour diameter, respectively, of some connected graph. This theorem can also be extended so that the connected vertex detour number can be prescribed when $a < b \leq 2a$.

Theorem 15 *For positive integers* R, D *and* $n \geq 3$ *with* $R < D \leq 2R$, *there exists a connected graph* G *with* $rad_D G = R, diam_D G = D$ *and* $cd_x(G) = n$ *for some vertex* x *in* G.

Proof. If $R = 1$, then $D = 2$. Let $G = K_{1,n}$. Then by Theorem 10 (ii), $cd_x(G) = n$ for an end vertex x in G. Now, let $R \geq 2$. We construct a graph G with the desired properties as follows:

Let $C_{R+1} : v_1, v_2, \ldots, v_{R+1}, v_1$ be a cycle of order $R + 1$ and let $P_{D-R+1} : u_0, u_1, \ldots, u_{D-R}$ be a path of order $D - R + 1$. Let H be the graph obtained

from C_{R+1} and P_{D-R+1} by identifying v_1 in C_{R+1} and u_0 in P_{D-R+1}. Now, add $n-2$ new vertices $w_1, w_2, \ldots, w_{n-2}$ to H and join each vertex $w_i (1 \leq i \leq n-2)$ to the vertex u_{D-R-1} to obtain the graph G of Figure 7.
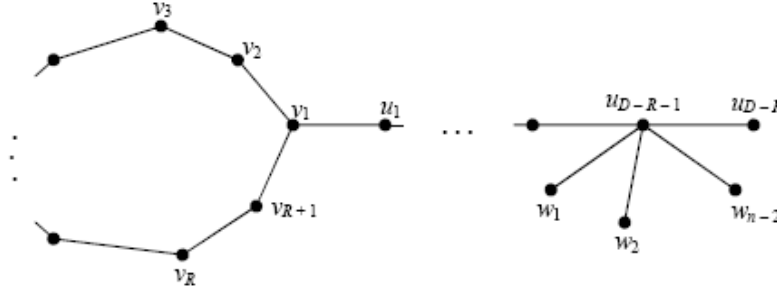

Figure 7

Now $\mathrm{rad}_D G = R$, $\mathrm{diam}_D G = D$ and G has $n - 1$ end vertices. Let $S = \{w_1, w_2, \ldots, w_{n-2}, u_{D-R}\}$ be the set of all end vertices of G. Then by Theorem 8, every connected x-detour set of G contains S for the vertex $x = v_2$. It is clear that S is an x-detour set of G and the induced subgraph $G[S]$ is not connected so that $cd_x(G) > n-1$. Let $S' = S \bigcup \{u_{D-R-1}\}$. Then S' is a connected x-detour set of G and so $cd_x(G) = n$. $\qquad\square$

The graph G of Figure 7 is the smallest graph with the properties described in Theorem 15. We leave the following problem as an open question.

Problem 1 *For positive integers R and $n \geq 3$, does there exist a connected graph G with $\mathrm{rad}_D G = \mathrm{diam}_D G = R$ and $cd_x(G) = n$ for some vertex x of G?*

In the following, we construct a graph of prescribed order, detour diameter and vertex detour number under suitable conditions.

Theorem 16 *For each triple D, n and p of integers with $4 \leq D \leq p - 1$ and $3 \leq n \leq p$, there is a connected graph G of order p, detour diameter D and $cd_x(G) = n$ for some vertex x of G.*

Proof. We prove this theorem by considering three cases.

Case 1. Suppose $3 \leq n \leq p - D + 2$. Let G be a graph obtained from the cycle $C_D : u_1, u_2, \ldots, u_D, u_1$ of order D by (i) adding $n - 2$ new vertices $v_1, v_2, \ldots, v_{n-2}$ and joining each vertex $v_i (1 \leq i \leq n-2)$ to u_1 and (ii) adding

$p - D - n + 2$ new vertices $w_1, w_2, \ldots, w_{p-D-n+2}$ and joining each vertex $w_i (1 \le i \le p - D - n + 2)$ to both u_1 and u_3. The graph G has order p and detour diameter D and is shown in Figure 8. Let $S = \{v_1, v_2, \ldots, v_{n-2}\}$ be the set of all end vertices of G. Then by Theorem 8, every connected x-detour set of G contains S for the vertex $x = u_1$. It is clear that S is not an x-detour set of G. Also any connected x-detour set of G must contain $S \bigcup \{u_1\}$. Since $S \bigcup \{u_1\}$ is not an x-detour set of G, $cd_x(G) > n - 1$. Let $S' = S \bigcup \{u_1, u_D\}$. Then S' is a connected x-detour set of G and so $cd_x(G) = n$.

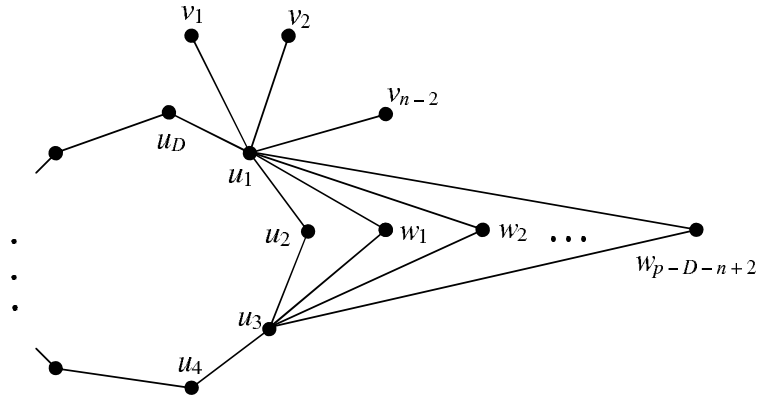Figure 8

Case 2. Suppose $p - D + 3 \le n \le p - 1$. Let $P_{D+1} : u_0, u_1, u_2, \ldots, u_D$ be a path of length D. Add $p - D - 1$ new vertices $v_1, v_2, \ldots, v_{p-D-1}$ to P_{D+1} and join each $v_i (1 \le i \le p - D - 1)$ to u_{p-n}, so by producing the graph G of Figure 9. The graph G has order p and detour diameter D. Then by Theorem 10 (ii), $cd_x(G) = p - (p - n) = n$ for the vertex $x = u_0$.
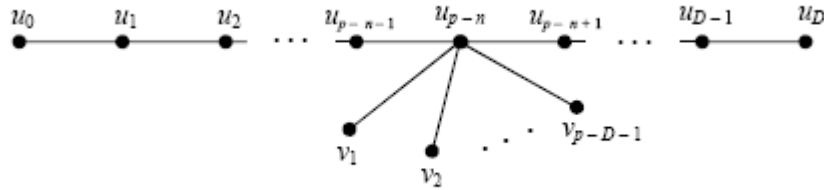

Figure 9

Case 3. Suppose $n = p$. Let G be any tree of order p and detour diameter

D. Then by Theorem 10 (i), $cd_x(G) = p$ for any cut vertex x in G.   □

Theorem 17 *For any two integers* n *and* p *with* $3 \leq n \leq p$, *there exists a connected graph* G *with order* p *and* $cd_x(G) = n$ *for some vertex* x *of* G.

Proof. We prove this theorem by considering two cases.

  Case 1. Let $3 \leq n \leq p - 2$. Then $p - n + 1 \geq 3$. Let G be the graph obtained from the cycle $C_{p-n+1} : u_1, u_2, \ldots, u_{p-n+1}, u_1$ by adding the $n - 1$ new vertices $v_1, v_2, \ldots, v_{n-1}$ and joining these to u_1. The graph G has order p and is shown in Figure 10. Let $S = \{v_1, v_2, \ldots, v_{n-1}\}$ be the set of all end vertices of G. Then by Theorem 8, every connected x-detour set of G contains S for the vertex $x = u_2$. It is clear that S is an x-detour set of G and the induced subgraph $G[S]$ is not connected so that $cd_x(G) > n - 1$. Let $S' = S \bigcup \{u_1\}$. It is clear that S' is a connected x-detour set of G and so $cd_x(G) = n$.

  Case 2: Let $n = p - 1$ or p. Let $G = K_{1,p-1}$. Then by Theorem 10, $cd_x(G) = p - 1$ or p according as x is an end vertex or the cut vertex.   □
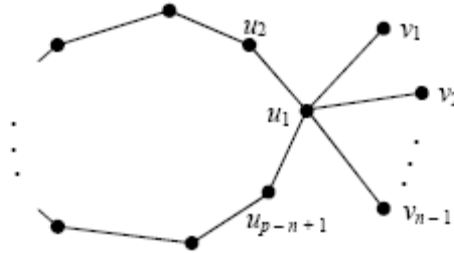

Figure 10

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