Large families of almost disjoint large subsets of $\mathbb{N}$

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Abstract. In the paper we study the question of possible cardinality of a family of almost disjoint subsets of positive integers each being large with respect to a given criterion. For example, it is shown that there are continuum many almost disjoint subsets of $\mathbb{N}$ where each set is large in both the sense of (R)-density and in the sense of the upper weighted density. On the other hand, when considering sets with positive lower weighted density, the result is completely different.

1 Introduction

There is a simple standard fact that any family of disjoint subsets of a given countable set, e.g. the set of all positive integers, can be at most countable. On the other hand, slightly relaxing the condition of disjointness so that any pair of sets in the considered family can overlap in a finite set, the possible cardinality of such a family is that of continuum. In this paper we will study the question of maximal possible cardinality of almost disjoint families of sets of integers, so that each set in the family is large with respect to some criterion.

Denote by $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$, the sets of all positive integers, rational numbers and real numbers, respectively. Two subsets of $\mathbb{N}$ are said to be *almost disjoint* if their intersection is finite. A family of subsets of $\mathbb{N}$ is said to be an *almost*

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disjoint family if it consists of pairwise almost disjoint sets. It is well known that there are almost disjoint families having cardinality of continuum. An easy way how to construct such a family may be described as follows.

- Map the set $\mathbb{N}$ by a one to one mapping b on the set $\mathbb{Q}$.
- For each real number r choose a sequence $\{s(r)\}$ of rational numbers converging to r.

Then the family $\{\{b^{-1}(s(r))\}\}_{r\in\mathbb{R}}$ consists of c almost disjoint sets. Here $c = 2^{\aleph_0}$ stands for the cardinality of continuum.

In fact, in this example the almost disjoint family is constructed on the set of rationals and all sets in the family are very small from the natural point of view the topological density, all they are nowhere dense sets.

A natural question arises: can such a large almost disjoint family consist of sets which are "large in some sense" as subsets of $\mathbb{N}$?

In this paper any family $\mathcal{F}$ of subsets of $\mathbb{N}$ satisfying the condition

$$\text{If} \quad A \in \mathcal{F} \quad \text{and} \quad A \subset B \quad \text{then} \quad B \in \mathcal{F}.$$

will be called a *family of large sets in* $\mathbb{N}$.

The purpose of this paper is to investigate the largest possible cardinality of almost disjoint families consisting of sets large in some natural sense.

2 Families of (R)-dense sets

Denote by $R(A) = \left\{\frac{a}{b}; a \in A,\ b \in A\right\}$ the *ratio set of* A and say that a set A is (R)-dense if $R(A)$ is (topologically) dense in the set $(0,\infty)$. It is manifest that the class $\mathcal{D}$ of all (R)-dense sets forms a family of large sets in $\mathbb{N}$.

Theorem 1 *There exists an almost disjoint family of* c *many (R)-dense sets.*

Proof. Let $\{J_n\}_{n=1}^{\infty}$ be a family of open subintervals of the interval $(0,\infty)$ forming a base for the ordinary topology on $(0,\infty)$. First, we will construct by induction a family $\{M_n\}_{n=1}^{\infty}$ of disjoint subsets of $\mathbb{N}$ such that each M_n is (R)-dense set. As a general rule used in the each step of the construction is the following:

Each element in the choice is different from all elements previously chosen.

Step 1. Choose $p_{11} \in \mathbb{N}$ and $q_{11} \in \mathbb{N}$ such that $\frac{p_{11}}{q_{11}} \in J_1$.

Step 2. Choose $p_{12} \in \mathbb{N}$ and $q_{12} \in \mathbb{N}$ such that $\frac{p_{12}}{q_{12}} \in J_2$. Then choose successively $p_{21} \in \mathbb{N}$ and $q_{21} \in \mathbb{N}$ such that $\frac{p_{21}}{q_{21}} \in J_1$ and $p_{22} \in \mathbb{N}$ and $q_{22} \in \mathbb{N}$ such that $\frac{p_{22}}{q_{22}} \in J_2$.

Step n. Choose for $i = 1, 2, 3, \dots n-1$ successively $p_{in} \in \mathbb{N}$ and $q_{in} \in \mathbb{N}$ such that $\frac{p_{in}}{q_{in}} \in J_n$. Then choose for $j = 1, 2, 3, \dots n$ successively $p_{nj} \in \mathbb{N}$ and $q_{nj} \in \mathbb{N}$ such that $\frac{p_{nj}}{q_{nj}} \in J_j$.

For each $n \in \mathbb{N}$ set $M_n = \{p_{n1}, q_{n1}, p_{n2}, q_{n2}, p_{n3}, q_{n3}, \dots\}$. Then, by construction, all sets M_n, $n = 1, 2, \dots$ are pairwise disjoint (R)-dense sets.

Now let $\mathcal{D}$ be any fixed almost disjoint family with cardinality of continuum. Let $D = \{d_1 < d_2 < d_3 < \dots\} \in \mathcal{D}$. Define

$$\varphi(D) = \{p_{d_1 1}, q_{d_1 1}, p_{d_2 2}, q_{d_2 2}, p_{d_3 3}, q_{d_3 3}, \dots\}.$$

We will show that for each $D \in \mathcal{D}$ the set $\varphi(D)$ is (R)-dense. Let U be an open set of real numbers. Then there exists a positive integer m such that $J_m \subset U$ and, by Step n, $\frac{p_{d_m m}}{q_{d_m m}} \in J_m \subset U$. Thus $\varphi(D)$ is an (R)-dense set.

Now let $D = \{d_1 < d_2 < d_3 < \dots\}$ and $E = \{e_1 < e_2 < e_3 < \dots\}$ be two sets in $\mathcal{D}$ and suppose that $k \in \varphi(D) \cap \varphi(E)$. Then there are positive integers m and n such that

$$k = p_{d_m m} = p_{e_n n} \qquad (\text{or } k = q_{d_m m} = q_{e_n n}),$$

consequently, by the above construction, we have $d_m = e_n$. As the sets D and E are almost disjoint, there are only finitely many such numbers $d_m = e_n$, $d_m \in D$, $e_n \in E$. Thus we have shown that $\{\varphi(D)\}_{D \in \mathcal{D}}$ is almost disjoint, so it is a required family. $\square$

3 Families of sets with large densities

For the rest of the paper let $f\colon \mathbb{N} \to (0,\infty)$. Denote by χ_A the characteristic function of a set A. For $A \subset \mathbb{N}$ define

$$\underline{d}_f(A) = \lim\inf_{x\to\infty} \frac{\sum\limits_{i\le x} f(i)\chi_A(i)}{\sum\limits_{i\le x} f(i)}, \qquad \overline{d}_f(A) = \lim\sup_{x\to\infty} \frac{\sum\limits_{i\le x} f(i)\chi_A(i)}{\sum\limits_{i\le x} f(i)}$$

$$d_f(A) = \lim_{x\to\infty} \frac{\sum\limits_{i\le x} f(i)\chi_A(i)}{\sum\limits_{i\le x} f(i)}$$

the *weighted lower* f*-density*, *weighted upper* f*-density*, and *weighted* f*-density* (if defined), respectively.

In this paper we will consider only functions f satisfying the condition

$$\sum_{n=1}^{\infty} f(n) = \infty. \tag{D}$$

Remark 1 *The most important special cases of weighted densities are those for* $f(i) \equiv 1$*, so called asymptotic density and for* $f(i) = \frac{1}{i}$*, so called logarithmic density. Also notice that the condition* (D) *guarantees that sets differing in a finite number of elements have the same upper and lower* f*-densities.*

Let $r \in (0,1]$. Then the classes $\mathbb{L}_f(r) = \{A \subset \mathbb{N}\ ;\ \underline{d}_f(A) \ge r\}$ and $\mathbb{U}_f(r) = \{A \subset \mathbb{N}\ ;\ \overline{d}_f(A) \ge r\}$ form families of large subsets in $\mathbb{N}$.

3.1 Sets with large lower f-densities

We will denote by $[x]$ the integer part of x, i.e. the largest integer less than or equal to x.

Theorem 2 *Let* f *fulfils the condition* (D) *and let* $\mathcal{S}$ *be an almost disjoint family. Then* $\sum\limits_{A\in\mathcal{S}} \underline{d}_f(A) \le 1$ *for every* $f\colon \mathbb{N} \to (0,\infty)$.

Proof. Suppose there exists an almost disjoint subfamily $\mathcal{S}$ of $\mathbb{L}_f(r)$ such that $\sum\limits_{A\in\mathcal{S}} \underline{d}_f(A) > 1$. Then it contains a finite subfamily $\mathcal{F} = \{F_1,\ F_2,\dots,F_n\}$ with $\sum\limits_{j=1}^{n} \underline{d}_f(F_j) = s > 1$. Since $\mathcal{F}$ is finite and almost disjoint, there is an integer

k_0 such that for every pair of distinct integers $i \in [1,n]$, $j \in [1,n]$ we have $F_i \cap F_j \cap [1,k_0] = \emptyset$. Let

$$G_j = F_j \cap (k_0, \infty) \quad \text{for each } j = 1, 2, \ldots n.$$

Then the sets $\{G_j\}_{j=1}^{n}$ are pairwise disjoint and

$$\underline{d}_f(G_j) = \underline{d}_f(F_j) \quad \text{for each } j = 1, 2, \ldots n.$$

Choose a positive number ε such that $s - n\varepsilon > 1$. Then there exists a positive integer m_0 such that for every $m > m_0$ and every $j = 1, 2, \ldots, n$ we have

$$\frac{\sum\limits_{i \le m} f(i)\chi_{G_j}(i)}{\sum\limits_{i \le m} f(i)} > \underline{d}(G_j) - \varepsilon$$

Denote by $G = \bigcup\limits_{j=1}^{n} G_j$ and calculate

$$1 \ge \frac{\sum\limits_{i \le m} f(i)\chi_G(i)}{\sum\limits_{i \le m} f(i)} = \sum_{j=1}^{n} \frac{\sum\limits_{i \le m} f(i)\chi_{G_j}(i)}{\sum\limits_{i \le m} f(i)} > \sum_{j=1}^{n} (\underline{d}(G_j) - \varepsilon) = s - n\varepsilon > 1,$$

a contradiction. □

The following statement is a straightforward corollary to the previous theorem.

Corollary 1 *Let $r \in (0,1]$. Then every almost disjoint subfamily of $\mathbb{L}_f(r)$ consists of at most $\left[\frac{1}{r}\right]$ sets.*

Theorem 3 *Every almost disjoint family consisting of subsets of $\mathbb{N}$ with positive lower f-densities is at most countable.*

Proof. Let $\mathcal{S}$ be an almost disjoint family of subsets of $\mathbb{N}$ and let $\underline{d}_f(S) > 0$ for every $S \in \mathcal{S}$. Then $\mathcal{S} = \bigcup\limits_{n=1}^{\infty} \left(\mathcal{S} \cap \mathbb{L}_f\left(\frac{1}{n}\right)\right)$. By Corollary 1 every set in the union on the right side is finite, so $\mathcal{S}$ is at most countable. □

Remark 2 *It is easy to find a countable disjoint family of subsets with positive lower f-densities. Thus in the class of sets with positive lower f-density the maximum cardinality of disjoint families is the same as the maximum cardinality of almost disjoint families.*

3.2 Sets with large upper f-densities

In the case of the upper f-density our considerations will substantially differ from those in the case of the lower f-density.

Theorem 4 *Let* f *satisfy the condition* (D). *Then there exists an almost disjoint family of* $\mathfrak{c}$ *many sets each of which has the upper* f*-density equal to* 1.

Proof. First notice that due to the condition (D) for every $p \in \mathbb{N}$ and for every $\varepsilon > 0$ there exists $q \in \mathbb{N}$ such that $\frac{\sum_{i=p+1}^{q} f(i)}{\sum_{i=1}^{q} f(i)} > 1 - \varepsilon$. Choose by induction a sequence $\{k_n\}_{n=1}^{\infty}$ of positive integers as follows

Step 1. Put $k_1 = 1$.

Step n. Suppose that positive integers $k_1 < k_2 < \cdots < k_{n-1}$ have already been chosen. Let k_n be the smallest positive integer

$$\text{such that} \qquad \frac{\sum_{i=k_{n-1}+1}^{k_n} f(i)}{\sum_{i=1}^{k_n} f(i)} > 1 - \frac{1}{n}. \tag{I}$$

Now for each $n \in \mathbb{N}$ put $I_n = [k_{n-1} + 1, k_n] \cap \mathbb{N}$. Let $\mathcal{D}$ be any almost disjoint family with cardinality of continuum. For every $D \in \mathcal{D}$ define $\psi(D) = \bigcup_{d \in D} I_d$. To see that $\mathcal{F} = \{\psi(D)\}_{D \in \mathcal{D}}$ is an almost disjoint family notice that the intersection of each pair of sets in $\mathcal{F}$ consists of union of finitely many finite intervals in $\mathbb{N}$, consequently it is finite. Let $D = \{d_1 < d_2 < \dots\} \in \mathcal{D}$ and calculate

$$\overline{d}_f(\psi(D)) = \limsup_{n \to \infty} \frac{\sum_{i=1}^{n} f(i)\chi_{\psi(D)}(i)}{\sum_{i=1}^{n} f(i)} \geq \limsup_{n \to \infty} \frac{\sum_{i=1}^{k_{d_n}} f(i)\chi_{\psi(D)}(i)}{\sum_{i=1}^{k_{d_n}} f(i)} \geq$$

$$\limsup_{n\to\infty} \frac{\sum\limits_{i=k_{d_{n-1}}+1}^{k_{d_n}} f(i)}{\sum\limits_{i=1}^{k_{d_n}} f(i)} = 1,$$

thus $\mathcal{F}$ is a required family. □

Remark 3 *Putting* $r = 1$ *in Corollary 1 for lower* f*-densities and comparing its statement to that of Theorem 4 for upper* f*-densities shows a huge difference between the lower and upper* f*-densities relative to the question in our investigation.*

Remark 4 *In [1] it is proved that every subset of* $\mathbb{N}$ *with the upper asymptotic density equal to* 1 *is necessary (R)-dense. By this result, Theorem 1 is a corollary to Theorem 4, even we can say more.*

Theorem 5 *Let a function* f *fulfil the condition* (D)*. Then there exists an almost disjoint family of* $\mathfrak{c}$ *many sets each of which is (R)-dense and at the same time it has the upper* f*-density equal to* 1*.*

Proof. In the proof of this theorem we will follow the same idea as in the previous one. The only difference is in the induction step where the condition (I) should be changed to the stronger one: Let k_n be the smallest positive integer

$$\text{greater than } n(k_{n-1}+1) \text{ such that } \quad \frac{\sum\limits_{i=k_{n-1}+1}^{k_n} f(i)}{\sum\limits_{i=1}^{k_n} f(i)} > 1 - \frac{1}{n}. \tag{II}$$

Using the previous proof, we need only to prove that each set in the family $\mathcal{F} = \{\psi(D)\}_{D\in\mathcal{D}}$ is (R)-dense. Let $D = \{d_1, d_2, \dots\} \in \mathcal{D}$ and let $1 < a < b$ be given real numbers. Choose an integer $d_l \in D$ so that

$$b < d_l \qquad \text{and} \qquad \frac{1}{d_l} < b - a. \tag{1}$$

Condition (II) guarantees $k_{d_l} > d_l(k_{d_l - 1} + 1)$, consequently

$$\frac{k_{d_l}}{k_{d_l-1}+1} > d_l > b. \tag{2}$$

Clearly $k_{d_{l-1}} + 1 \geq d_l$, thus we also have

$$\frac{1}{k_{d_l - 1} + 1} \leq \frac{1}{d_l}. \tag{3}$$

As $I_{d_l} = [k_{d_l - 1} + 1, k_{d_l}] \cap \mathbb{N} \subset \psi(D)$, by (1), (2) and (3) the set

$$\left\{ \frac{k_{d_l - 1} + 1}{k_{d_l - 1} + 1} < \frac{k_{d_l - 1} + 2}{k_{d_l - 1} + 1} < \cdots < \frac{k_{d_l} - 1}{k_{d_l - 1} + 1} < \frac{k_{d_l}}{k_{d_l - 1} + 1} \right\} \subset R(\psi(D))$$

intersects (a, b), thus $\psi(D)$ is (R)-dense. $\square$

Remark 5 *In the case when* $\sum_{n=1}^{\infty} f(n) < \infty$ *the statement corresponding to that in Theorem 4 does not hold. In this case the statement corresponding to that in Theorem 3 for lower* f*-densities takes place.*

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References

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