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The discrete Fourier transform of r-even functions

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Abstract. We give a detailed study of the discrete Fourier transform (DFT) of r-even arithmetic functions, which form a subspace of the space of r-periodic arithmetic functions. We consider the DFT of sequences of r-even functions, their mean values and Dirichlet series. Our results generalize properties of the Ramanujan sum. We show that some known properties of r-even functions and of the Ramanujan sum can be obtained in a simple manner via the DFT.

1 Introduction

The discrete Fourier transform (DFT) of periodic functions is an important tool in various branches of pure and applied mathematics. For instance, in number theory, the DFT of a Dirichlet character $\chi \pmod r$ is the Gauss sum (character sum) given by

$$G(\chi, n) = \sum_{k \pmod r} \chi(k) \exp(2\pi i k n/r), \tag{1}$$

and if $\chi = \chi_0$ is the principal character $\pmod r$, then (1) reduces to the Ramanujan sum $c_r(n)$.

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For the history, properties and various applications, including signal and image processing, of the DFT see for example the books of Briggs and Henson [6], Broughton and Bryan [7], Sundararajan [25], Terras [26]. For recent number theoretical papers concerning the DFT see [4, 13, 21].

It is the aim of the present paper to give a detailed study of the DFT of r-even arithmetic functions, to be defined in Section 2, which form a subspace of the space of r-periodic arithmetic functions.

Some aspects of the DFT of r-even functions were given by Haukkanen [13], Lucht [15] and were considered also by Samadi, Ahmad and Swamy [20] in the context of signal processing methods. Schramm [21] investigated the DFT of certain special r-even functions, without referring to this notion.

Our results generalize and complete those of [13, 15, 20, 21]. Note that the Ramanujan sum $c_r(n)$ is r-even and it is the DFT of χ_0, which is also r-even. Therefore, our results generalize properties of the Ramanujan sum.

The paper is organized as follows. Section 2 presents an overview of the basic notions and properties needed throughout the paper. In Section 3 we give a new simple characterization of r-even functions. Section 4 contains properties of the DFT of r-even functions, while in Sections 5 and 6 we consider sequences of r-even functions and their DFT, respectively. Mean values and Dirichlet series of the DFT of r-even functions and their sequences are investigated in Sections 7 and 8.

We also show that some known properties of r-even functions and of the Ramanujan sum can be obtained in a simple manner via the DFT.

2 Preliminaries

In this section we recall some known properties of arithmetic functions, periodic arithmetic functions, even functions, Ramanujan sums and the DFT. We also fix the notations, most of them being those used in the book by Schwarz and Spilker [22].

2.1 Arithmetic functions

Consider the $\mathbb{C}$-linear space $\mathcal{F}$ of arithmetic functions $f\colon \mathbb{N} = \{1, 2, \ldots\} \to \mathbb{C}$ with the usual linear operations. It is well known that with the Dirichlet convolution defined by

$$(f * g)(n) = \sum_{d|n} f(d)g(n/d) \tag{2}$$

the space $\mathcal{F}$ forms a unital commutative $\mathbb{C}$-algebra. The unity is the function ε given by $\varepsilon(1) = 1$ and $\varepsilon(n) = 0$ for $n > 1$. The group of invertible functions is $\mathcal{F}^* = \{f \in \mathcal{F} : f(1) \neq 0\}$. The Möbius function μ is defined as the inverse of the function $\mathbf{1} \in \mathcal{F}^*$ (constant 1 function). The divisor function is $\tau = \mathbf{1} * \mathbf{1}$, Euler's function is $\varphi = \mu * \mathrm{id}$ and $\sigma = \mathbf{1} * \mathrm{id}$ is the sum-of-divisors function, where $\mathrm{id}(n) = n$ ($n \in \mathbb{N}$). A function $f \in \mathcal{F}$ is called multiplicative if $f(1) = 1$ and $f(mn) = f(m)f(n)$ for any $m, n \in \mathbb{N}$ such that $\gcd(m, n) = 1$. The set $\mathcal{M}$ of multiplicative functions is a subgroup of $\mathcal{F}^*$ with respect to the Dirichlet convolution. Note that $\mathbf{1}, \mathrm{id}, \mu, \tau, \sigma, \varphi \in \mathcal{M}$. For an $f \in \mathcal{F}$ we will use the notation $f' = \mu * f$.

2.2 Periodic functions

A function $f \in \mathcal{F}$ is called r-periodic if $f(n + r) = f(n)$ for every $n \in \mathbb{N}$, where $r \in \mathbb{N}$ is a fixed number (this periodicity extends f to a function defined on $\mathbb{Z}$). The set $\mathcal{D}_r$ of r-periodic functions forms an r-dimensional subspace of $\mathcal{F}$. A function $f \in \mathcal{F}$ is called periodic if $f \in \bigcup_{r \in \mathbb{N}} \mathcal{D}_r$. The functions δ_k with $1 \leq k \leq r$ given by $\delta_k(n) = 1$ for $n \equiv k \pmod r$ and $\delta_k(n) = 0$ for $n \not\equiv k \pmod r$ form a basis of $\mathcal{D}_r$ (standard basis).

The functions e_k with $1 \leq k \leq r$ defined by $e_k(n) = \exp(2\pi i k n/r)$ (additive characters) form another basis of the space $\mathcal{D}_r$. Therefore, every r-periodic function f has a Fourier expansion of the form

$$f(n) = \sum_{k \pmod r} g(k) \exp(2\pi i k n/r) \qquad (n \in \mathbb{N}), \tag{3}$$

where the Fourier coefficients $g(k)$ are uniquely determined and are given by

$$g(n) = \frac{1}{r} \sum_{k \pmod r} f(k) \exp(-2\pi i k n/r) \qquad (n \in \mathbb{N}) \tag{4}$$

and the function g is also r-periodic.

For a function $f \in \mathcal{D}_r$ its discrete (finite) Fourier transform (DFT) is the function $\widehat{f} \in \mathcal{F}$ defined by

$$\widehat{f}(n) = \sum_{k \pmod r} f(k) \exp(-2\pi i k n/r) \qquad (n \in \mathbb{N}), \tag{5}$$

where by (5) and (4) one has $\widehat{f} = rg$.

For any $r \in \mathbb{N}$ the DFT is an automorphism of $\mathcal{D}_r$ satisfying $\widehat{\widehat{f}} = rf$. The inverse discrete Fourier transform (IDFT) is given by

$$f(n) = \frac{1}{r} \sum_{k \pmod r} \widehat{f}(k) \exp(2\pi i k n/r) \qquad (n \in \mathbb{N}). \tag{6}$$

If $f \in \mathcal{D}_r$, then

$$\sum_{n=1}^{r} |\widehat{f}(n)|^2 = r \sum_{n=1}^{r} |f(n)|^2, \tag{7}$$

which is a version of Parseval's formula.

Let $f, h \in \mathcal{D}_r$. The Cauchy convolution of f and h is given by

$$(f \otimes h)(n) = \sum_{a \pmod r} f(a) h(n-a) \qquad (n \in \mathbb{N}), \tag{8}$$

where $(\mathcal{D}_r, \otimes)$ is a unital commutative semigroup, the unity being the function ε_r given by $\varepsilon_r(n) = 1$ for $r \mid n$ and $\varepsilon_r(n) = 0$ otherwise. Also, $\widehat{f \otimes h} = \widehat{f}\,\widehat{h}$ and $\widehat{f} \otimes \widehat{h} = r\widehat{fh}$.

For the proofs of the above statements and for further properties of r-periodic functions and the DFT we refer to the books by Apostol [3, Ch. 8], Montgomery and Vaughan [17, Ch. 4], Schwarz and Spilker [22].

2.3 Even functions

A function $f \in \mathcal{F}$ is said to be an r-even function if $f(\gcd(n, r)) = f(n)$ for all $n \in \mathbb{N}$, where $r \in \mathbb{N}$ is fixed. The set $\mathcal{B}_r$ of r-even functions forms a $\tau(r)$ dimensional subspace of $\mathcal{D}_r$, where $\tau(r)$ is the number of positive divisors of r. A function $f \in \mathcal{F}$ is called even if $f \in \bigcup_{r \in \mathbb{N}} \mathcal{B}_r$. The functions g_d with $d \mid r$ given by $g_d(n) = 1$ if $\gcd(n, r) = d$ and $g_d(n) = 0$ if $\gcd(n, r) \neq d$ form a basis of $\mathcal{B}_r$. This basis can be replaced by the following one. The functions c_q with $q \mid r$ form a basis of the subspace $\mathcal{B}_r$, where c_q are the Ramanujan sums, quoted in the Introduction, defined explicitly by

$$c_q(n) = \sum_{\substack{k \pmod q \\ \gcd(k,q)=1}} \exp(2\pi i k n/q) \qquad (n, q \in \mathbb{N}). \tag{9}$$

Consequently, every r-even function f has a (Ramanujan-)Fourier expansion of the form

$$f(n) = \sum_{d \mid r} h(d) c_d(n) \qquad (n \in \mathbb{N}), \tag{10}$$

where the (Ramanujan-)Fourier coefficients $h(d)$ are uniquely determined and are given by

$$h(d) = \frac{1}{r} \sum_{e|r} f(e) c_{r/e}(r/d) \qquad (d \mid r) \tag{11}$$

and the function h is also r-even. Notation: $h(d) = \alpha_f(d)$ $(d \mid r)$. Note that $(\mathcal{B}_r, \otimes)$ is a subsemigroup of $(\mathcal{D}_r, \otimes)$ and $\alpha_{f\otimes h}(d) = r\alpha_f(d)\alpha_h(d)$ $(d \mid r)$, cf. Application 4.

Recall the following properties of Ramanujan sums $c_r(n)$. They can be represented as

$$c_r(n) = \sum_{d|\gcd(n,r)} d\mu(r/d) \qquad (n, r \in \mathbb{N}), \tag{12}$$

and as

$$c_r(n) = \frac{\mu(m)\varphi(r)}{\varphi(m)}, \qquad m = r/\gcd(n,r), \qquad (n, r \in \mathbb{N}), \tag{13}$$

where (13) is Hölder's identity. It follows that $c_r(n) = \varphi(r)$ for $r \mid n$ and $c_r(n) = \mu(r)$ for $\gcd(n, r) = 1$.

Let $\eta_r(n) = r$ if $r \mid n$ and $\eta_r(n) = 0$ otherwise. For any fixed $n \in \mathbb{N}$, $c_\bullet(n) = \mu * \eta_\bullet(n)$ and $r \mapsto c_r(n)$ is a multiplicative function. On the other hand, $n \mapsto c_r(n)$ is multiplicative if and only if $\mu(r) = 1$.

As it was already mentioned, $c_r(\bullet)$ is the DFT of the principal character (mod r) to be denoted in what follows by ρ_r and given explicitly by $\rho_r(n) = 1$ if $\gcd(n, r) = 1$ and $\rho_r(n) = 0$ otherwise. Note that $\rho_r = g_1$ with the notation of above (for r fixed). Thus

$$\widehat{\rho}_r = c_r, \qquad \widehat{c}_r = r\rho_r. \tag{14}$$

The concept of r-even functions originates from Cohen [8] and was further studied by Cohen in subsequent papers [9, 10, 11]. General accounts of r-even functions and of Ramanujan sums can be found in the books by McCarthy [16], Schwarz and Spilker [22], Sivaramakrishnan [23], Montgomery and Vaughan [17, Ch. 4]. See also the papers [12, 24, 27].

3 Characterization of r-even functions

For an $r \in \mathbb{N}$ let $\mathcal{B}'_r = \{f \in \mathcal{F} : f(n) = 0 \text{ for any } n \nmid r\}$. We have

Proposition 1 *Let $f \in \mathcal{F}$ and $f' = \mu * f$. Then the following assertions are equivalent:*

i) $f \in \mathcal{B}_r$,
ii) $f(n) = \sum_{d|\gcd(n,r)} f'(d) \quad (n \in \mathbb{N})$,
iii) $f' \in \mathcal{B}'_r$.

Proof. If $f' \in \mathcal{B}'_r$, then for any $n \in \mathbb{N}$,

$$f(n) = \sum_{d|n} f'(d) = \sum_{d|n,\, d|r} f'(d) = \sum_{d|\gcd(n,r)} f'(d)$$

$$= (f' * \mathbf{1})(\gcd(n,r)) = f(\gcd(n,r)).$$

This shows that iii) $\Rightarrow$ ii) $\Rightarrow$ i).

Now we show that i) $\Rightarrow$ iii). Assume that $f \in \mathcal{B}_r$ and $f' \notin \mathcal{B}'_r$, i.e., $f'(n) \neq 0$ for some $n \in \mathbb{N}$ with $n \nmid r$. Consider the minimal $n \in \mathbb{N}$ with this property. Then all proper divisors d of n with $f'(d) \neq 0$ divide r so that

$$f(n) = \sum_{d|n} f'(d) = \sum_{d|\gcd(n,r)} f'(d) + f'(n) = f(\gcd(n,r)) + f'(n) \neq f(\gcd(n,r)),$$

which gives $f \notin \mathcal{B}_r$. □

Remark 1 Let $f \in \mathcal{B}_r$. Assume that $f(n) = \sum_{d|\gcd(n,r)} g(d)$ $(n \in \mathbb{N})$ for a function $g \in \mathcal{F}$. Then $f = g\varepsilon_\bullet(r) * \mathbf{1}$ and $f = f' * \mathbf{1}$, by Proposition 1. Hence $g\varepsilon_\bullet(r) = f'$ and obtain that $g(n) = f'(n)$ for any $n \mid r$.

For $f = c_r$ (Ramanujan sum) we have by (12), Proposition 1 and Remark 1 the next identity, which can be shown also directly.

Application 1 *For any* $n, r \in \mathbb{N}$,

$$\sum_{d|n} c_r(d)\mu(n/d) = \begin{cases} n\mu(r/n), & n \mid r, \\ 0, & n \nmid r. \end{cases} \tag{15}$$

4 The DFT of r-even functions

We investigate in this section general properties of the DFT of r-even functions.

Proposition 2 *For each* $r \in \mathbb{N}$ *the DFT is an automorphism of* $\mathcal{B}_r$. *For any* $f \in \mathcal{B}_r$,

$$\widehat{f}(n) = \sum_{d|r} f(d) c_{r/d}(n) \qquad (n \in \mathbb{N}) \tag{16}$$

and the IDFT is given by

$$f(n) = \frac{1}{r} \sum_{d|r} \widehat{f}(d) c_{r/d}(n) \qquad (n \in \mathbb{N}). \tag{17}$$

Proof. By the definition of r-even functions and grouping the terms according to the values $d = \gcd(k, r)$,

$$\widehat{f}(n) = \sum_{d|r} f(d) \sum_{\substack{1 \le j \le r/d \\ \gcd(j, r/d)=1}} \exp(-2\pi i j n/(r/d)) = \sum_{d|r} f(d) c_{r/d}(n)$$

giving (16) and also that $\widehat{f} \in \mathcal{B}_r$. Now applying (16) for $\widehat{f}$ (instead of f) and using that $\widehat{\widehat{f}} = rf$ we have (17). □

Proposition 2 is given by Lucht [15, Th. 4]. Formulas (16) and (17) are implicitly given by Haukkanen [13, Th. 3.2 and Eq. (9)], Samadi, Ahmad and Swamy [20, Eq. (18)] for r-even functions, and by Schramm [21] for functions $n \mapsto F(\gcd(n, r))$, where $F \in \mathcal{F}$ is arbitrary, without referring to the notion of even functions.

Remark 2 By Proposition 2, for a function $f \in \mathcal{D}_r$ one has $f \in \mathcal{B}_r$ if and only if $\widehat{f} \in \mathcal{B}_r$. This can be used to show that a given function is r-even, cf. Application 4. Furthermore, it follows that the Fourier coefficients $\alpha_f(d)$ of $f \in \mathcal{B}_r$ can be represented as

$$\alpha_f(d) = \frac{1}{r} \widehat{f}(r/d) \qquad (d \mid r). \tag{18}$$

Corollary 1 *Let $f \in \mathcal{B}_r$. Then*

$$\widehat{f}(n) = \sum_{d|r} f(d) \varphi(r/d) \qquad (r \mid n), \tag{19}$$

$$\widehat{f}(n) = \sum_{d|r} f(d) \mu(r/d) \qquad (\gcd(n, r) = 1). \tag{20}$$

Corollary 2 *If f is a real (integer) valued r-even function, then $\widehat{f}$ is also real (integer) valued.*

Proof. Use that $c_r(n) \in \mathbb{Z}$ for any $n, r \in \mathbb{N}$. □

Corollary 3 *Let* f *be an* r*-even function. Then*

$$\widehat{f}(n) = \sum_{d \mid \gcd(n,r)} d\, f'(r/d) \qquad (n \in \mathbb{N}), \tag{21}$$

and $(\widehat{f})'(n) = nf'(r/n)$ *for any* $n \mid r$ *and* $(\widehat{f})'(n) = 0$ *otherwise.*

Proof. Recall that $c_{\bullet}(n) = \mu * \eta_{\bullet}(n)$, see (12). We obtain $\widehat{f}(n) = (f * c_{\bullet}(n))(r)$ $= (f * \mu * \eta_{\bullet}(n))(r) = (f' * \eta_{\bullet}(n))(r)$, and apply Remark 1. □

Note that by (21) the DFT of any $f \in \mathcal{B}_r$ can be written in the following forms:

$$\widehat{f}(n) = (f' * \eta_{\bullet}(n))(r), \tag{22}$$

and

$$\widehat{f} = h * \mathbf{1}, \tag{23}$$

where $h(n) = nf'(r/n)$ for $n \mid r$ and $h(n) = 0$ otherwise.

Proposition 3 *Let* f *be an* r*-even function. Then*

$$\sum_{d \mid n} \widehat{f}(d) = \sum_{d \mid \gcd(n,r)} d\, f'(r/d)\tau(n/d) \qquad (n \in \mathbb{N}). \tag{24}$$

Proof. Using (23),

$$\sum_{d \mid n} \widehat{f}(d) = (\widehat{f} * \mathbf{1})(n) = (h * \mathbf{1} * \mathbf{1})(n) = (h * \tau)(n) = \sum_{d \mid n} h(d)\tau(n/d)$$

$$= \sum_{d \mid \gcd(n,r)} d\, f'(r/d)\tau(n/d).$$

□

In the special case $f = \rho_r$ we reobtain (cf. [2, Th. 1] – where σ should be replaced by τ, [16, p. 91]),

$$\sum_{d \mid n} c_r(d) = \sum_{d \mid \gcd(n,r)} d\mu(r/d)\tau(n/d) \qquad (n \in \mathbb{N}). \tag{25}$$

The DFT can be used to obtain short direct proofs of certain known properties for Ramanujan sums and special r-even functions. We give the following examples.

Application 2 By $\widehat{\rho}_r = c_r$, cf. (14), we obtain $\widehat{\widehat{\rho}}_r = r\rho_r$. Therefore, by Proposition 2,

$$\sum_{d|r} c_r(r/d)c_d(n) = \begin{cases} r, & \gcd(n,r) = 1, \\ 0, & \text{otherwise}, \end{cases} \tag{26}$$

see [16, p. 94].

Application 3 Let $f(n) = (-1)^n$, which is r-even for any even number r. Its DFT is

$$\widehat{f}(n) = \sum_{k=1}^{r} (-1)^k \exp(-2\pi i k n/r) = \sum_{k=1}^{r} (-\exp(-2\pi i n/r))^k, \tag{27}$$

which is r for $n = r/2 + mr$ ($m \in \mathbb{Z}$) and 0 otherwise. Using Proposition 2 we obtain for any even number r,

$$\sum_{d|r} (-1)^d c_{r/d}(n) = \begin{cases} r, & n \equiv r/2 \pmod r, \\ 0, & \text{otherwise}, \end{cases} \tag{28}$$

cf. [18, Th. IV], [16, p. 90].

Application 4 Let $f, h \in \mathcal{B}_r$. We show that their Cauchy product $f \otimes h \in \mathcal{B}_r$ and the Fourier coefficients of $f \otimes h$ are given by $\alpha_{f\otimes h}(d) = r\alpha_f(d)\alpha_h(d)$ for any $d \mid r$, cf. Section 2.3.

To obtain this use that $\widehat{(f \otimes h)}(n) = \widehat{f}(n)\widehat{h}(n)$ ($n \in \mathbb{N}$), valid for functions $f, h \in \mathcal{D}_r$, cf. Section 2.2. Hence for any $n \in \mathbb{N}$,

$$\widehat{(f \otimes h)}(\gcd(n,r)) = \widehat{f}(\gcd(n,r))\widehat{h}(\gcd(n,r)) = \widehat{f}(n)\widehat{h}(n) = \widehat{(f \otimes h)}(n),$$

showing that $\widehat{f \otimes h}$ is r-even. It follows that $f \otimes h$ is also r-even. Furthermore, by (18), for every $d \mid r$,

$$\alpha_{f\otimes h}(d) = \frac{1}{r}\widehat{(f \otimes h)}(r/d) = \frac{1}{r}\widehat{f}(r/d)\widehat{h}(r/d) = r\alpha_f(d)\alpha_h(d).$$

Application 5 Let $N_r(n,k)$ denote the number of (incongruent) solutions (mod r) of the congruence $x_1 + \ldots + x_k \equiv n \pmod r$ with $\gcd(x_1, r) = \ldots = \gcd(x_k, r) = 1$. Then it is immediate from the definitions that

$$N_r(\,\cdot\,, k) = \underbrace{\rho_r \otimes \cdots \otimes \rho_r}_{k}. \tag{29}$$

Therefore, $\widehat{N_r}(\bullet, k) = (\widehat{\rho_r})^k = (c_r)^k$. Now the IDFT formula (17) gives at once

$$N_r(n,k) = \frac{1}{r}\sum_{d|r}((c_r(r/d))^k c_d(n) \qquad (n \in \mathbb{N}), \tag{30}$$

formula which goes back to the work of H. Rademacher (1925) and A. Brauer (1926) and has been recovered several times. See [16, Ch. 3], [22, p. 41], [24].

Application 6 We give a new proof of the following inversion formula of Cohen [9, Th. 3]: If f and g are r-even functions and if f is defined by

$$f(n) = \sum_{d|r} g(d)c_d(n) \qquad (n \in \mathbb{N}), \tag{31}$$

then

$$g(m) = \frac{1}{r}\sum_{d|r} f(r/d)c_d(n), \quad m = r/\gcd(n,r), \qquad (n \in \mathbb{N}). \tag{32}$$

To show this consider the function $G(n) = g(r/\gcd(n,r))$ which is also r-even. By Proposition 2,

$$\widehat{G}(n) = \sum_{d|r} G(r/d)c_d(n) = \sum_{d|r} g(d)c_d(n) = f(n). \tag{33}$$

Hence

$$rg(m) = rG(n) = \widehat{\widehat{G}}(n) = \widehat{f}(n) = \sum_{d|r} f(r/d)c_d(n). \tag{34}$$

Application 7 Anderson and Apostol [1] and Apostol [2] investigated properties of r-even functions $S_{g,h}$ given by

$$S_{g,h}(n) = \sum_{d|\gcd(n,r)} g(d)h(r/d) \qquad (n \in \mathbb{N}), \tag{35}$$

where $g, h \in \mathcal{F}$ are arbitrary functions.

For $f = S_{g,h}$ we have according to (21) and Remark 1, $f'(n) = g(n)h(r/n)$ $(n \mid r)$ and obtain at once

$$\widehat{S_{g,h}}(n) = \sum_{d|\gcd(n,r)} df'(r/d) = \sum_{d|\gcd(n,r)} dg(r/d)h(d), \tag{36}$$

which is proved in [1, Th. 4] by other arguments.

Application 8 If f is any r-even function, then

$$\sum_{n=1}^{r} |\widehat{f}(n)|^2 = r \sum_{d|r} |f(d)|^2 \varphi(r/d). \tag{37}$$

This follows by the Parseval formula (7) and grouping the terms of the right hand side according to the values $\gcd(n, r)$. For $f = \rho_r$ we reobtain the familiar formula

$$\sum_{n=1}^{r} (c_r(n))^2 = r\varphi(r) \qquad (r \in \mathbb{N}). \tag{38}$$

5 Sequences of r-even functions

In this section we consider sequences of functions $(f_r)_{r\in\mathbb{N}}$ such that $f_r \in \mathcal{B}_r$ for any $r \in \mathbb{N}$. Note that the sequence $(f_r)_{r\in\mathbb{N}}$ can be viewed also as a function of two variables: $f\colon \mathbb{N}^2 \to \mathbb{C}$, $f(n, r) = f_r(n)$.

We recall here the following concept: A function $f\colon \mathbb{N}^2 \to \mathbb{C}$ of two variables is said to be multiplicative if $f(mn, rs) = f(m, r)f(n, s)$ for every $m, n, r, s \in \mathbb{N}$ such that $\gcd(mr, ns) = 1$. For example, the Ramanujan sum $c(n, r) = c_r(n)$ is multiplicative, viewed as a function of two variables.

The next result includes a generalization of this property of the Ramanujan sum.

Proposition 4 *Let $(f_r)_{r\in\mathbb{N}}$ be a sequence of functions. Assume that*

i) $f_r \in \mathcal{B}_r$ $(r \in \mathbb{N})$,

ii) $r \mapsto f_r(n)$ is multiplicative $(n \in \mathbb{N})$.

Then

1) the function $f\colon \mathbb{N}^2 \to \mathbb{C}$, $f(n, r) = f_r(n)$ is multiplicative as a function of two variables,

2) $f_r(m)f_r(n) = f_r(1)f_r(mn)$ holds for any $m, n, r \in \mathbb{N}$ with $\gcd(m, n) = 1$,

3) $n \mapsto f_r(n)$ is multiplicative if and only if $f_r(1) = 1$.

Proof. 1) For any $m, n, r, s \in \mathbb{N}$ such that $\gcd(mr, ns) = 1$ we have by i) and ii),

$$\begin{aligned} f_{rs}(mn) &= f_r(mn)f_s(mn) = f_r(\gcd(mn, r))f_s(\gcd(mn), s) \\ &= f_r(\gcd(m, r))f_s(\gcd(n, s)) = f_r(m)f_s(n). \end{aligned}$$

2) By the definition of multiplicative functions of two variables $f\colon \mathbb{N}^2 \to \mathbb{C}$ it is immediate that $f(n, r) = \prod_p f(p^a, p^b)$ for $n = \prod_p p^a$, $r = \prod_p p^b$, and the given quasi-multiplicative property is a direct consequence of this equality.

3) Follows by 2). □

Part 1) of Proposition 4 is given also in [14] and for parts 2) and 3) cf. [23, Th. 80, 81].

We say that the sequence $(f_r)_{r\in\mathbb{N}}$ of functions is completely even if there exists a function $F \in \mathcal{F}$ of a single variable such that $f_r(n) = F(\gcd(n,r))$ for any $n, r \in \mathbb{N}$. This concept originates from Cohen [9] (for a function of two integer variables $f(n,r)$ satisfying $f(n,r) = F(\gcd(n,r))$ for any $n, r \in \mathbb{N}$ he used the term completely r-even function, which is ambiguous).

If the sequence $(f_r)_{r\in\mathbb{N}}$ is completely even, then $f_r \in \mathcal{B}_r$ for any $r \in \mathbb{N}$, but the converse is not true. For example, the Ramanujan sums $c_r(n)$ do not form a completely even sequence. To see this, assume the contrary and let p be any prime. Then for $n = r = p$, $F(p) = c_p(p) = p - 1$ and for $n = p, r = p^2$, $F(p) = c_{p^2}(p) = -p$, a contradiction.

If $(f_r)_{r\in\mathbb{N}}$ is completely even, then $f_r(n) = F(\gcd(n,r)) = \sum_{d|\gcd(n,r)} F'(d)$ $(n, r \in \mathbb{N})$ and by Remark 1 we have $f'_r(n) = F'(n)$ for any $n \mid r$, where $F' = \mu * F$.

6 The DFT of sequences of r-even functions

First we consider multiplicative properties of the DFT of sequences of r-even functions.

Proposition 5 *Let $(f_r)_{r\in\mathbb{N}}$ be a sequence of functions. Assume that*

i) $f_r \in \mathcal{B}_r$ $(r \in \mathbb{N})$,

ii) $r \mapsto f_r(n)$ is multiplicative $(n \in \mathbb{N})$.

Then

1) the function $r \mapsto \widehat{f}_r(n)$ is multiplicative $(n \in \mathbb{N})$,

2) the function $\widehat{f}\colon \mathbb{N}^2 \to \mathbb{C}$, $\widehat{f}(n,r) = \widehat{f}_r(n)$ is multiplicative as a function of two variables,

3) $\widehat{f}_r(m)\widehat{f}_r(n) = f'_r(r)\widehat{f}_r(mn)$ holds for any $m, n, r \in \mathbb{N}$ with $\gcd(m,n) = 1$,

4) $n \mapsto \widehat{f}_r(n)$ is multiplicative if and only if $f'_r(r) = 1$.

Proof. 1) Let $r, s \in \mathbb{N}$, $\gcd(r,s) = 1$. Then, for any fixed $n \in \mathbb{N}$, by Proposition 2 and using that $c_r(n)$ is multiplicative in r,

$$\widehat{f}_{rs}(n) = \sum_{d|rs} f_{rs}(d)c_{rs/d}(n) = \sum_{\substack{a|r\\ b|s}} f_{rs}(ab)c_{(r/a)(s/b)}(n)$$

$$= \sum_{\substack{a|r\\ b|s}} f_r(a) f_s(b) c_{r/a}(n) c_{s/b}(n) = \sum_{a|r} f_r(a) c_{r/a}(n) \sum_{b|s} f_s(b) c_{s/b}(n)$$

$$= \widehat{f}_r(n) \widehat{f}_s(n).$$

2), 3), 4) If $f_r \in \mathcal{B}_r$, then $\widehat{f}_r \in \mathcal{B}_r$ ($r \in \mathbb{N}$) and by 1) we know that the function $r \mapsto \widehat{f}_r(n)$ is multiplicative ($n \in \mathbb{N}$). Now apply Proposition 4 for the sequence $(\widehat{f}_r)_{r\in\mathbb{N}}$ and use that $\widehat{f}_r(1) = f'_r(r)$. □

Proposition 6 *Let $(f_r)_{r\in\mathbb{N}}$ be a sequence of functions such that $f_r \in \mathcal{B}_r$ ($r \in \mathbb{N}$). Then*

$$\sum_{d|r} \widehat{f}_d(n) = \sum_{d|\gcd(n,r)} d\, f_r(r/d) \qquad (n, r \in \mathbb{N}), \tag{39}$$

which is also r-even ($r \in \mathbb{N}$). Furthermore,

$$\sum_{d|n} \sum_{e|r} \widehat{f}_e(d) = \sum_{d|\gcd(n,r)} d\, f_r(r/d)\tau(n/d) \qquad (n, r \in \mathbb{N}). \tag{40}$$

Proof. Similar to the proof of Proposition 3. □

In the special case $f_r = \rho_r$ we reobtain the following known identities for the Ramanujan sum:

$$\sum_{d|r} c_d(n) = \begin{cases} r, & r \mid n, \\ 0, & r \nmid n, \end{cases} \tag{41}$$

$$\sum_{d|n} \sum_{e|r} c_e(d) = \begin{cases} r\,\tau(n/r), & r \mid n, \\ 0, & r \nmid n, \end{cases} \tag{42}$$

(41) being a familiar one and for (42) see [16, p. 91].

Consider in what follows the DFT of completely even sequences, defined in Section 5. Note that formulae (16) and (17) for the DFT and IDFT, respectively of such sequences (that is, functions with values $F(\gcd(n, r))$) were given by Schramm [21]. He considered also special cases of F.

Corollary 4 *Let $(f_r)_{r\in\mathbb{N}}$ be a sequence of functions. Assume that*
i) $(f_r)_{r\in\mathbb{N}}$ is completely even with $f_r(n) = F(\gcd(n, r))$ ($n, r \in \mathbb{N}$),
ii) F is multiplicative.
Then

1) the function $f\colon \mathbb{N}^2 \to \mathbb{C}$, $f(n,r) = f_r(n)$ *is multiplicative in both variables, with the other variable fixed, and is multiplicative as a function of two variables,*

2) the function $r \mapsto \widehat{f}_r(n)$ *is multiplicative* ($n \in \mathbb{N}$),

3) the function $\widehat{f}\colon \mathbb{N}^2 \to \mathbb{C}$, $\widehat{f}(n,r) = \widehat{f}_r(n)$ *is multiplicative as a function of two variables.*

4) $n \mapsto \widehat{f}_r(n)$ *is multiplicative if and only if* $F'(r) = 1$.

Proof. Follows from the definitions and from Proposition 5. □

The results of Section 4 can be applied for completely even sequences.

Corollary 5 *Let* $(f_r)_{r\in\mathbb{N}}$ *be a completely even sequence with* $f_r(n) = F(\gcd(n,r))$ $(n,r \in \mathbb{N})$. *Then*

$$\widehat{f}_r(n) = \sum_{d\mid\gcd(n,r)} d\,F'(r/d) \qquad (n,r \in \mathbb{N}), \tag{43}$$

$$\sum_{d\mid r} \widehat{f}_{r/d}(d) = \sum_{e^2 k=r} e\,F(k) \qquad (r \in \mathbb{N}). \tag{44}$$

Proof. Here (43) follows at once by Corollary 3, while (44) is a simple consequence of it. □

In particular, for $f_r = \rho_r$ (44) gives

$$\sum_{d\mid r} c_{r/d}(d) = \begin{cases} \sqrt{r}, & r \text{ is a square,} \\ 0, & \text{otherwise,} \end{cases} \tag{45}$$

see [16, p. 91].

It follows from (43) that the DFT of a completely even sequence of functions is a special case of the functions $S_{g,h}$ defined by (35), investigated by Anderson and Apostol [1], Apostol [2].

The example of $c_r(n)$ shows that the DFT sequence of a completely even sequence is, in general, not completely even ($c_r(n) = \widehat{\rho}_r(n)$, where $\rho_r(n) = \varepsilon(\gcd(n,r))$).

Consider now the completely even sequence $f_r(n) = \tau(\gcd(n,r))$. Then using (43),

$$\widehat{f}_r(n) = \sum_{d\mid\gcd(n,r)} d(\mu * \tau)(r/d) = \sum_{d\mid\gcd(n,r)} d = \sigma(\gcd(n,r)) \tag{46}$$

is completely even.

Next we characterize the completely even sequences such that their DFT is also a completely even sequence.

Proposition 7 *Let* $(f_r)_{r\in\mathbb{N}}$ *be a completely even sequence of functions with* $f_r(n) = F(\gcd(n,r))$. *Then the DFT sequence* $(\widehat{f_r})_{r\in\mathbb{N}}$ *is completely even if and only if* $F = c\,\tau$, *where* $c \in \mathbb{C}$. *In this case* $\widehat{f_r}(n) = c\,\sigma(\gcd(n,r))$.

Proof. Assume that there is a function $G \in \mathcal{F}$ such that

$$\widehat{f_r}(n) = \sum_{d\mid\gcd(n,r)} d\,F'(r/d) = G(\gcd(n,r)).$$

Then for any $n = r \in \mathbb{N}$, $G(r) = \widehat{f_r}(r) = \sum_{d\mid r} d\,F'(r/d) = (\mathrm{id} * F')(r)$, hence G has to be $G = \mathrm{id} * F'$. Now for $n = 1$ and any $r \in \mathbb{N}$, $G(1) = \widehat{f_r}(1) = F'(r)$. Denoting $G(1) = c$ we obtain that F' is the constant function c. Therefore, $F = c\,\mathbf{1} * \mathbf{1} = c\,\tau$.

Conversely, for $F = c\,\tau$ we have $F' = \mu * c\,\tau = c\,\mathbf{1}$ and $\widehat{f_r}(n) = c\sum_{d\mid\gcd(n,r)} d = c\,\sigma(\gcd(n,r))$. □

We now give a Hölder-type identity, see (13), for the DFT of completely even sequences, which is a special case of [1, Th. 2], adopted to our case. We recall that a function $F \in \mathcal{F}$ is said to be strongly multiplicative if F is multiplicative and $F(p^a) = F(p)$ for every prime p and every $a \in \mathbb{N}$.

Proposition 8 *Let* $(f_r)_{r\in\mathbb{N}}$ *be a completely even sequence with* $f_r(n) = F(\gcd(n,r))$ *(*$n, r \in \mathbb{N}$*). Suppose that*

i) F *is strongly multiplicative,*

ii) $F(p) \neq 1 - p$ *for any prime* p.

Then

$$\widehat{f_r}(n) = \frac{(F*\mu)(m)(F*\varphi)(r)}{(F*\varphi)(m)}, \qquad m = r/\gcd(n,r), \qquad (n,r \in \mathbb{N}). \tag{47}$$

Furthermore, for every prime power p^a *(*$a \in \mathbb{N}$*),*

$$\widehat{f_{p^a}}(n) = \begin{cases} p^{a-1}(p + F(p) - 1), & p^a \mid n, \\ p^{a-1}(F(p) - 1), & p^{a-1} \parallel n, \\ 0, & p^{a-1} \nmid n. \end{cases} \tag{48}$$

Proof. Here for any prime p, $(F*\mu)(p) = F(p) - 1$, $(F*\mu)(p^a) = 0$ for any $a \geq 2$ and $(F*\varphi)(p^a) = p^{a-1}(F(p) + p - 1)$ for any $a \geq 1$. The function F is multiplicative, thus $\widehat{f_r}(n)$ is multiplicative in r, cf. Corollary 4. Therefore, it is sufficient to verify the given identity for $r = p^a$, a prime power. Consider

three cases: Case 1) $p^a \mid n$, where $\gcd(n, p^a) = p^a$; Case 2) $p^a \parallel n$, where $\gcd(n, p^a) = p^{a-1}$; Case 3) $p^a \mid n$, where $\gcd(n, p^a) = p^\delta$ with $\delta \le a - 2$. □

Recall that a function $f \in \mathcal{F}$ is said to be semi-multiplicative if $f(m)f(n) = f(\gcd(m,n))f(\mathrm{lcm}[m,n])$ for any $m, n \in \mathbb{N}$. For example, $r \mapsto c_r(n)$ is semi-multiplicative for any $n \in \mathbb{N}$. As a generalization of this property we have:

Corollary 6 *Let* $(f_r)_{r\in\mathbb{N}}$ *be a completely even sequence with* $f_r(n) = F(\gcd(n,r))$ *(*$n, r \in \mathbb{N}$*) satisfying conditions i) and ii) of Proposition 8. Then* $r \mapsto \widehat{f}_r(n)$ *is semi-multiplicative for any* $n \in \mathbb{N}$.

Proof. If $g \in \mathcal{F}$ is multiplicative, then it is known that for any constant C and any $r \in \mathbb{N}$, the function $n \mapsto C\, g(r/\gcd(n,r))$ is semi-multiplicative, cf. [19], and apply (47). □

7 Mean values of the DFT of r-even functions

The mean value of a function $f \in \mathcal{F}$ is $M(f) = \lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} f(n)$ if this limit exists. It is known that $\sum_{n\le x} c_r(n) = \mathcal{O}(1)$ for any $r > 1$. It follows from (10) that the mean value of any r-even function f exists and is given by $M(f) = \alpha_f(1) = \frac{1}{r}\widehat{f}(r) = \frac{1}{r}(f * \varphi)(r)$, using (18), (19) (see also [27, Prop. 1]). Therefore, if f is r-even, then the mean value of $\widehat{f}$ exists and is given by $M(\widehat{f}) = \frac{1}{r}\widehat{\widehat{f}}(r) = f(r)$. This follows also by Proposition 2. More exactly, we have

Proposition 9 *Let* $f \in \mathcal{B}_r$ *(with* $r \in \mathbb{N}$ *fixed).*
i) If $x \in \mathbb{N}$ *and* $r \mid x$*, then*

$$\sum_{n=1}^{x} \widehat{f}(n) = f(r)x. \tag{49}$$

ii) For any real $x \ge 1$*,*

$$\sum_{n\le x} \widehat{f}(n) = f(r)x + T_f(x), \quad |T_f(x)| \le \sum_{d\mid r} d|f'(r/d)|. \tag{50}$$

iii) The mean value of the DFT function $\widehat{f}$ *is* $f(r)$.

Proof. For any $x \geq 1$, by Corollary 3,

$$\sum_{n\leq x} \widehat{f}(n) = \sum_{\substack{n\leq x \\ d\mid \gcd(n,r)}} d\, f'(r/d) = \sum_{d\mid r} d\, f'(r/d)[x/d] = \sum_{d\mid r} d\, f'(r/d)(x/d - \{x/d\})$$

$$= x\sum_{d\mid r} f'(r/d) - \sum_{d\mid r} d\, f'(r/d)\{x/d\} = xf(r) + T_f(x),$$

where $T_f(x)$ is identically zero for $x \in \mathbb{N}$, $r \mid x$. Furthermore, $T_f(x) = \mathcal{O}(1)$ for $x \to \infty$. □

Now we generalize Ramanujan's formula

$$\sum_{n=1}^{\infty} \frac{c_r(n)}{n} = -\Lambda(r) \qquad (r > 1), \tag{51}$$

where Λ is the von Mangoldt function.

Proposition 10 *Let f be an r-even function ($r \in \mathbb{N}$).*
i) Then uniformly for x and r,

$$\sum_{n\leq x} \frac{\widehat{f}(n)}{n} = f(r)(\log x + C) - (f*\Lambda)(r) + \mathcal{O}\left(x^{-1}V_f(x)\right), \quad V_f(x) = \sum_{d\mid r} d\,|f'(r/d)|, \tag{52}$$

where C is Euler's constant.
ii) If $f(r) = 0$, then

$$\sum_{n=1}^{\infty} \frac{\widehat{f}(n)}{n} = -(f * \Lambda)(r). \tag{53}$$

Proof. i) By Corollary 3,

$$\sum_{n\leq x} \frac{\widehat{f}(n)}{n} = \sum_{n\leq x} \frac{1}{n} \sum_{d\mid (n,r)} d\, f'(r/d) = \sum_{d\mid r} f'(r/d) \sum_{j\leq x/d} \frac{1}{j}$$

$$= \sum_{d\mid r} f'(r/d)\big(\log(x/d) + C + \mathcal{O}(d/x)\big)$$

$$= (\log x + C)\sum_{d\mid r} f'(r/d) - \sum_{d\mid r} f'(r/d)\log d + \mathcal{O}\left(x^{-1}\sum_{d\mid r} d|f'(r/d)|\right)$$

$$= (\log x + C)f(r) - (f * \mu * \log)(r) + \mathcal{O}\left(x^{-1}\sum_{d\mid r} d|f'(r/d)|\right).$$

ii) Part ii) follows from i) with $x \to \infty$. □

Remark 3 There is no simple general formula for $\sum_{r\le x} \widehat{f}_r(n)$, where $n \in \mathbb{N}$ is fixed and $(f_r)_{r\in\mathbb{N}}$ is a sequence of r-even functions (for example, $c_r(0) = \varphi(r)$ and $c_r(1) = \mu(r)$ have different asymptotic behaviors). For asymptotic formulae concerning special functions of type $\sum_{k=1}^{n} F(\gcd(k,n))$ see the recent papers [5, 28].

8 Dirichlet series of the DFT of sequences of r-even functions

We consider the Dirichlet series of the DFT of sequences $(f_r)_{r\in\mathbb{N}}$ such that $f_r \in \mathcal{B}_r$ for any $r \in \mathbb{N}$. By $\widehat{f}_r(n) = (\eta_\bullet(n) * \mu * f_r)(r)$, cf. (22), we have formally,

$$\sum_{r=1}^{\infty} \frac{\widehat{f}_r(n)}{r^s} = \sum_{r=1}^{\infty} \frac{\eta_r(n)}{r^s} \sum_{r=1}^{\infty} \frac{(f_r * \mu)(r)}{r^s} = \frac{\sigma_{s-1}(n)}{n^{s-1}} \sum_{r=1}^{\infty} \frac{1}{r^s} \sum_{k\ell=r} \mu(k) f_r(\ell) \tag{54}$$
$$= \frac{\sigma_{s-1}(n)}{n^{s-1}} \sum_{k=1}^{\infty} \frac{\mu(k)}{k^s} \sum_{\ell=1}^{\infty} \frac{f_{k\ell}(\ell)}{\ell^s},$$

where $\sigma_k(n) = \sum_{d|n} d^k$. This can be written in a simpler form by considering the DFT of completely even sequences of functions.

Proposition 11 *Let $(f_r)_{r\in\mathbb{N}}$ be a completely even sequence of functions with $f_r(n) = F(\gcd(n,r))$ and let a_F denote the absolute convergence abscissa of the Dirichlet series of F. Then*

$$\sum_{r=1}^{\infty} \frac{\widehat{f}_r(n)}{r^s} = \frac{\sigma_{s-1}(n)}{n^{s-1}\zeta(s)} \sum_{r=1}^{\infty} \frac{F(r)}{r^s} \tag{55}$$

for any $n \in \mathbb{N}$, absolutely convergent for $\operatorname{Re} s > \max\{1, a_F\}$,

$$\sum_{n=1}^{\infty} \frac{\widehat{f}_r(n)}{n^s} = \zeta(s)(F * \phi_{1-s})(r) \tag{56}$$

for any $r \in \mathbb{N}$, absolutely convergent for $\operatorname{Re} s > 1$, where $\phi_k(r) = \sum_{d|r} d^k \mu(r/d)$ is a generalized Euler function,

$$\sum_{n=1}^{\infty} \sum_{r=1}^{\infty} \frac{\widehat{f}_r(n)}{n^s r^t} = \frac{\zeta(s)\zeta(s+t-1)}{\zeta(t)} \sum_{n=1}^{\infty} \frac{F(n)}{n^t} \tag{57}$$

absolutely convergent for $\operatorname{Re} s > 1$, $\operatorname{Re} t > \max\{1, a_F\}$.

Proof. Apply (22) and (23). □

For $F = \varepsilon$ we reobtain the known formulae for the Ramanujan sum.

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